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Weighted Convolution Measure Algebras Characterized by Convolution Algebras

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Abstract

The weighted semigroup algebra $M_b(S, w)$ is studied via its identification with $M_b(S)$ together with a weighted algebra product $*_w$ so that $(M_b(S, w), *)$ is isometrically isomorphic to $(M_b(S), *_w)$. This identification enables us to study the relation between regularity and amenability of $M_b(S, w)$ and $M_b(S)$, and improve some old results from discrete to general case.

Keywords: Weighted semigroup algebra; Arens product; Amenable Banach algebra

1. Introduction and Preliminary Results

For a locally compact Hausdorff topological semigroup S, let $M_b(S)$ be the Banach algebra of all complex regular Borel measures on S, and let $w: S \to (0,\infty)$ be a Borel-measurable weight function, such that w^{-1} is bounded on compacta. Then $M_b(S,w) \cong C_0(S,w)^*$, where $C_0(S,w) = \{f : f/w \in C_0(S)\}$ was defined in [6]. In this paper we consider $M_b(S, w)$ as the Banach algebra $M_b(S)$ together with a weighted algebra product $*_w$ so that $(M_b(S,w), *, \| \|_w) \cong (M_b(S), *_w, \| \|)$ (hereafter, "$\cong$" is used for Banach algebra isometrical isomorphism), where

$$\mu *_w \nu(E) = \int_S \int_S \chi_E(xy)\Omega(x,y)\,d\mu(x)\,d\nu(y),$$

$$\mu * \nu(E) = \int_S \int_S \chi_E(xy)\,d\mu(x)\,d\nu(y),$$

for each Borel subset E in S, and $\Omega(x,y) = \frac{w(xy)}{w(x)w(y)}$ for $\mu, \nu \in M_b(S)$ and $x, y \in S$.

We define $M_b(S,w)$ so that the Riesz representation theorem holds for it, that is: $M_b(S,w) \cong C_0(S,w)^*$. It should be noted that the elements of $M_b(S,w)$ need not to be a measure. Let $M_b{}^+(S,w)$ be the set of all positive regular measures μ on S such that $\mu w \in M_b(S)^+$, where for every Borel set E, $\mu w(E) = \int_E w(x)\,d\mu(x)$. Now define the equivalence relation "~" on $M_b{}^+(S,w) \times M_b{}^+(S,w)$ by $(\mu_1, \nu_1) \sim (\mu_2, \nu_2)$ if and only if $\mu_1 + \nu_2 = \mu_2 + \nu_1$. We denote $[\mu, \nu]$ as the equivalence class of (μ, ν). The set of all $[\mu, \nu]$ such that $\mu, \nu \in M_b{}^+(S,w)$ is denoted by $M_b(S,w)$. Note that $I(f) =$

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$\int_S\int_S f(xy)d\mu(x)d\nu(y)$ for $f \in C_0(S,w)$ defines a linear functional on $C_0(S,w)$ which corresponds to the equivalence class $[\mu,\nu]$, see [6]. In other words $M_b(S,w) \cong C_0(S,w)^*$.

Let $[\mu_1,\nu_1],[\mu_2,\nu_2]$ be in $M_b(S, w)$. We define convolution product "$*$" so that ($M_b(S, w)$, $*$) with the norm $\|[\mu,\nu]\|_w = \|\mu w - \nu w\|$ turns into a Banach algebra, where $[\mu_1,\nu_1]*[\mu_2,\nu_2] = [\mu_1*\mu_2+\nu_1*\nu_2,\ \mu_1*\nu_2+\nu_1*\mu_2]$. Also we define the weighted convolution product $*_w$ by:

$$[\mu_1,\nu_1]*_w[\mu_2,\nu_2]$$
$$=[\mu_1*_w\mu_2+\nu_1*_w\nu_2,\mu_1*_w\nu_2+\nu_1*_w\mu_2].$$

Some authors consider *M(S, w)* as the set of all complex measures μ such that $\mu w \in M_b(S)$, for example see [2], [3] and [4]. It has been shown that, in this case *M (S, w)* need not be complete in general, see [5]; Nevertheless, as the next lemma demonstrates, if $w \geq 1$ then there is no difference between *M (S, w)* and $M_b(S, w)$; moreover each of them can be considered as a subspace of $M_b(S)$.

Lemma (1.1). Let w be a weight function on S with $w \geq 1$. Then $M(S,w) \subseteq M_b(S)$, and $(M_b(S,w),*,\|\ \|_w) \cong (M(S,w),*,\|\ \|_w)$. Moreover, $(M(S,w), *_w, \|\ \|)$ is a normed algebra. If w is bounded on compacta, then $M(S,w)$ is norm-dense in $M_b(S)$.

Proof. Since $w \geq 1$, for $\mu \in M(S,w)$ we have $\|\mu\| \leq \|\mu\|_w$, which implies $M(S, w) \subseteq M_b(S)$. One may verify that the mapping $[\mu,\nu] \mapsto \mu-\nu$ defines an isometric isomorphism from ($M_b(S, w)$, $*$, $\|\ \|_w$) onto ($M(S, w)$, $*$, $\|\ \|_w$).

Clearly for each $\eta \in M(S,w) \subseteq M_b(S)$ there exist unique $\eta^+,\eta^- \in M_b(S)$ such that $\eta = \eta^+ - \eta^-$ and $\eta^+ \perp \eta^-$. Since $\eta^+ << |\eta|$ and $\overline{\eta} << |\eta|$, where $|\eta| = \eta^+ + \eta^-$ and $M(S,w)$ is solid, so η^+, $\eta^- \in M^+(S,w)$. Hence the map is onto. For every $\mu,\nu \in M_b{}^+(S,w)$, it is obvious that

$$\|\mu *_w \nu\| \leq \|\mu *_w \nu\|_w \leq \|\mu\|_w \|\nu\|_w,$$

which means $\mu *_w \nu \in M_b(S,w)$.

Also, the inequality

$$\|\mu *_w \nu\| = \mu *_w \nu(S)$$
$$= \int_S\int_S \Omega(x,y)d\mu(x)d\nu(y) \leq \|\mu\|.\|\nu\|$$

Implies that ($M_b(S,w)$, $*_w$, $\|\ \|$) is a normed algebra. Let $\mu \in M_b(S)$ has compact support and w is bounded on compacta. Then $\mu \in M(S,w)$, so $M(S,w)$ is dense in $M_b(S)$.

Example (1.2). (i) Let $S=(\mathbb{R}^+, +)$ and $w(x)=e^{-x}$ for $x \in S$. Since w is multiplicative $*_w = *$. Now for the Lebesgue measure μ on $\mathbb{R}^+$, $\mu \in M_b^+(S,w)$ but $\|\mu *_w \mu\| = \mu(S)\mu(S) = \infty$. Note that $w \leq 1$.

(ii) Let $S=(\mathbb{Z}, +)$ and $w(n)=1+|n|$, for $n \in \mathbb{Z}$. Then $l_1(\mathbb{Z}, w) \subsetneq l_1(\mathbb{Z})$, so $(l_1(\mathbb{Z}, w), \|.\|_1)$ is not complete. In fact $\sum_{n=1}^{\infty}\frac{1}{n^2}\delta_n \in l_1(\mathbb{Z}) \setminus l_1(\mathbb{Z}, w)$.

Lemma (1.3). Let w_1 and w_2 be weight functions on S. Then $*_{w_1} = *_{w_2}$ if and only if $\Omega_1 = \Omega_2$.

Proof. Assume that $*_{w_1} = *_{w_2}$. Then $\Omega_1(x,y)\delta_{xy} = \delta_x *_{w_1} \delta_y = \delta_x *_{w_2} \delta_y = \Omega_2(x,y)\delta_{xy}$, for all $x,y \in S$. Thus

$$\Omega_1(x,y) = \Omega_1(x,y)\delta_{xy}(S)$$
$$= \Omega_2(x,y)\delta_{xy}(S)$$
$$= \Omega_2(x,y)$$

Conversely, let $\Omega_1 = \Omega_2$. Then $*_{w_1}$ and $*_{w2}$ coincide on $l_1(S)$ which is weak $*$-dense in $M_b(S)$. Thus $*_{w_1} = *_{w_2}$, for $M_b(S)$.

We now state our key lemma as follows:

Lemma (1.4). Let w_1 and w_2 be two weight functions on S. Then:

(i) ($M_b(S, w_1)$, $*_{w_2}$, $\|\ \|_{w_1}$) is a Banach algebra.

(ii) The map

$$\phi : (M_b(S,w_1), *_{w_2}, \|\ \|_{w_1})$$
$$\to (M_b(S,w_2), *_{w_1}, \|\ \|_{w_2})$$

defined by $\phi([\mu,\nu])\mapsto[\frac{w_1}{w_2}\mu,\frac{w_1}{w_2}\nu]$ is an isometric algebra isomorphism.

Proof. Parts (i) and (ii) follow from the fact that for every $\mu,\nu\in M_b^{+}(S,w_1)$

$$\|\mu *_{w_2} \nu\|_{w_1} = \mu *_{w_2} \nu(w_1)$$

$$=\int_S\int_S w_1(xy)\Omega_2(x,y)d\mu(x)d\nu(y)$$

$$\le\int_S\int_S w_1(x)w_1(y)d\mu(x)d\nu(y)=\|\mu\|_{w_1}.\|\nu\|_{w_1},$$

Also:

$$\frac{w_1}{w_2}(\mu *_{w_2} \nu)=(\frac{w_1}{w_2}\mu)*_{w_1}(\frac{w_1}{w_2}\nu)\text{, and}$$

$$\left\|[\frac{w_1}{w_2}\mu,\frac{w_1}{w_2}\nu]\right\|_{w_2}=\|w_1\mu-w_2\nu\|=\|[\mu,\nu]\|_{w_1}.$$

Corollary (1.5). Let w be a weight function on S, then:

(i) $(M_b(S,w),*_w,\|\ \|_w)$ and $(M_b(S,w),*,\|\ \|_w)$ are Banach algebras and $(M_b(S,w),*,\|\ \|_w)\cong(M_b(S),*_w,\|\ \|)$.

(ii) If w is multiplicative, then $(M_b(S,w),*,\|\ \|_w)\cong(M_b(S),*,\|\ \|)$.

Proof. Part (i) follows trivially from 1.4 and (ii) is obtained from (i) and the fact that, if w is multiplicative then $\Omega=1$ and so $*_w=*$.

Two weights w_1 and w_2 on S are said to be equivalent (in symbol, $w_1\sim w_2$) if $\alpha w_2\le w_1\le\beta w_2$ for some $\alpha,\beta\in\mathbb{R}^{+}$.

Lemma (1.6). For every pair of weight functions w_1 and w_2 on S, $w_1\sim w_2$ if and only if $M_b(S,w_1)=M_b(S,w_2)$ and the norms $\|\ \|_{w_1}$ and $\|\ \|_{w_2}$ are equivalent.

Proof. If $w_1\sim w_2$, then there exist $\alpha,\beta\in\mathbb{R}^{+}$ such that $\alpha w_2\le w_1\le\beta w_2$. Hence $\mu\in M_b^{+}(S,w_1)$ if and only if $\mu\in M_b^{+}(S,w_2)$, indeed, $\alpha\|\mu\|_{w_2}\le\|\mu\|_{w_1}\le\beta\|\mu\|_{w_2}$, so $M_b(S,w_1)=M_b(S,w_2)$.

Conversely: there exist $\alpha,\beta\in\mathbb{R}^{+}$ such that $\alpha B_{w_2}\subseteq B_{w_1}\subseteq\beta B_{w_2}$, where B_w is the unit ball of $M_b(S,w)$. In particular, $\frac{\delta_x}{w_1(x)}\in\beta B_{w_2}$ and $\alpha\frac{\delta_x}{w_2(x)}\in B_{w_1}$ for all $x\in S$. Thus $\left\|\frac{\delta_x}{w_1(x)}\right\|_{w_2}\le\beta$ and $\left\|\frac{\alpha\delta_x}{w_2(x)}\right\|_{w_1}\le 1$; *i.e.* $\frac{1}{\beta}w_2(x)\le w_1(x)\le\frac{1}{\alpha}w_2(x)$.

It should be mentioned that all properties involving the Arens regularity and amenability of Banach algebras are unchanged if we move to an equivalent norm on it.

2. The Relation between $M_b(S,w)$ and $M_b(S)$ and Their Second Duals

Baker and Rejali in [1] studied the relation between the (Arens) regularity of $l_1(S)$ and $l_1(S,w)$. They showed that $l_1(S,w)$ is regular, whenever $l_1(S)$ is regular. Let $S=(\mathbb{Z},+)$ and $w(n)=|n|+1$, for $n\in\mathbb{Z}$. Then $l_1(S,w)$ is regular, but $l_1(S)$ is not regular, see [1].

In this section, we generalize the above result for non-discrete semigroups.

Lemma (2.1). $(M_b(S,w)^{**},\circledast)\cong(M_b(S)^{**},\circledast_w)$, where $\circledast$ and $\circledast_w$ are the first Arens products induced by $*$ and $*_w$, respectively.

Proof. As we have seen earlier (see corollary 1.5), $\phi:(M_b(S,w),*)\to(M_b(S),*_w)$ which is defined by $[\mu,\nu]\mapsto\mu w-\nu w$ is an isometric algebra-isomorphism. It is not difficult to verify that the second adjoint ϕ^{**} is an isometric algebra-isomorphism from $(M_b(S)^{**},\circledast)$, onto $(M_b(S)^{**},\circledast_w)$.

J.C.S. Wong [8] was shown that the dual $M_b(S)^{*}$ is isometrically order isomorphic to the space $GL(S)$ of all generalized functions on S. An element $f=(f_\mu)_{\mu\in M_b(S)}$ in the product linear space $\Pi\{L_\infty(|\mu|):\mu\in M_b(S)\}$ is called a generalized function on S, if the following conditions are satisfied:

(a) $\|f\|:=\operatorname{Sup}\{\|f_\mu\|_{\mu,\infty}:\mu\in M_b(S)\}<\infty$.

(b) If $\mu,\nu\in M_b(S)$ and $\mu<<\nu$, then $f_\mu=f_\nu$,

$|\mu| - a.e.$

Let $\eta \in M_b(S, w)$, there exist unique η^+, $\eta^- \in M_b^+(S, w)$ such that $\eta = [\eta^+, \eta^-]$ and $\eta^+ \perp \eta^-$. Put $|\eta| = \eta^+ + \eta^-$. One can define $f = (f_\eta) \in GL(S, w)$ if and only if $f_\eta \in L_\infty(|\eta|, w)$ such that $\|f\|_w$:= Sup $\{\|f_\eta\|_{\eta,w} : \eta \in M_b(S, w)\} < \infty$ and if $\eta, \xi \in M_b(S, w)$ and $|\eta| << |\xi|$, then $f_\eta = f_\xi$ $|\eta| - a.e.$, where $\|f_\eta\|_{\eta,w} := \|f_\eta / w\|_{\eta,\infty}$ see [6].

Let $GL(S, w)$ be the space of all w-generalized functions on S. Then it has been shown in [4], (see also [8], [6]) that $M_b(S,w)^*$ is isometrically isomorphic to $GL(S,w)$ whose duality is given in [6], explicitly.

Lemma (2.2). For every F, $G \in M_b(S)^{**}$ and $f \in GL(S)$ there exists an $\eta \in M_b(S)$ such that, $F \circledast_w G(f) = (F \circledast G)\, \Omega(\tilde{f}_\eta)$; where $\tilde{f}_\eta(x, y) = f_\eta(xy)$ for $x, y \in S$. The same equality holds for the second Arens product.

Proof. Let $\{\mu_\alpha\}$ and $\{\nu_\beta\}$ be two nets in $M_b(S)$ such that $\mu_\alpha \xrightarrow{w^*} F$ and $\nu_\beta \xrightarrow{w^*} G$. Then there exists a subnet (μ_n) [resp. (ν_m)] of net (μ_α) [resp. (ν_β)] so that,

$$
\begin{aligned}
F \circledast_w G(f) &= \lim_\alpha \lim_\beta f(\mu_\alpha *_w \nu_\beta) \\
&= \lim_n \lim_m f(\mu_n *_w \nu_m) \\
&= \lim_n \lim_m (\mu_n \times \nu_m)\Omega(\tilde{f}_{\mu_n * \nu_m}) \\
&= \lim_n \lim_m (\mu_n * \nu_m)\Omega(\tilde{f}_\eta) \\
&= (F \circledast G)\Omega(\tilde{f}_\eta)
\end{aligned}
$$

where $\eta \in M_b(S)$ is so that $\mu_n * \nu_m << \eta$, for all n and m, for example $\eta = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{2^{n+m}} . \frac{|\mu_n|}{1+\|\mu_n\|} * \frac{|\nu_m|}{1+\|\nu_m\|}$. Note that, if $f = (f_\mu) \in GL(S)$ then for every $\xi \in M_b(S), f(\xi) = \int_S f_\xi(z)\, d\xi(z)$. Also $f(\xi_1) = f(\xi_2)$, whenever $\xi_1 << \xi_2$, see [8]).

We now state the main result of this section. Hereafter, our mean by $M_b(S,w)$ is the Banach algebra $(M_b(S, w), *, \|\ \|_w)$, which is identified with $(M_b(S), *_w, \|\ \|)$.

Theorem (2.3). Let $M_b(S)$ [resp. $M_b(S)^{**}$] be regular. Then $M_b(S, w)$ [resp. $M_b(S, w)^{**}$] is regular.

Proof. Let $\circledast$ and $\circledast_w$ denote the second Arens products induced by * and $*_w$, respectively. If $M_b(S)$ is regular, then for every $F, G \in M_b(S)^{**}$, and $f \in GL(S)$ by the above lemma we have

$$
\begin{aligned}
F \circledast_w G(f) &= F \circledast G(\Omega\tilde{f}_\eta), \text{ for some } \eta \in M_b(S) \\
&= F \circledcirc G(\Omega\tilde{f}_\eta) \\
&= F \circledcirc_w G(f).
\end{aligned}
$$

Therefore $F \circledast_w G = F \circledcirc_w G$, so $M_b(S,w)$ is regular.

Suppose $M_b(S)^{**}$ is regular. Let A, B $\in M_b(S)^{**}$. Then by a similar argument as is used in (2.2), one can show that:

$$
\begin{aligned}
A \mathbin{\overline{\circledast}_w} B(h) &= A \mathbin{\overline{\circledast}} (\Omega h_\xi), \text{ for some } \xi \in M_b(S)^{**} \\
&= A \mathbin{\overline{\circledcirc}} B(\Omega h_\xi) \\
&= A \mathbin{\overline{\circledcirc}_w} B(h),
\end{aligned}
$$

where $\overline{\circledast}_w$ [resp. $\overline{\circledcirc}_w$] is the first [resp. second] Arens product.

In [1], the authors showed that $l_1(S, w)$ is regular, whenever Ω is 0-cluster, *i.e.* for all sequences (x_n), (y_m) of distinct elements in S,

$$\lim_n \lim_m \Omega(x_n, y_m) = 0 = \lim_m \lim_n \Omega(x_n, y_m),$$

Whenever the both iterated limits exist.

The next statement extend this for non-discrete case.

Proposition (2.4). Let Ω be 0-cluster. Then $M_b(S,w)^{**}$ is regular.

Proof. Let $A, B \in M_b(S)^{****}$ and $h \in M_b(S)^{***}$. Then there exist sequences (F_n), (G_m) in $M_b(S)^{**}$ such that,

$$A \mathbin{\overline{\circledast}_w} B(h) = \lim_n \lim_m \hat{F}_n \circledast_w \hat{G}_m(h).$$

The assumption of Ω being 0-cluster implies that $A \mathbin{\overline{\circledast}_w} B = 0 = A \mathbin{\overline{\circledcirc}_w} B$, for all A,B $\in M_b(S)^{****} \setminus M_b(S)^{**}$; Indeed, as in the proof of 2.2, there exists a $\xi \in M_b(S)^{**}$ such that $A \mathbin{\overline{\circledast}_w} B(h) = A \mathbin{\overline{\circledast}} B\ (\Omega h_\xi)$

$= 0$, for each $h \in M_b(S)^{***}$.

Corollary (2.5). Let Ω be 0-cluster. Then $M_b(S,w)$ is regular.

Proof. This is an immediate consequence of 2.4.

Theorem (2.6). Let $M_b(S,w)$ [resp. $M_b(S,w)^{**}$] be amenable and $w \geq 1$. Then $M_b(S)$ [resp. $M_b(S)^{**}$] is amenable.

Proof. Let $\phi : (M_b(S), *_w) \to (M_b(S), *)$ be defined by $\mu \mapsto \mu/w$. Then $\frac{\mu *_w \nu}{w} = \mu/w * \nu/w$, for $\mu, \nu \in M_b(S)$, *i.e.* ϕ is a continuous homomorphism with $\phi(M_b(S)) = M_b(S,w)$ which is dense in $M_b(S)$. Therefore amenability of $M_b(S, w)$ implies that of $M_b(S)$, (see [7], for the notion of amenable Banach algebra). The same argument may be used for the second dual ϕ^{**} of ϕ to show that, amenability of $M_b(S,w)^{**}$ implies that of $M_b(S)^{**}$.

Remark (2.7). If we define $M_a^l(S,w) = \{[\mu,\nu] \in M_b(S,w) : \mu w - \nu w \in M_a^l(S)$, (where $M_a^l(S) = \{\mu \in M_b(S) : x \to \bar{x} * \mu$ is weak-continuous$\}$, see [3]) then it has been shown that $M_a^l(S,w)$ is a closed solid left ideal of $M_b(S,w)$; (for more details see [5]). In particular, $(M_a^l(S,w), *, \| \|_w)$ is a Banach algebra. Similar to what we have seen in corollary (1.5), it can be identified with $(M_a^l(S,w), *_w, \| \|)$. And also $(M_a^l(S,w)^{**}, \circledast) \cong (M_a^l(S)^{**}, \circledast_w)$, see Lemma (2.1). So, one can repeat the results (2.3), (2.4), (2.5) and (2.6) with M_a^l in stead of M_b, which of course gives a new proof for corollary 9 of [6].

Question. Does the conclusion of 2.6 hold without $w \geq 1$?

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