Weighted Convolution Measure Algebras Characterized by Convolution Algebras

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Abstract

The weighted semigroup algebra $M_b(S, w)$ is studied via its identification with $M_b(S)$ together with a weighted algebra product $*$ so that $(M_b(S, w), *)$ is isometrically isomorphic to $(M_b(S), *)$. This identification enables us to study the relation between regularity and amenability of $M_b(S, w)$ and $M_b(S)$, and improve some old results from discrete to general case.

Keywords: Weighted semigroup algebra; Arens product; Amenable Banach algebra

1. Introduction and Preliminary Results

For a locally compact Hausdorff topological semigroup $S$, let $M_b(S)$ be the Banach algebra of all complex regular Borel measures on $S$, and let $w : S \rightarrow (0, \infty)$ be a Borel-measurable weight function, such that $w^{-1}$ is bounded on compacta. Then $M_b(S, w) \cong C_0(S, w)^*$, where $C_0(S, w) = \{ f : f/w \in C_0(S) \}$ was defined in [6]. In this paper we consider $M_b(S, w)$ as the Banach algebra $M_b(S)$ together with a weighted algebra product $*$ so that $(M_b(S, w), *, ) \cong (M_b(S), *)^*$ (hereafter, "\cong" is used for Banach algebra isometrical isomorphism), where

$$\mu^*v(E) = \int_S \int_S \chi_E(xy) \Omega(x, y) \tilde{w}(x) \tilde{w}(y) \ d\mu(x) \ d\nu(y),$$

for each Borel subset $E$ in $S$, and

$\Omega(x, y) = \frac{w(xy)}{w(x)w(y)}$ for $\mu, \nu \in M_b(S)$ and $x, y \in S$.

We define $M_b(S, w)$ so that the Riesz representation theorem holds for it, that is: $M_b(S, w) \cong C_0(S, w)^*$. It should be noted that the elements of $M_b(S, w)$ need not to be a measure. Let $M_b^+(S, w)$ be the set of all positive regular measures $\mu$ on $S$ such that $\mu + v \in M_b(S)$, for every Borel set $E$. Now define the equivalence relation "\&" on $M_b^+(S, w) \times M_b^+(S, w)$ by $(\mu, v_1) \& (\mu, v_2)$ if and only if $\mu + v_2 = \mu + v_1$. We denote $[\mu, v]$ as the equivalence class of $(\mu, v)$. The set of all $[\mu, v]$ such that $\mu, v \in M_b^+(S, w)$ is denoted by $M_b(S, w)$. Note that $I(f) = \int_S f(x) \ d\mu(x)$. No...
\[ \int \int f(\mu_{x},y) d\mu(x) d\nu(y) \quad \text{for} \quad f \in C_0(S,w) \] defines a linear functional on \( C_0(S,w) \) which corresponds to the equivalence class \([\mu,\nu]\), see [6]. In other words \( M_b(S,w) \cong C_0(S,w) \).

Let \([\mu_1,v_1],[\mu_2,v_2]\) be in \( M_b(S,w) \). We define convolution product "*" so that \( (M_b(S,w),*)\) with the norm \( \|\mu,\nu\|_w = \|\mu v - w\| \) turns into a Banach algebra, where \([\mu_1,v_1]*[\mu_2,v_2] = [\mu_1*\mu_2 + v_1*v_2, \mu_1*v_2 + v_1*\mu_2]\). Also we define the weighted convolution product \(*_w\) by:

\[
[\mu_1,v_1] *_w [\mu_2,v_2] = [\mu_1*_w \mu_2 + v_1*_w v_2, \mu_1*_w v_2 + v_1*_w \mu_2].
\]

Some authors consider \( M(S,w) \) as the set of all complex measures \( \mu \) such that \( \mu w \in M_b(S) \), for example see [2], [3] and [4]. It has been shown that, in this case \( M(S,w) \) need not be complete, see [5]; Nevertheless, as the next lemma demonstrates, if \( w \geq 1 \) then there is no difference between \( M(S,w) \) and \( M_b(S,w) \); moreover each of them can be considered as a subspace of \( M_b(S) \).

**Lemma (1.1).** Let \( w \) be a weight function on \( S \) with \( w \geq 1 \). Then \( M(S,w) \subseteq M_b(S) \), and \( (M_b(S,w),*), \|\| \) is a normed algebra. If \( w \) is bounded on compacta, then \( M(S,w) \) is norm-dense in \( M_b(S) \).

**Proof.** Since \( w \geq 1 \), for \( \mu \in M(S,w) \) we have \( \|\mu\| = \|\mu\|_w \), which implies \( M(S,w) \subseteq M_b(S) \). One may verify that the mapping \( [\mu,\nu] \mapsto \mu * \nu \) defines an isometric isomorphism from \( (M_b(S,w),*,\|\|) \) onto \( (M(S,w),*),\|\| \) isometrically.

Clearly for each \( \eta \in M(S,w) \subseteq M_b(S) \) there exist unique \( \eta^+ \) and \( \eta^- \) such that \( \eta = \eta^+ - \eta^- \) and \( \eta^+ \perp \eta^- \). Since \( \eta^+ << \eta \) and \( \eta^- << \eta \), where \( \|\eta\| = \|\eta^+\| + \|\eta^-\| \), the map is onto. For every \( \mu,\nu \in M_b(S,w) \), it becomes clear that

\[ \|\mu*v\| \leq \|\mu\|_w \|v\|_w \leq \|\mu\| \|v\|, \]

which means \( \mu * v \in M_b(S,w) \).

Also, the inequality

\[ \|\mu * v\| \leq \|\mu\| \|v\| \]

implies that \( (M_b(S,w),*,\|\|) \) is a normed algebra.

**Example (1.2).** (i) Let \( S = (\mathbb{R}^+,+) \) and \( w(x) = e^{-x} \) for \( x \in S \). Since \( w \) is multiplicative \( \mu w = \mu \). Now for the Lebesgue measure \( \mu \) on \( \mathbb{R}^+ \), \( \mu \in M_b(S,w) \) but

\[ \min(\mu,\nu) = \mu(S) \mu(S) = \infty. \]

(ii) Let \( S = (\mathbb{Z},+) \) and \( w(n) = 1 + \frac{1}{n} \), for \( n \in \mathbb{Z} \). Then \( l_1(\mathbb{Z},w) \subseteq l_1(\mathbb{Z},w) \), so \( (l_1(\mathbb{Z},w),\|\|_1) \) is not complete. In fact \( \sum_{n=1}^{\infty} \frac{1}{n^2} \delta_{x} \in l_1(\mathbb{Z}), l_1(\mathbb{Z},w) \).

**Lemma (1.3).** Let \( \omega_1 \) and \( \omega_2 \) be weight functions on \( S \). Then \( \omega_1 * \omega_2 = \omega_2 \) if and only if \( \Omega_1 = \Omega_2 \).

**Proof.** Assume that \( \omega_1 * \omega_2 = \omega_2 \). Then \( \Omega_1(x,y) \delta_{\omega_1} = \delta_{\omega_1} \delta_{\omega_2} = \Omega_2(x,y) \delta_{\omega_1} \delta_{\omega_2} = \Omega_2(x,y) \delta_{\omega_1} \delta_{\omega_2} \), for all \( x,y \in S \). Thus

\[ \Omega_1(x,y) = \Omega_2(x,y) \delta_{\omega_1} \]

Conversely, let \( \Omega_1 = \Omega_2 \). Then \( \omega_1 \) and \( \omega_2 \) coincide on \( l_1(S) \) which is weak \( *\)-dense in \( M_b(S) \). Thus \( \omega_1 * \omega_2 = \omega_2 \), for \( M_b(S) \).

We now state our key lemma as follows:

**Lemma (1.4).** Let \( \omega_1 \) and \( \omega_2 \) be two weight functions on \( S \). Then:

(i) \( (M_b(S,w_1),*_{w_1},\|\|_{w_1}) \) is a Banach algebra.

(ii) The map

\[ \phi: (M_b(S,w_1),*_{w_1},\|\|_{w_1}) \rightarrow (M_b(S,w_2),*_{w_2},\|\|_{w_2}) \]

is an isometric isomorphism.
defined by \( \phi([\mu, \nu]) \mapsto \left[ \frac{w_1}{w_2} \mu, \frac{w_1}{w_2} \nu \right] \) is an isometric algebra isomorphism.

**Proof.** Parts (i) and (ii) follow from the fact that for every \( \mu, \nu \in M_b^+(S, w_1) \)

\[
\left\| \mu_{w_2}^* v \right\|_{w_2} = \mu_{w_2} v (w_1)
\]

\[
= \int_S w_1(x y) \Omega_2(x, y) d \mu(x) d \nu(y)
\]

\[
\leq \int_S w_1(x) w_1(y) d \mu(x) d \nu(y) = \left\| \mu \right\|_{w_1} \left\| \nu \right\|_{w_1},
\]

Also:

\[
\frac{w_1}{w_2} (\mu_{w_2}^* v) = \left( \frac{w_1}{w_2} \mu \right)_{w_1} \left( \frac{w_1}{w_2} \nu \right),
\]

Corollary (1.5). Let \( w \) be a weight function on \( S \), then:

(i) \( (M_b(S, w), *)_w \| \|_w \) and \( (M_b(S, w), *)_w \| \|_w \)

are Banach algebras and \( (M_b(S, w), *)_w \| \|_w \equiv (M_b(S), *)_w \| \|_w \).

(ii) If \( w \) is multiplicative, then \( (M_b(S, w), *)_w \| \|_w \equiv (M_b(S), *)_w \| \|_w \).

**Proof.** Part (i) follows trivially from 1.4 and (ii) is obtained from (i) and the fact that, if \( w \) is multiplicative then \( \Omega = 1 \) and so \( *_{w^w} = *_w \).

Two weights \( w_1 \) and \( w_2 \) on \( S \) are said to be equivalent (in symbol, \( w_1 \sim w_2 \) if \( \alpha w_2 \leq w_1 \leq \beta w_2 \) for some \( \alpha, \beta \in \mathbb{R}^+ \).

**Lemma (1.6).** For every pair of weight functions \( w_1 \) and \( w_2 \) on \( S \), \( w_1 \sim w_2 \) if and only if \( M_b(S, w_1) = M_b(S, w_2) \) and the norms \( \| \|_{w_1} \) and \( \| \|_{w_2} \) are equivalent.

**Proof.** If \( w_1 \sim w_2 \), then there exist \( \alpha, \beta \in \mathbb{R}^+ \) such that \( \alpha w_2 \leq w_1 \leq \beta w_2 \). Hence \( \mu \in M_b^+(S, w_1) \) if and only if \( \mu \in M_b^+(S, w_2) \), indeed, \( \alpha \mu \|_{w_1} \leq \mu \|_{w_1} \leq \beta \mu \|_{w_1} \), so \( M_b(S, w_1) = M_b(S, w_2) \).

Conversely: there exist \( \alpha, \beta \in \mathbb{R}^+ \) such that \( \alpha B_{w_1} \subseteq B_{w_2} \subseteq \beta B_{w_1} \), where \( B_w \) is the unit ball of \( M_b(S, w) \). In particular, \( \frac{\delta_x}{w_1(x)} \in \beta B_{w_1} \) and \( \alpha \frac{\delta_x}{w_2(x)} \in B_{w_2} \) for all \( x \in S \). Thus \( \left\| \frac{\delta_x}{w_1(x)} \right\|_{w_1} \leq \beta \) and \( \left\| \alpha \frac{\delta_x}{w_2(x)} \right\|_{w_1} \leq 1 \); i.e. \( \frac{1}{\beta} w_2(x) \leq w_1(x) \) \leq \frac{1}{\alpha} w_2(x).

It should be mentioned that all properties involving the Arens regularity and amenability of Banach algebras are unchanged if we move to an equivalent norm on it.

2. The Relation between \( M_b(S, w) \) and \( M_b(S) \) and Their Second Duals

Baker and Rejali in [1] studied the relation between the (Arens) regularity of \( l_1(S) \) and \( l_1(S, w) \). They showed that \( l_1(S, w) \) is regular, whenever \( l_1(S) \) is regular. Let \( S = (\mathbb{Z}, +) \) and \( w(n) = n + 1, \) for \( n \in \mathbb{Z} \). Then \( l_1(S, w) \) is regular, but \( l_1(S) \) is not regular, see [1].

In this section, we generalize the above result for non-discrete semigroups.

**Lemma (2.1).** \( (M_b(S, w)^{**}, @) \equiv (M_b(S)^{**}, @) \), where \( @, \oplus \) and \( @_w \) are the first Arens products induced by \( * \) and \( *_w \), respectively.

**Proof.** As we have seen earlier (see corollary 1.5), \( \phi : (M_b(S, w), *) \to (M_b(S), *)_w \) which is defined by \( (\mu, \nu) \mapsto \mu \nu - w \) is an isometric algebra-isomorphism.

It is not difficult to verify that the second adjoint \( \phi^* \) is an isometric algebra-isomorphism from \( (M_b(S)^{**}, @) \) onto \( (M_b(S)^{**}, @) \).

J.C.S. Wong [8] was shown that the dual \( M_b(S)^* \) is isometrically order isomorphic to the space \( GL(S) \) of all generalized functions on \( S \). An element \( f = (f_\mu)_{\mu \in M_b(S)} \) in the product linear space

\( \Pi(L_w : \mu \in M_b(S)) \) is called a generalized function on \( S \), if the following conditions are satisfied:

(a) \( \| f \| := \text{Sup} \left\{ \| f_{\mu} \|_{w, \infty} : \mu \in M_b(S) \} < \infty. \)

(b) If \( \mu, \nu \in M_b(S) \) and \( \mu \ll \nu \), then \( f_\mu = f_\nu \).
Let $\eta \in M_b(S \wedge w)$, there exist unique $\eta^+$, $\eta^- \in M_b(S \wedge w)$ such that $\eta = [\eta^+, \eta^-]$ and $\eta^+ \perp \eta^-$. Put $[\eta] = \eta^+ + \eta^-$. One can define $f = (f_\eta) \in GL(S \wedge w)$ if and only if $f_\eta \in L_\infty([\eta]w)$ such that $\|f_\eta\|_\infty : \eta \in M_b(S \wedge w) < \infty$ and if $\eta, \xi \in M_b(S \wedge w)$ and $|\eta| << |\xi|$, then $f_\eta = f_\xi$ $[\eta] = \eta^+ + \eta^-$, where $\|f_\eta\|_\infty : \eta \in M_b(S \wedge w)$ see [6].

Let $GL(S \wedge w)$ be the space of all $w$-generalized functions on $S$. Then it has been shown in [4], (see also [8], [6]) that $M_b(S \wedge w)$ is isometrically isomorphic to $GL(S \wedge w)$ whose duality is given in [6], explicitly.

**Lemma (2.2).** For every $F, G \in M_b(S \wedge w)$ and $f \in GL(S \wedge w)$ there exists an $\eta \in M_b(S \wedge w)$ such that $F \circ \eta \wedge w G(f) = (F \circ \eta \wedge w G) \Omega(f_\eta)$; where $f_\eta(x, y) = f_\eta(xy)$ for $x, y \in S$. The same equality holds for the second Arens product.

**Proof.** Let $\{\mu_\alpha\}$ and $\{\nu_\beta\}$ be two nets in $M_b(S \wedge w)$ such that $\mu_\alpha \wedge w F$ and $\nu_\beta \wedge w G$. Then there exists a subnet $\{\mu_\alpha\}$ [resp. $\{\nu_\beta\}$] of net $\{\mu_\alpha\}$ [resp. $\{\nu_\beta\}$] so that

$$F \circ \eta \wedge w G(f) = \lim_\alpha \lim_\beta F_\Delta \mu_\alpha \circ \nu_\beta = \lim_\alpha \lim_\beta F_\Delta \mu_\alpha \circ \nu_\beta = \lim_\alpha \lim_\beta F_\Delta \mu_\alpha \circ \nu_\beta \Omega(f_\eta) = \lim_\alpha \lim_\beta (F_\Delta \eta \wedge w G) \Omega(f_\eta)$$

where $\eta \in M_b(S \wedge w)$ is so that $\mu_\alpha \wedge w \eta < \eta$, for all $n$ and $m$, for example $\eta = \sum \sum n, m 1 \frac{1}{2}^m \mu_\alpha \wedge w \nu_\beta$. Note that, if $f = (f_\eta) \in GL(S \wedge w)$ then for every $\xi \in M_b(S \wedge w), f_\xi(\xi) = \sum f_\xi(z) d_\xi(z)$. Also $f_\xi(\xi_1) = f(\xi_2)$, whenever $\xi_1 < \xi_2$, see [8].

We now state the main result of this section. Hereafter, our mean by $M_b(S \wedge w)$ is the Banach algebra $(M_b(S \wedge w), \eta \wedge w \eta, \|\|)$, which is identified with $(M_b(S \wedge w), \eta \wedge w \eta, \|\|)$.

**Theorem (2.3).** Let $M_b(S \wedge w)$ be regular. Then $M_b(S \wedge w)$ is regular.

**Proof.** Let $\ast$ and $\ast w$ denote the second Arens products induced by $\ast$ and $\ast w$, respectively. If $M_b(S \wedge w)$ is regular, then for every $F \ast G \in M_b(S \wedge w)$, and $f \in GL(S \wedge w)$ by the above lemma we have

$$F \ast w G(f) = F \ast w G(\Omega f_\eta), \text{ for some } \eta \in M_b(S \wedge w)$$

$$= F \ast G(\Omega f_\eta)$$

$$= F \ast w G(f).$$

Therefore $F \ast w G = F \ast w G \ast w$, so $M_b(S \wedge w)$ is regular.

Suppose $M_b(S \wedge w)$ is regular. Let $A, B \in M_b(S \wedge w)$. Then by a similar argument as is used in (2.2), one can show that:

$$A \ast \omega w B(h) = A \ast \omega w (\Omega h_\xi), \text{ for some } \xi \in M_b(S \wedge w)$$

$$= A \ast B(\Omega h_\xi)$$

$$= A \ast \omega w B(h),$$

where $\ast \omega w$ [resp. $\ast w$] is the first [resp. second] Arens product.

In [1], the authors showed that $l_1(S \wedge w)$ is regular, whenever $\Omega$ is 0-cluster, i.e. for all sequences $(x_n), (y_m)$ of distinct elements in $S$,

$$\lim_n \lim_m \Omega(x_n, y_m) = 0 = \lim_m \lim_n \Omega(x_n, y_m),$$

Whenever the both iterated limits exist.

The next statement extend this for non-discrete case.

**Proposition (2.4).** Let $\Omega$ be 0-cluster. Then $M_b(S \wedge w)$ is regular.

**Proof.** Let $A \ast B \in M_b(S \wedge w)$ and $h \in M_b(S \wedge w)$. Then there exist sequences $(F_n), (G_m)$ in $M_b(S \wedge w)$ such that

$$A \ast \omega w B(h) = \lim_n \lim_m F_n \ast \omega w G_m(h).$$

The assumption of $\Omega$ being 0-cluster implies that $A \ast \omega w B = 0 = A \ast w B$, for all $A \ast B \in M_b(S \wedge w)$. Indeed, as in the proof of 2.2, there exists a $\xi \in M_b(S \wedge w)$ such that $A \ast \omega w B(h) = A \ast B(\Omega h_\xi)$.
= 0, for each $h \in M_b(S)^{**}$.

**Corollary (2.5).** Let $\Omega$ be 0-cluster. Then $M_\theta(S,w)$ is regular.

**Proof.** This is an immediate consequence of 2.4.

**Theorem (2.6).** Let $M_\theta(S,w)$ [resp. $M_\theta(S,w)^{**}$] be amenable and $w \geq 1$. Then $M_\delta(S)$ [resp. $M_\delta(S)^{**}$] is amenable.

**Proof.** Let $\phi : (M_\theta(S), \ast_w) \to (M_\theta(S), \ast)$ be defined by $\mu \mapsto \mu / w$. Then $w^{-1} \ast \mu \ast v / w = \mu / w \ast v / w$, for $\mu, v \in M_\theta(S)$, i.e. $\phi$ is a continuous homomorphism with $\phi(M_\theta(S)) = M_\delta(S,w)$ which is dense in $M_\delta(S)$. Therefore amenability of $M_\delta(S,w)$ implies that of $M_\delta(S)$, (see [7], for the notion of amenable Banach algebra). The same argument may be used for the second dual $\phi^{**}$ of $\phi$ to show that, amenability of $M_\delta(S,w)^{**}$ implies that of $M_\delta(S)^{**}$.

**Remark (2.7).** If we define $M_\mu^{\prime \prime}(S,w) = \{ [\mu, v] \in M_\mu(S,w) : wv - v w \in M_\mu(S), \} \cap M_\mu^{\prime \prime}(S)$, (where $M_\mu(S) = \{ \mu \in M_\mu(S) : \mu = \pi^\ast \mu \}$ is weak-continuous), see [3]) then it has been shown that $M_\mu^{\prime \prime}(S,w)$ is a closed solid left ideal of $M_\delta(S,w)$; (for more details see [5]). In particular, $(M_\mu^{\prime \prime}(S,w), \ast, \infty, \infty)$ is a Banach algebra. Similar to what we have seen in corollary (1.5), it can be identified with $(M_\mu(S), \ast, \infty, \infty)$ and also in $(M_\mu^{\prime \prime}(S,w), \ast, \infty) \cong (M_\mu^{\prime \prime}(S), \ast, \infty)$, see Lemma (2.1). So, one can repeat the results (2.3), (2.4), (2.5) and (2.6) with $M_\mu^{\prime \prime}$ in stead of $M_\mu$, which of course gives a new proof for corollary 9 of [6].

**Question.** Does the conclusion of 2.6 hold without $w \geq 1$?

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**References**