Journal of Sciences, Islamic Republic of Iran 19(2): 145-152 (2008) http://jsciences.ut.ac.ir
University of Tehran, ISSN 1016-1104

On Ideals Which Have the Weakly Insertion of Factors Property

E. Hashemi*

Department of Mathematics, Shahrood University of Thechnology, Shahrood, P.O. Box 316-3619995161, Islamic Republic of Iran

Abstract

A one-sided ideal I of a ring R has the insertion of factors property (or simply, IFP) if $ab \in I$ implies r$aRb \subseteq I$ for $a,b \in R$. We say a one-sided ideal I of R has the weakly IFP if for each $a,b,r \in R$, $ab \in I$ implies $(arb)^n \in I$, for some non-negative integer n. We give some examples of ideals which have the weakly IFP but have not the IFP. Connections between ideals of R which have the IFP and related ideals of some ring extensions are also shown.

Keywords: Insertion of factors property; $(\alpha,\delta)-$compatible ideals; $\alpha-$rigid ideals; Ore extensions; Skew Laurent polynomials ring

0. Introduction

Throughout this paper R denotes an associative ring with identity. $R[x;\alpha,\delta]$ will stands for the Ore extension of R, where α is an endomorphism and δ an $\alpha-$ derivation of R, that is, δ is an additive map such that $\delta(ab)=\delta(a)b+\alpha(a)\delta(b)$ for all $a,b \in R$. Recall from [14] that a one-sided ideal I of a ring R has the insertion of factors property (or simply, IFP) if $ab \in I$ implies $aRb \subseteq I$ for $a,b \in R$. (H.E. Bell [2] in 1973 introduced this notion for $I=0$). Observe that every completely semiprime ideal I (*i.e.*, $a^2 \in I$ implies $a \in I$) of R has the IFP [14, Lemma 3.2(a)]. If $I=0$ has the IFP, then we say R has the IFP. A ring R is called reduced if it has no non-zero nilpotent element. By [5], reduced rings have the IFP. If R has the IFP, then it is Abelian (*i.e.*, all idempotents are central).

Recall that an ideal I of R is called $\alpha-$ ideal if $\alpha(I) \subseteq I$; I is called $\alpha-$ invariant if $\alpha^{-1}(I)=I$; I is called $\delta-$ideal if $\delta(I) \subseteq I$; I is called $(\alpha,\delta)-$ ideal if it is both α and $\delta-$ ideal.

According to Hong, Kawak and Rizvi [4], for an endomorphism α of a ring R, a $\alpha-$ ideal I is called to be $\alpha-$rigid if $a\alpha(a) \in I$ implies $a \in I$ for $a \in R$. Hong, Kawak and Rizvi [4] studied connections between $\alpha-$ rigid ideals of R and related ideals of some ring extensions. Motivated by the above facts, for an endomorphism α of a ring R, we define $\alpha-$compatible ideals in R which are a generalization of $\alpha-$ rigid ideals. For an ideal I, we say that I is compatible ideal if for each $a,b \in R$, $ab \in I \Leftrightarrow a\alpha(b) \in I$. Moreover, I is said to be $\delta-$ compatible ideal if for each $a,b \in R$, $ab \in I \Rightarrow a\delta(b) \in I$. If I is both $\alpha-$compatible and $\delta-$compatible, we say that I is a $(\alpha,\delta)-$compatible ideal. If $I=0$ is $(\alpha,\delta)-$ compatible ideal, we say that

* Corresponding author, Tel.: +98(0273)3332012, Fax: +98(0273)3332012, E-mail: eb_hashemi@shahroodut.ac.ir

R is a compatible ring. The definition is quite natural, in the light of its similarity with the notion of $\alpha-$ rigid ideals, where in Proposition 2.2, we will show that I is a $\alpha-$ rigid ideal if and only if I is $\alpha-$ compatible ideal and completely semiprime.

In this paper, we will show that for each $n \geq 2$, there exists a non-zero ideal of the $n \times n$ upper triangular matrix ring over the ring of integers Z such that has not IFP. Connections between ideals of R which has the IFP and related ideals of some ring extensions are also shown. In section 2, we will show that: (1) If I is a $(\alpha,\delta)-$compatible ideal of R and has the IFP, then ideal $I[x;\alpha,\delta]$ of $R[x;\alpha,\delta]$ has the weakly IFP. (2) For a monomorphism α of R, if I is $\alpha-$ compatible ideal of R and has the IFP, then ideal $I[x,x^{-1};\alpha]$ of skew Laurent polynomials ring $R[x,x^{-1};\alpha]$ has the weakly IFP. As a corollary, we show that if R is $(\alpha,\delta)-$ compatible ring and has the IFP, then $R[x;\alpha,\delta]$ has the weakly IFP. Also, for a monomorphism α of R, if R is $\alpha-$compatible ring and has the IFP, then $R[x,x^{-1};\alpha]$ has the weakly IFP.

In [13], Li Liang, Limin Wang and Zhongkui Liu show that if R is a $\alpha-$compatible ring and has the IFP, then $R[x;\alpha]$ has the weakly IFP. For a ring R, we denote by $T_n(R)$ the $n-$by$-n$ upper triangular matrix ring over R. Clearly

$$R_n(R) = \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}$$

is a subring of $T_n(R)$.

1. Examples

For an ideal I of R, put $\sqrt{I} = \{a \in R \mid a^n \in I$ for some non-negative integers $n \geq 0\}$.

Definition 1.1. A one-sided ideal I of a ring R has the weakly insertion of factors property (or simply, weakly IFP) if $ab \in I$ implies $arb \in \sqrt{I}$ for each $r \in R$. If $I = 0$ has the weakly IFP, then we say R has the weakly IFP.

Clearly, if I has the IFP, then it has the weakly IFP. In the following we will see the converse is not true.

Example 1.2. Let

$$J = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \mid a_{ij} \in 2pZ \right\},$$

where p is a prime number and Z is the set of integers. Then

$$\begin{pmatrix} p & p & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & p \end{pmatrix} = \begin{pmatrix} 0 & 0 & 4p \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in J,$$

but

$$\begin{pmatrix} p & p & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & p \end{pmatrix} = \begin{pmatrix} 0 & 0 & 7p \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \notin J.$$

Hence J has not IFP, but J has the weakly IFP, by Corollary 1.12.

By a similar way as used in Example 1.2, we can construct numerous ideals of $T_n(Z)$ such that has weakly IFP but have not IFP, for $n \geq 2$.

Example 1.3. Let

$$J = \left\{ \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & 0 & 0 & a_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid a_{ij} \in 2pZ \right\}$$

be an ideal of $R_4(Z)$, where $p \neq 2$ is a prime number and Z is the set of integers. Then

$$\begin{pmatrix} 0 & p & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 2p \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in J,$$

but

$$\begin{pmatrix} 0 & P & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & P \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 3p^3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \notin J.$$

Hence J has not IFP, but J has weakly IFP, by Corollary 1.6.

By a similar way as used in Example 1.3, we can construct numerous ideals of $R_n(Z)$ such that have weakly IFP but have not IFP, for $n \geq 4$.

Lemma 1.4. Let

$$J = \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \mid a \in I, a_{ij} \in I_{ij}, 1 \leq i < j \leq n \right\},$$

where I, I_{ij} are ideals of R, such that $I \subseteq I_{ij} \subseteq I_{is}$ for $1 \leq i < j \leq s \leq n$ and $I_{sj} \subseteq I_{ij}$ for $j = 3, \cdots, n$, $2 \leq i \leq s \leq n$. Then J is an ideal of $R_n(R)$.

Proof. It is clear.

In Propositions 1.5, 1.8 and Theorem 1.6, I and J are ideals that mentioned in Lemma 1.4.

Proposition 1.5. Let

$$A = \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \in R_n(R)$$

such that $a^k \in I$ for a non-negative integer k. Then $A^{nk} \in J$.

Proof. We proceed by induction on n. Let $n = 2$. For a positive integer k, $A^k = \begin{pmatrix} a^k & b_{12} \\ 0 & a^k \end{pmatrix}$ and that $A^{2k} = \begin{pmatrix} a^{2k} & a^k b_{12} + b_{12} a^k \\ 0 & a^{2k} \end{pmatrix}$. Hence $A^{2k} \in J$, since $a^{2k}, a^k b_{12} + b_{12} a^k \in I$. Now, let

$$A = \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \in R_n(R)$$

such that $a^k \in I$ for a non-negative integer k. Consider

$$A^{(n-1)k} = \begin{pmatrix} a^{(n-1)k} & b_{12} & \cdots & b_{1n} \\ 0 & a^{(n-1)k} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a^{(n-1)k} \end{pmatrix}$$

and

$$A^k = \begin{pmatrix} a^k & c_{12} & \cdots & c_{1n} \\ 0 & a^k & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a^k \end{pmatrix}.$$

By the induction hypothesis all b_{ij} 's, except b_{1n}, are in I. Let $x = a^k b_{1n} + c_{12} b_{2n} + \cdots + c_{1n} a^{(n-1)k}$. Hence

$$A^{nk} = \begin{pmatrix} a^{nk} & y_{12} & \cdots & y_{1n-1} & x \\ 0 & a^{nk} & \cdots & y_{2n-1} & y_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a^{nk} & y_{n-1n} \\ 0 & 0 & \cdots & 0 & a^{nk} \end{pmatrix} \in J,$$

since a^{nk}, x and all y_{ij} 's are in I.

Theorem 1.6. Let I has the weakly IFP. Then J has the weakly IFP.

Proof. Let

$$A = \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix}, \ B = \begin{pmatrix} b & b_{12} & \cdots & b_{1n} \\ 0 & b & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b \end{pmatrix}$$

and

$$C = \begin{pmatrix} c & c_{12} & \cdots & c_{1n} \\ 0 & c & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c \end{pmatrix} \in R_n(R)$$

such that $AB \in J$. Then $ab \in I$ and that $(acb)^k \in I$ for some non-negative integer k, since I has the weakly IFP. Thus $(ACB)^{nk} \in J$, by Proposition 1.5. Therefore J has the weakly IFP.

Corollary 1.7. Let R has the weakly IFP. Then $R_n(R)$ has the weakly IFP, for each $n \geq 2$.

Proof. It follows from Theorem 1.6.

Proposition 1.8. Let J has the weakly IFP. Then I has the weakly IFP.

Proof. It is clear.

Corollary 1.9. Let $R_n(R)$ has the weakly IFP, for some n. Then R has the weakly IFP.

Proof. It follows from Proposition 1.8.

Lemma 1.10. Let

$$J=\left\{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}\end{pmatrix} \mid a_{ij}\in I_{ij}, 1\le i\le j\le n\right\},$$

where I_{ij} are ideals of R, such that $I_{ij}\subseteq I_{is}$ for $1\le i\le j\le s\le n$ and $I_{sj}\subseteq I_{ij}$ for $j=1,\dots,n$, $1\le i\le s\le n$. Then J is an ideal of $T_n(R)$.

Proof. It is clear.

In Propositions 1.11, 1.14 and Theorem 1.12, I_{ii} are ideals that mentioned in Lemma 1.10, for $1\le i\le n$.

Proposition 1.11. Let

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}\end{pmatrix}\in T_n(R)$$

such that $a_{ii}{}^k\in I_{ii}$ for some non-negative integer k and $i=1,\dots,n$. Then $(A^{2k+1})^{n-1}\in J$.

Proof. We proceed by induction on n. For $n=2$, let $A=\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22}\end{pmatrix}$. Since $A^{2k+1}=\begin{pmatrix} a_{11}{}^{2k+1} & x \\ 0 & a_{11}{}^{2k+1}\end{pmatrix}$, where $x=\sum a_{11}{}^{i}a_{12}a_{22}{}^{j}$, $i+j=2k$, $i,j\ge 0$, we have $A^{2k+1}\in J$. Now, assume $n\ge 3$ and $A\in T_n(R)$. Consider

$$(A^{2k+1})^{n-2}=\begin{pmatrix} a_{11}{}^{(2k+1)(n-2)} & b_{12} & \cdots & b_{1n} \\ 0 & a_{22}{}^{(2k+1)(n-2)} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}{}^{(2k+1)(n-2)}\end{pmatrix}$$

and

$$A^{2k+1}=\begin{pmatrix} a_{11}{}^{2k+1} & c_{12} & \cdots & c_{1n} \\ 0 & a_{22}{}^{2k+1} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}{}^{2k+1}\end{pmatrix}.$$

By the induction hypothesis all b_{ij}'s, except b_{1n}, are in I. Hence (1,n)-entry of $(A^{2k+1})^{n-1}$ is $x=a_{11}{}^{(2k+1)}b_{1n}+c_{12}b_{12}+\cdots+c_{1n-1}b_{n-1n}+c_{1n}a_{nn}{}^{(2k+1)(n-2)}\in I$, since $a_{11}{}^{(2k+1)},a_{nn}{}^{(2k+1)},b_{2n},\cdots,b_{n-1n}\in I$. Therefore $(A^{2k+1})^{n-1}\in J$.

Theorem 1.12. If each $I_{ii}, 1\le i\le n$ has the weakly IFP, then J has the weakly IFP.

Proof. It follows from Proposition 1.11.

Corollary 1.13. Let R has the weakly IFP. Then $T_n(R)$ has the weakly IFP for each $n\ge 2$.

Proof. It follows from Theorem 1.12.

Proposition 1.14. If J has the weakly IFP, then each $I_{ii}, 1\le i\le n$, has the weakly IFP.

Proof. It is clear.

Corollary 1.15. Let $T_n(R)$ has the weakly IFP for some $n\ge 2$. Then R has the weakly IFP.

2. Extensions of Ideals Which Have the IFP

In this section α is an endomorphism and δ an α-derivation of R. For an ideal I, we say that I is a $\alpha-$compatible ideal if for each $a,b\in R$, $ab\in I\Leftrightarrow a\alpha(b)\in I$. Moreover, I is said to be $\delta-$compatible ideal if for each $a,b\in R$, $ab\in I\Rightarrow a\delta(b)\in I$. If I is both $\alpha-$compatible and $\delta-$compatible, we say that I is a $(\alpha,\delta)-$compatible ideal. If $I=0$ is both $(\alpha,\delta)-$compatible ideal, we say that R is a $(\alpha,\delta)-$compatible ring.

In [5, Example 2], the authors show that there exists a non-zero ideal I of a ring R such that has IFP but ideal $I[x]$ of $R[x]$ has not IFP. We will show that if I has the IFP then $I[x]$ has the weakly IFP. More generally, we will show that: (1) If I is a $(\alpha,\delta)-$compatible ideal of R and has the IFP, then ideal

$I[x;\alpha,\delta]$ of $R[x;\alpha,\delta]$ has the weakly IFP. (2) For a monomorphism α of R, if I is $\alpha-$compatible ideal of R and has the IFP, then ideal $I[x,x^{-1};\alpha]$ of skew Laurent polynomials ring $R[x,x^{-1};\alpha]$ has the weakly IFP.

For non-empty subsets A,B of R and $r\in R$, put $AB=\{ab\mid a\in A, b\in B\}$, $A^0=\{1\}$ and $rA=\{ra\mid a\in A\}$.

The following proposition extends [3, Lemma 2.1].

Proposition 2.1. Let I be a $(\alpha,\delta)-$compatible ideal of R and $a,b\in R$.

(i) If $ab\in I$, then $a\alpha^n(b)\in I$, $\alpha^n(a)b\in I$ for each positive integer n. Conversely, if $a\alpha^k(b)\in I$ or $\alpha^k(a)b\in I$ for some positive integer k, then $ab\in I$.

(ii) If $ab\in I$, then $\alpha^m(a)\delta^n(b), \delta^n(a)\alpha^m(b)\in I$ for each non-negative integers m,n.

Proof. (i) If $ab\in I$, then $\alpha^n(a)\alpha^n(b)\in I$, since I is $\alpha-$ideal. Hence $\alpha^n(a)b\in I$, since I is $\alpha-$compatible. Conversely, let $\alpha^k(a)b\in I$. Then $\alpha^k(a)\alpha^k(b)\in I$, since I is $\alpha-$compatible. Hence $\alpha^k(ab)\in I$ and that $ab\in I$, since I is $\alpha-$compatible.

(ii) It is enough to show that $\delta(a)\alpha(b)\in I$. If $ab\in I$, then by (i) and $\delta-$ compatibility of I, $\alpha(a)\delta(b)\in I$. Hence $\delta(a)b=\delta(ab)-\alpha(a)\delta(b)\in I$. Thus $\delta(a)b\in I$ and that $\delta(a)\alpha(b)\in I$, since I is $\alpha-$compatible.

Proposition 2.2. Let R be a ring, I be an ideal of R and $\alpha: R\to R$ be an endomorphism of R. Then the following conditions are equivalent:

(1) I is a $\alpha-$ rigid ideal of R;

(2) I is $\alpha-$compatible, semiprime and has the IFP;

(3) I is $\alpha-$compatible and completely semiprime.

If δ is a $\alpha-$ derivation of R, then the following are equivalent:

(4) I is a $\alpha-$ rigid $\delta-$ ideal of R;

(5) I is $(\alpha,\delta)-$compatible, semiprime and has the IFP;

(6) I is $(\alpha,\delta)-$compatible and completely semiprime.

Proof. $(1)\Rightarrow(2)$. It follows from [4, Propositions 2.2 and 2.4].

$(2)\Rightarrow(1)$. Let $a\alpha(a)\in I$. Then $a^2\in I$, since I is $\alpha-$compatible. Hence $aRa\subseteq I$, since I has the IFP. Thus $a\in I$, since I is semiprime. Similarly we can prove $(1)\Rightarrow(3)$.

$(4)\Rightarrow(6)$. By $(1)\Rightarrow(3)$, I is $\alpha-$compatible and completely semiprime. We show that $a\alpha(b)\in I$, when $ab\in I$. If $ab\in I$, then $\delta(ab)=\delta(a)b+\alpha(a)\delta(b)\in\delta(I)\subseteq I$. Thus $(\alpha(a)\delta(b))^2=\delta(ab)\alpha(a)\delta(b)-\delta(a)b\alpha(a)\delta(b)\in I$, because $\delta(ab), b\alpha(a)\in I$. Since I is completely semiprime, we have $\alpha(a)\delta(b)\in I$ and so $a\delta(b)\in I$, by Proposition 2.1.

$(6)\Rightarrow(4)$ and $(4)\Rightarrow(5)$ are clear.

Note that there exists a ring R for which every non-zero proper ideals are $\alpha-$compatible. For example, consider the ring $R=\begin{pmatrix}F & F\\ 0 & F\end{pmatrix}$, where F is a field, and the endomorphism α of R is defined by $\alpha(\begin{pmatrix}a & b\\ 0 & c\end{pmatrix})=\begin{pmatrix}a & 0\\ 0 & c\end{pmatrix}$ for $a,b,c\in F$.

Lemma 2.3. Let I be a $(\alpha,\delta)-$compatible ideal of R and has the IFP. If $(ab)^k\in I$, for some $k\geq 0$, then $(a\alpha(b))^k, (a\delta(b))^k\in I$.

Proof. Since I is $\alpha-$compatible and $(ab)^k=(ab)\cdots(ab)\in I$, we have $a\alpha(b)\alpha(ab\cdots ab)=a\alpha(bab\cdots ab)\in I$. Hence $a\alpha(b)\alpha(ab\cdots ab)\in I$, since I is $\alpha-$compatible. Now, $a\alpha(b)a\alpha(b\cdots ab)\in I$ and that $a\alpha(b)a\alpha(b)\alpha(ab\cdots ab)\in I$. Continuing this procedure yields $(a\alpha(b))^k\in I$. Since I is $\delta-$compatible and $(ab)^k=(ab)\cdots(ab)\in I$, we have $a\delta(bab\cdots ab)=a\delta(b)(ab\cdots ab)+a\alpha(b)\delta(ab\cdots ab)\in I$. Since $a\alpha(b)(ab\cdots ab)\in I$ and I is $\delta-$compatible, we have $a\alpha(b)\delta(ab\cdots ab)\in I$. Thus $a\delta(b)(ab\cdots ab)\in I$. Continuing this procedure yields $(a\delta(b))^k\in I$.

Lemma 2.4. If I is a $(\alpha,\delta)-$compatible ideal of R, then $I[x;\alpha,\delta]$ is an ideal of $R[x;\alpha,\delta]$.

Proof. It follows from Lemma 2.3.

Lemma 2.5. Let I be an ideal of R and has the IFP. Then $\sqrt{I}$ is an ideal of R and has the IFP.

Proof. Let $a, b \in \sqrt{I}$. So $a^n, b^m \in I$ for some $m, n \geq 0$. Hence $(a+b)^{m+n+1} = \sum (a^{i_1} b^{j_1}) \cdots (a^{i_{m+n+1}} b^{j_{m+n+1}})$, such that $i_k + j_k = 1$, $0 \leq i_k \leq 1$, $0 \leq j_k \leq 1$. It can be easily checked that a more than n or b more than m in $(a^{i_1} b^{j_1}) \cdots (a^{i_{m+n+1}} b^{j_{m+n+1}})$. Since $a^n, b^m \in I$ and I has the IFP, we have $(a^{i_1} b^{j_1}) \cdots (a^{i_{m+n+1}} b^{j_{m+n+1}}) \in I$. Therefore $(a+b)^{m+n+1} \in I$ and $(a+b) \in \sqrt{I}$. Clearly $\sqrt{I}$ has the IFP.

Lemma 2.6. Let I be a $(\alpha, \delta)-$compatible ideal of R and has the IFP. Then $\sqrt{I}$ is a $(\alpha, \delta)-$compatible ideal of R.

Proof. It follows from Lemmas 2.3, 2.5.

Remark. Given α and δ as above and integers $0 \leq i \leq j$ and $a \in R$, let us write $f_i^{\ j}$ for the set of all "words" in α and δ in which there are i factors of α and $j-i$ factors of δ. For instance, $f_j^{\ j}(a) = \{\alpha^j(a)\}$, $f_0^{\ j}(a) = \{\delta^j(a)\}$ and $f_{j-1}^{\ \ j} = \{\alpha^{j-1}\delta(a), \alpha^{j-2}\delta\alpha(a), \cdots, \delta\alpha^{j-1}(a)\}$.

Lemma 2.7. Let I be a $(\alpha, \delta)-$compatible ideal of R and has the IFP. Let $f(x) = a_0 + \cdots + a_n x^n$, $g(x) = b_0 + \cdots + b_m x^n \in R[x; \alpha, \delta]$ with $f(x)g(x) \in I[x; \alpha, \delta]$. Then $a_i b_j \in \sqrt{I}$ for each i, j.

Proof. Note that $f(x)g(x) = \sum_{i=0}^{n} \sum_{j=0}^{m} (a_i x^i)(b_j x^j)$. Then $a_n \alpha^n(b_m) \in I$, since it is the leading coefficient of $f(x)g(x)$. Hence $a_n b_m \in I$, since I is $\alpha-$compatible. Thus $a_n f_i^{\ j}(b_m) \subseteq I$, for each $0 \leq i \leq j$, by Proposition 2.1. Since the coefficient of x^{m+n-1} is $a_n \alpha^n(b_{m-1}) + a_{n-1}\alpha^{n-1}(b_m) + a_n \delta(\alpha^{n-1}(b_m)) \in I$ and $a_n \delta(\alpha^{n-1}(b_m)) \in I$, we have $a_n \alpha^n(b_{m-1}) + a_{n-1}\alpha^{n-1}(b_m) \in I$. Hence $a_n \alpha^n(b_{m-1}) b_m + a_{n-1}\alpha^{n-1}(b_m) b_m \in I$ and that $a_{n-1}\alpha^{n-1}(b_m) b_m \in \sqrt{I}$, since $a_n \alpha^n(b_{m-1}) b_m \in \sqrt{I}$. Thus $a_{n-1} b_m \in \sqrt{I}$, by Lemmas 2.5 and 2.6, and that $a_n b_{m-1} \in \sqrt{I}$. Consequently, $a_{n-1} f_i^{\ j}(b_m) \cup a_n f_i^{\ j}(b_{m-1}) \subseteq \sqrt{I}$, for each $0 \leq i \leq j$. Coefficient of x^{m+n-2} is $a_n \alpha^n(b_{m-2}) + a_{n-1}\alpha^{n-1}(b_{m-1}) + a_{n-2}\alpha^{n-2}(b_m) + t$, where t is a sum of elements of $\bigcup_{0 \leq i \leq j} [a_n f_i^{\ j}(b_m) \cup a_{n-1} f_i^{\ j}(b_m) \cup a_n f_i^{\ j}(b_{m-1})]$. By a similar way as above, we can show that $a_n b_{m-2}, a_{n-1} b_{m-1}, a_{n-2} b_m \in \sqrt{I}$. Continuing this process, we can prove $a_i b_j \in \sqrt{I}$ for each i, j.

Lemma 2.8. Let I be a $(\alpha, \delta)-$compatible ideal of and has the IFP. Let $f(x) = a_0 + \cdots + a_n x^n \in R[x; \alpha, \delta]$. If $a_0, \cdots, a_n \in \sqrt{I}$, then $f(x) \in \sqrt{I[x; \alpha, \delta]}$.

Proof. Suppose that $a_i^{m_i} \in I$, for $i = 0, \cdots, n$. Let $k = m_0 + \cdots + m_n + 1$. Then $(f(x))^k = \sum (a_0^{\ i_{01}} (a_1 x)^{i_{11}} \cdots (a_n x^n)^{i_{n1}}) \cdots (a_0^{\ i_{0k}} (a_1 x)^{i_{1k}} \cdots (a_n x^n)^{i_{nk}})$, $i_{0r} + \cdots + i_{nr} = 1$, $r = 1, \cdots k$, $0 \leq i_{rs} \leq 1$. Each coefficient of $(a_0^{\ i_{01}} (a_1 x)^{i_{11}} \cdots (a_n x^n)^{i_{n1}}) \cdots (a_0^{\ i_{0k}} (a_1 x)^{i_{1k}} \cdots (a_n x^n)^{i_{nk}})$ is a sum of such elements $\gamma \in ((f_{r_{01}}^{\ \ s_{01}}(a_0))^{i_{01}} \cdots (f_{r_{n1}}^{\ \ s_{n1}}(a_n))^{i_{n1}}) \cdots ((f_{r_{0k}}^{\ \ s_{0k}}(a_0))^{i_{0k}} \cdots (f_{r_{nk}}^{\ \ s_{nk}}(a_n))^{i_{nk}})$. It can be easily checked that there exists $a_k \in \{a_0, \cdots, a_n\}$ such that $i_{t1} + \cdots + i_{tk} \geq m_t$. Since $a_t^{\ m_t} \in I$ and I has the IFP and is $(\alpha, \delta)-$compatible, we have $\gamma \in I$. Thus each coefficient of $(f(x))^k$ belong to I. Therefore $f(x) \in \sqrt{I[x; \alpha, \delta]}$.

Theorem 2.9. Let I be a $(\alpha, \delta)-$compatible ideal of R and has the IFP. Then $I[x; \alpha, \delta]$ has the weakly IFP.

Proof. Let $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x; \alpha, \delta]$ such that $f(x)g(x) \in I[x; \alpha, \delta]$ and let $h(x) = \sum_{s=0}^{k} c_s x^s \in R[x; \alpha, \delta]$ be any element. By Lemma 2.7, $a_i b_j \in \sqrt{I}$ for each i, j. Hence $a_i f_p^{\ q}(c_s) f_r^{\ t}(b_j) \subseteq \sqrt{I}$ for each i, j, $0 \leq p \leq q$, $0 \leq r \leq t$, by Lemma 2.5 and Proposition 2.1. Note that each coefficient of $f(x)h(x)g(x)$ is a sum of such elements $\gamma \in \bigcup_{0 \leq p \leq q} \bigcup_{0 \leq r \leq t} a_i f_p^{\ q}(c_s) f_r^{\ t}(b_j)$. Thus $f(x)h(x)h(x) \in \sqrt{I[x; \alpha, \delta]}$ by Lemma 2.8. This means that $I[x; \alpha, \delta]$ has the weakly IFP.

Corollary 2.10. Let R be a $(\alpha, \delta)-$compatible ring and has the IFP. Then $R[x; \alpha, \delta]$ has the weakly IFP.

Corollary 2.11. ([13], Theorem 3.1) Let R be a $\alpha-$compatible ring and has the IFP. Then $R[x;\alpha]$ has the weakly IFP.

Recall that for a ring R with an injective ring endomorphism $\alpha: R\to R$, $R[x;\alpha]$ is the Ore extension of R. The set $\{x^i\}_{i\geq 0}$ is easily seen to be a left Ore subset of $R[x;\alpha]$, so that one can localize $R[x;\alpha]$ and form the skew Laurent polynomials ring $R[x,x^{-1};\alpha]$. Elements of $R[x,x^{-1};\alpha]$ are finite sum of elements of the form $x^{-j}rx^i$ where $r\in R$ and i,j are non-negative integers. Multiplication is subject to $xr=\alpha(r)x$ and $rx^{-1}=x^{-1}\alpha(r)$ for all $r\in R$.

Now we consider D.A. Jordan's construction of the ring $A(R,\alpha)$ (See [8], for more details). Let $A(R,\alpha)$ or A be the subset $\{x^{-i}rx^i \mid r\in R, i\geq 0\}$ of the skew Laurent polynomials ring $R[x,x^{-1};\alpha]$. For each $j\geq 0$, $x^{-i}rx^i=x^{-(i+j)}rx^{(i+j)}$. It follows that the set of all such elements forms a subring of $R[x,x^{-1};\alpha]$ with $x^{-i}rx^i+x^{-j}sx^j=x^{-(i+j)}(\alpha^j(r)+\alpha^i(s))x^{(i+j)}$ and $(x^{-i}rx^i)(x^{-j}sx^j)=x^{-(i+j)}\alpha^j(r)\alpha^i(s)x^{(i+j)}$ for $r,s\in R$ and $i,j\geq 0$. Note that α is actually an automorphism of $A(R,\alpha)$. We have $R[x,x^{-1};\alpha]\cong A[x,x^{-1};\alpha]$, by way of an isomorphism which maps $x^{-i}rx^j$ to $\alpha^{-i}(r)x^{j-i}$. For an $\alpha-$ ideal I of R, put $\Delta(I)=\bigcup_{i\geq 0}x^{-i}Ix^i$. Hence $\Delta(I)$ is $\alpha-$ ideal of A. The constructions $I\to\Delta(I)$, $J\to J\bigcap R$ are inverses, so there is an order-preserving bijection between the sets of $\alpha-$ invariant ideals of R and $\alpha-$ invariant ideals of A. For an ideal I of R, put $J_i=\{r\in R \mid x^{-i}rx^i\in J\}$ for $i\geq 0$.

Theorem 2.12. Let α be a monomorphism of a ring R.

(i) If I is a $\alpha-$compatible ideal of R and has the (weakly) IFP, then $\Delta(I)$ is a $\alpha-$compatible ideal of A and has the (weakly) IFP.

(ii) If J is a $\alpha-$compatible ideal of A and has the (weakly) IFP, then $J=\Delta(J_0)$ and J_0 is a $\alpha-$compatible ideal of R and has the (weakly) IFP.

Proof. (i) Let I be a $\alpha-$compatible ideal of R. Hence $\Delta(I)$ is an ideal of A. Now, let $(x^{-i}rx^i)(x^{-j}sx^j)\in\Delta(I)$. Hence $x^{-(i+j)}\alpha^j(r)\alpha^i(s)x^{(i+j)}\in\Delta(I)$ and that $\alpha^j(r)\alpha^i(s)\in I$. Thus $\alpha^j(r)\alpha^{i+1}(s)\in I$, since I is $\alpha-$compatible. Consequently $(x^{-i}rx^i)\alpha(x^{-j}sx^j)\in\Delta(I)$. Therefore $\Delta(I)$ is $\alpha-$compatible. Now, assume $(x^{-i}rx^i)(x^{-j}sx^j)\in\Delta(I)$. Then $\alpha^j(r)\alpha^i(s)\in I$. Hence $\alpha^{j+t}(r)\alpha^{i+t}(s)\in I$ for each $t\geq 0$. Since I has the weakly IFP, so for each $a\in R$ and each $t\geq 0$, there exists $n>0$ such that $(\alpha^{j+t}(r)\alpha^{i+j}(a)\alpha^{i+t}(s))^n\in I$. Therefore $((x^{-i}rx^i)(x^{-t}ax^t)(x^{-j}sx^j))^n\in\Delta(I)$. Consequently $\Delta(I)$ has the weakly IFP.

(ii) Let $r\in J_0$. Then $\alpha^n(r)\in J_0$ for each $n\geq 0$. Hence $\alpha^n(x^{-n}rx^n)=r\in J$ for each $n\geq 0$. Thus $x^{-n}rx^n\in J$, since J is $\alpha-$compatible. Therefore $\Delta(J_0)\subseteq J_0$. Now, let $x^{-m}rx^m\in J$. Then $\alpha^m(x^{-m}rx^m)\in J$ and that $r\in J$, since J is $\alpha-$compatible. Thus $J\subseteq\Delta(J_0)$. Consequently, $J=\Delta(J_0)$. Clearly J_0 has the weakly IFP.

Note that if I is a $\alpha-$ ideal of R, then $I[x,x^{-1};\alpha]$ is an ideal of the skew Laurent polynomials ring $R[x,x^{-1};\alpha]$. By a similar way as in the proof of Lemmas 2.7, 2.8 and Theorem 2.9 one can prove the following results.

Lemma 2.13. Let I be a $\alpha-$compatible ideal of R and has the IFP. Let α be an automorphism of R. Let $f(x)=a_rx^r+\cdots+a_nx^n$, $g(x)=b_sx^s+\cdots+b_mx^m\in R[x,x^{-1};\alpha]$ with $f(x)g(x)\in I[x,x^{-1};\alpha]$. Then $a_ib_j\in\sqrt{I}$ for each i,j.

Lemma 2.14. Let I be a $\alpha-$compatible ideal of R and has the IFP. Let α be an automorphism of R. Let $f(x)=a_rx^r+\cdots+a_nx^n\in R[x,x^{-1};\alpha]$. If $a_r,\cdots,a_n\in\sqrt{I}$, then $f(x)\in\sqrt{I[x,x^{-1};\alpha]}$.

Proposition 2.15. Let I be a $\alpha-$compatible ideal of R and has the IFP. Let α be an automorphism of R. Then $I[x,x^{-1};\alpha]$ has the weakly IFP.

Theorem 2.16. Let I be a $\alpha-$compatible ideal of R and has the IFP. Let α be a monomorphism of R. Then $I[x,x^{-1};\alpha]$ has the weakly IFP.

Proof. Since I is $\alpha-$compatible and has the IFP, so $\Delta(I)$ is $\alpha-$compatible and has the IFP, by Theorem

2.12. Since $I[x, x^{-1}; \alpha] \cong \Delta(I)[x, x^{-1}; \alpha]$, $R[x, x^{-1}; \alpha] \cong A[x, x^{-1}; \alpha]$ and α is an automorphism of A, the result follows from Proposition 2.15.

Corollary 2.17. Let R be a $\alpha-$compatible ring and has the IFP. Let α be a monomorphism of a ring R. Then the skew Laurent polynomials ring $R[x, x^{-1}; \alpha]$ has the weakly IFP.

Proof. It follows from Theorem 2.16.

Corollary 2.18. If R has the IFP, then Laurent polynomials ring $R[x, x^{-1}]$ has the weakly IFP.

Acknowledgements

The author thanks the referee for his/her helpful suggestions. This research is supported by the Shahrood University of Technology.

References

1. Anderson D.D. and Camillo V. Armendariz rings and Gaussian rings. *Comm. Algebra*, **26**(7): 2265-2275 (1998).
2. Bell H. E. Near-Rings in which each element is a power of itself. *Bull. Australian Math. Soc.*, **2**: 363-368 (1973).
3. Hashemi E. and Moussavi A. Polynomial extensions of quasi-Baer rings. *Acta Math. Hungar.*, **107**(3): 207-224 (2005).
4. Hong C.Y., Kwak T.K., and Rizvi S.T. Rigid Ideals and Radicals of Ore Extensions. *Algebra Colloq.*, **12**(3): 399-412 (2005).
5. Hong C.Y., Kim N.Y., Kwak T.K., and Lee Y. Extensions of zip rings. *J. Pure Appl. Algebra*, **195**: 231-242 (2005).
6. Huh C., Lee Y., and Smoktunowicz A. Armendariz rings and semicommutative rings. *Comm. Algebra*, **30**(2): 751-761 (2002).
7. Huh C., Kim H.K., and Lee Y. P.P.-rings and generalized P.P.-rings. *J. Pure Appl. Algebra*, **167**: 37-52 (2002).
8. Jordan D.A. Bijective extension of injective ring endomorphisms. *J. London Math. Soc.*, **35**(2): 435-488 (1982).
9. Kim N.K. and Lee Y. Armendariz rings and reduced rings. *J. Algebra*, **223**: 477-488 (2000).
10. Kim N.K. and Lee Y. Extensions of reversible rings. *J. Pure Appl. Algebra*, **185**: 207-223 (2003).
11. Krempa J. Some examples of reduced rings. *Algebra Colloq.*, **3**(4): 289-300 (1996).
12. Krempa J. and Neiwieczerzal D. Rings in which annihilators are ideal and their application to semigroup ring. *Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys.*, **25**: 851-856 (1977).
13. Liang L., Wang L., and Liu Z. On a generalization of semicommutative rings, to appear in Taiwanese Journal of Mathematics.
14. Mason G. Reflexive ideals. *Comm. Algebra*, **9**: 1709-1724 (1981).
15. Passman D.S. *The Algebraic Structure of Group Rings*. Wiley, New York (1977).