Journal of Sciences, Islamic Republic of Iran 19(3): 265-272 (2008) http://jsciences.ut.ac.ir
University of Tehran, ISSN 1016-1104

A Note on Artinianess of Certain Generalized Local Cohomology Modules

S.H. Tahamtan[1,*] and H. Zakeri[2]

[1]*Department of Science, Science and Research Branch Islamic Azad University of Tehran, Tehran, Islamic Republic of Iran*
[2]*Department of Mathematics, Faculty of Mathematical Sciences and Computer Engineering, Teacher Training University, Tehran 15618, Islamic Republic of Iran*

Abstract

Let φ: $R_0 \to R$ be a ring homomorphism and suppose that a and a_0, respectively, are ideals of R and R_0 such that $\frac{R}{a_0R+a}$ is an Artinian ring. Let M and N be two finitely generated R-modules and suppose that (R_0,m_0) is a local ring. In this note we prove that the R-modules $\frac{R_0}{a_0}\otimes_{R_0} H_a^i(M,N)$ and $H_{a_0R}^j(H_a^i(M,N))$ are Artinian for all integers i and j, whenever $\sqrt{(a_0+a\bigcap R_0)}=m_0$ and $\dim(\frac{R_0}{a\bigcap R_0})\le 1$. Also we will show that if a is principal, then the R-modules $\frac{R_0}{a_0}\otimes_{R_0} H_a^i(M,N)$ and $H_{a_0R}^j(H_a^i(M,N))$ are Artinian, for all integers i and j. Moreover, we will show that if $t\ge 0$ is the largest integer i such that $H_a^i(M,N)$ is not Artinian, then the R-modules $\frac{R_0}{a_0}\otimes_{R_0} H_a^q(M,N)$ and $H_{a_0R}^p(H_a^q(M,N))$ are Artinian, for $\dim R_0-1\le p\le \dim R_0$ and all $q\ge t$. Also as a consequence of this result we get that the R-modules $\frac{R_0}{a_0}\otimes_{R_0} H_a^c(M,N)$ and $H_{a_0R}^j(H_a^c(M,N))$ are artinian, for j=d, d−1, in which $c=cd_a(M,N)$ is the cohomological dimension of M and N with respect to a. Our results recover the corresponding known ones.

Keywords: Artinian modules; Local cohomology modules; Generalized local cohomology functor

Introduction

Generalized local cohomology was given in the local case by J. Herzog [6] and in the more general case by Bijan-Zadeh [1]. Let R be a commutative Noetherian ring with identity, a an ideal of R and let M, N be

2000 Mathematics Subject Classification. 13D45, 13E10
* Corresponding author, Tel.: +98(662)3503388, Fax: +98(662)3512548, E-mail: taham_sh@yahoo.com

two R-modules. For an integer $i \geq 0$, the i-th generalized local cohomology module $H_a^i(M,N)$ is defined by

$$H_a^i(M,N) = \varinjlim_{n \in N} Ext_R^i\left(\frac{M}{a^n M}, N\right).$$

With $M = R$, we obtain the ordinary local cohomology module $H_a^i(N)$ of N with respect to a which was introduced by Grothendieck. It is well known that the generalized local cohomology modules have some similar properties as ordinary local cohomology modules. We recall some properties of generalized local cohomology modules which we need in this note. For any ideal a of R and two finitely generated R-modules M and N the following statements hold:

(i) Let $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$ be an exact sequence of R-modules, then we have two long exact sequences

$$0 \longrightarrow H_a^0(M,N') \longrightarrow H_a^0(M,N) \longrightarrow H_a^0(M,N'') \longrightarrow H_a^1(M,N') \longrightarrow \ldots$$

and

$$0 \longrightarrow H_a^0(N'',M) \longrightarrow H_a^0(N,M) \longrightarrow H_a^0(N',M) \longrightarrow H_a^1(N'',M) \longrightarrow \ldots$$

of generalized local cohomology modules.

(ii) If N is an a-torsion R-module, then for each non-negative integer i we have $H_a^i(M,N) \cong Ext_R^i(M,N)$.

(iii) If N is an a-torsion R-module, then, for any ideal b of R and each i, we have $H_b^i(M,N) \cong H_{a+b}^i(M,N)$.

(iv) Let R' be a second commutative Noetherian ring with identity and let $f : R \longrightarrow R'$ be a flat ring homomorphism. Then we have an isomorphism $H_a^i(M,N) \otimes R' \cong H_{aR'}^i(M \otimes_R R', N \otimes_R R')$.

(v) Let $f : R \longrightarrow R'$ be a ring homomorphism. Suppose that M and N are two R' modules. Our next example shows that the isomorphism $H_a^i(M,N) \cong H_{aR'}^i(M,N)$ is not hold in general. Hence the statement (4) (i) in section 2 of [8] dos not hold either.

Example 1.1. Let K be a field and let x be an indeterminate. Set $R = K[[x]]$, $R' = \frac{K[[x]]}{(x^2)}$ and $m = (x)$. Since R' is not regular, the projective dimension $pd_{R'}\left(\frac{R}{m}\right)$ of $\frac{R}{m}$ is infinite. Therefore, the isomorphism $H_{mR'}^i\left(\frac{R}{m}, \frac{R}{m}\right) \cong Ext_{R'}^i\left(\frac{R}{m}, \frac{R}{m}\right)$ implies that $H_{mR'}^i\left(\frac{R}{m}, \frac{R}{m}\right) \neq 0$ for infinite values of i; while $H_m^i\left(\frac{R}{m}, \frac{R}{m}\right) \cong Ext_R^i\left(\frac{R}{m}, \frac{R}{m}\right) = 0$ for all $i > 1$.

In this note, we show that under some assumption, generalized local cohomology modules are Artinian. Throughout this paper R and R_0 are two commutative Noetherian rings such that R_0 is local with maximal ideal m_0 and that there is a ring homomorphism $\varphi : R_0 \longrightarrow R$. We shall use M and N to denote two finitely generated $R-$ modules. Furthermore, we assume that a and a_0 are ideals of R and R_0, respectively, such that $\frac{R}{a_0 R + a}$ is Artinian. An interesting problem in commutative algebra is determining when generalized local cohomology modules are Artinian. In this note we are interested to the following question: When the R-modules $\frac{R_0}{a_0} \otimes_{R_0} H_a^i(M,N)$ and $H_{a_0 R}^j\left(H_a^i(M,N)\right)$ are Artinian?

There are several papers which answer this question in graded case. To be more precise, let R be a homogeneous graded ring with local base ring (R_0, m_0) and let N be a finitely generated and graded R-module. Moreover let R_+ be the irrelevant ideal of R. As shown by recent investigations, the R-modules $\frac{R_0}{m_0} \otimes_{R_0} H_{R+}^i(N)$ and $H_{m_0 R}^j\left(H_{R+}^i(N)\right)$ are Artinian in some special cases. For example, in [3,2.5], the authors have shown that if $\dim(R_0) \leq 1$, then $\frac{R_0}{m_0} \otimes_{R_0} H_{R+}^i(N)$ and $H_{m_0 R}^j\left(H_{R+}^i(N)\right)$ are Artinian for all $i \geq 0$ and $j = 0,1$. Also if M is another finitely generated graded R-module, then, in this specific case, Zamani, in [12, 2.6], proved that $\frac{R_0}{m_0} \otimes_{R_0} H_{R+}^i(M,N)$

is Artinian for all $i, j \geq 0$. On the other hand, in [11,2.6], it has been proved that if R_+ is principal, then $H^j_{m_0R}\left(H^i_{R+}(N)\right)$ is Artinian for all $i, j \geq 0$. Moreover, for any ideal b_0 of R_0 with $\dim\frac{R_0}{b_0} \leq 1$, the authors in [7,2.5,2.1] showed that $H^j_{m_0R}\left(H^i_{b_0+R_+}(N)\right)$ is Artinian for all $i, j \geq 0$.

With the above notation assume that $c = cd_{R_+}(N)$ is the largest integer i, such that $H^i_{R_+}(N) \neq 0$. Then, it is proved in [9, 2.1] and [11, 2.8] that the R-modules $\frac{R_0}{m_0} \otimes_{R_0} H^c_{R+}(N)$ and $H^{\dim(R_0)}_{m_0R}\left(H^c_{R_+}(N)\right)$ are Artinian R-module.

The aim of this paper is to provide a partial answer to the previous question in the non-graded case. We will show that whenever $\dim\left(\frac{R_0}{a \cap R_0}\right) \leq 1$ or a is principal, the R-modules $\frac{R_0}{a_0} \otimes_{R_0} H^i_a(M, N)$ and $H^j_{a_0R}\left(H^i_a(M, N)\right)$ are Artinian for all $i, j \geq 0$. Also, We will assume that the projective dimension of M is finite and that t is the largest integer i such that $H^i_a(M, N)$ is not Artinian. Then, under this assumption, we will show that if $\dim R_0 - 1 \leq p \leq \dim R_0$ then the R-modules $\frac{R_0}{a_0} \otimes_{R_0} H^q_a(M, N)$ and $H^p_{a_0R}\left(H^q_a(M, N)\right)$ are Artinian for all $q \geq t$.

Of course the results of this paper recover the corresponding previously known results as mentioned above.

Results

We keep the previous notations and hypothesis and we prove that certain quotients and sub-modules of the generalized local cohomology module $H^i_a(M, N)$ are Artinian. We begin this section by the following two lemmas which are needed in the sequel.

Lemma 2.1. (i) ([12, 2.2]) Let b be an ideal of R such that $\frac{R}{b}$ is Artinian. Then $H^i_b(M, N)$ is Artinian for all $i \geq 0$.

(ii) ([3,2.2]) Let A be an Artinian R-module and let M_0 be a finitely generated R_0-module. Then the R-module $Tor^{R_0}_i(M_0, A)$ is Artinian for all $i \geq 0$. □

Lemma 2.2. Let i and j be two non-negative integers. Then

(i) $\frac{R_0}{a_0} \otimes_{R_0} H^i_a(M, N)$ is Artinian if and only if $\frac{R_0}{a_0} \otimes_{R_0} H^i_a\left(M, \frac{N}{\Gamma_{a_0R}(N)}\right)$ is Artinian, and

(ii) $H^j_{a_0R}\left(H^i_a(M, N)\right)$ is Artinian if and only if $H^j_{a_0R}\left(H^i_a\left(M, \frac{N}{\Gamma_{a_0R}(N)}\right)\right)$ is Artinian.

Proof. Consider the exact sequence $0 \to \Gamma_{a_0R}(N) \to N \to \frac{N}{\Gamma_{a_0R}(N)} \to 0$. Application of the functor $H^i_a(M, -)$ to this exact sequence induces the following exact sequence

$$H^i_a\left(M, \Gamma_{a_0R}(N)\right) \longrightarrow H^i_a(M, N)$$

$$\xrightarrow{u} H^i_a\left(M, \frac{N}{\Gamma_{a_0R}(N)}\right) \longrightarrow H^{i+1}_a\left(M, \Gamma_{a_0R}(N)\right)$$

which in turn, yields two short exact sequences

$$0 \longrightarrow \ker u \longrightarrow H^i_a(M, N) \longrightarrow \operatorname{Im} u \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Im} u \longrightarrow H^i_a\left(M, \frac{N}{\Gamma_{a_0R}(N)}\right)$$

$$\longrightarrow co\ker u \longrightarrow 0$$

Now we will use these exact sequences to prove (i) and (ii).

(i) The above short exact sequences induce the exact sequences

$$\frac{R_0}{a_0} \otimes_{R_0} \ker u \longrightarrow \frac{R_0}{a_0} \otimes_{R_0} H^i_a(M, N)$$

$$\longrightarrow \frac{R_0}{a_0} \otimes_{R_0} \operatorname{Im} u \longrightarrow 0$$

and

$$Tor_1^{R_0}\left(\frac{R_0}{a_0}, co\ker u\right) \longrightarrow \frac{R_0}{a_0}\otimes_{R_0} \operatorname{Im} u \longrightarrow$$

$$\frac{R_0}{a_0}\otimes_{R_0} H_a^i\left(M, \frac{N}{\Gamma_{a_0R}(N)}\right) \longrightarrow \frac{R_0}{a_0}\otimes_{R_0} co\ker u.$$

Note that, in view of 2.1 (i), $H_a^i\left(M, \Gamma_{a_0R}(N)\right)$ is Artinian for all $i \geq 0$. So, $\ker u$ and co $\ker u$ are Artinian and the assertion follows immediately from the above exact sequences by 2.1 (ii).

(ii) Again, consider the previous short exact sequences. It should be noted that if A is an Artinian R-module and b an ideal of R, then $H_b^i(A) = 0$ for all $i \geq 1$. Application of the functor $H_{a_0R}^j(-)$ induces the exact sequences

$$H_{a_0R}^j(\ker u) \longrightarrow H_{a_0R}^j\left(H_a^i(M, N)\right)$$

$$\longrightarrow H_{a_0R}^j(\operatorname{Im} u) \longrightarrow H_{a_0R}^{j+1}(\ker u)$$

and

$$H_{a_0R}^{j-1}(co\ker u) \longrightarrow H_{a_0R}^j(\operatorname{Im} u) \longrightarrow$$

$$H_{a_0R}^i\left(H_a^i\left(M, \frac{N}{\Gamma_{a_0R}(N)}\right)\right) \longrightarrow H_{a_0R}^i(co\ker u).$$

Now, since $\ker u$ and co $\ker u$ are Artinian, the result follows from these exact sequences. □

The next theorem recovers the results 2.1 and 2.5 of [7] and 2.5 of [3].

Theorem 2.3. Let $\sqrt{(a_0 + a \cap R_0)} = m_0$ and $\dim\left(\frac{R_0}{a \cap R_0}\right) \leq 1$. Let i and j be two non-negative integers. Then the R-modules $\frac{R_0}{a_0}\otimes_{R_0} H_a^i(M, N)$ and $H_{a_0R}^j\left(H_a^i(M, N)\right)$ are Artinian.

Proof. Let $\dim\left(\frac{R_0}{a \cap R_0}\right) = 0$. We can easily show that $\sqrt{a_0R + a} = \sqrt{a}$. So, by 2.1 (i), the R-module $H_a^i(M, N)$ is Artinian; therefore the claim follows in this case. Now, let $\dim\left(\frac{R_0}{a \cap R_0}\right) = 1$. If $a_0 \subseteq \ker\varphi$, then $a_0R = 0$. Hence $\frac{R}{a}$ is an Artinian ring and there is nothing to proof in this case. So let $a_0 \not\subset \ker\varphi$. In view of lemma 2.2, we may assume that N is an a_0R-torsion free R-module. Hence, there is an element $x \in a_0$ such that

$$x \notin \bigcup_{q \in Ass_R(N)} (q \cap R_0) \bigcup \Big(\bigcup_{q_0 \in Assh_{R_0}\frac{R_0}{(a\cap R_0)}} q_0 \Big) \bigcup \ker\varphi,$$

in which,

$$Assh_{R_0}\frac{R_0}{(a \cap R_0)} = \left\{ q_0 \in Ass_{R_0}\frac{R_0}{(a \cap R_0)} : \dim\frac{R_0}{q_0} = 1 \right\}.$$

It is straightforward to see that $\sqrt{xR + a} = \sqrt{a_0R + a}$. Now it follows from the isomorphism $H_{a_0R}^j\left(H_a^i(M, N)\right) \cong H_{xR}^j\left(H_a^i(M, N)\right)$ that $H_{a_0R}^j\left(H_a^i(M, N)\right) = 0$ for all $j \geq 2$. Therefore, we may assume that $j \leq 1$. Since x is a non-zero divisor on N, we have an exact sequence

$$H_a^{i-1}\left(M, \frac{N}{xN}\right) \longrightarrow H_a^i(M, N)$$

$$\xrightarrow{x} H_a^i(M, N) \longrightarrow H_a^i\left(M, \frac{N}{xN}\right)$$

of generalized local cohomology modules. Set $L := H_a^i(M, N)$. It should be noted that, by 2.1 (i), the R-module $H_a^i\left(M, \frac{N}{xN}\right) \cong H_{a+Rx}^i\left(M, \frac{N}{xN}\right)$ is Artinian for all $i \leq 0$. So, it follows from the previous exact sequence that both of the R-modules $0 :_L x$ and $\frac{L}{xL}$ are Artinian. Now we apply the functor $\frac{R_0}{a_0}\otimes_{R_0}(-)$ on the exact sequence $0 \longrightarrow L \xrightarrow{x} L \longrightarrow \frac{L}{xL} \longrightarrow 0$ to see that the R-module $\frac{R_0}{a_0}\otimes_{R_0} L \cong \frac{R_0}{a_0}\otimes_{R_0}\frac{L}{xL}$ is Artinian. On the other hand there is an exact sequence $0 \longrightarrow H_{a_0R}^0(0 :_L x) \longrightarrow H_{a_0R}^0(L) \xrightarrow{x} H_{a_0R}^0(L)$ Which, by [4, 7.1.2] shows that $H_{a_0R}^0(L)$ is Artinian.

Finally in order to complete the proof, we only need to show that $H_{a_0R}^1(L) \cong H_{xR}^1(L)$ is Artinian. To do this, consider the well-known exact sequence $0 \longrightarrow \Gamma_{xR}(L) \longrightarrow L \xrightarrow{v} L_x \longrightarrow H_{xR}^1(L) \longrightarrow 0$

to obtain the isomorphism $H^1_{xR}(L) \cong \frac{L_x}{\mathrm{Im}v}$. Therefore, in view of [4, 7.1.2], it is enough for us to show that $0:_{\frac{L_x}{\mathrm{Im}v}} x$ is Artinian. Let $f : \frac{L}{\Gamma_{xR}(L)} \longrightarrow 0:_{\frac{L_x}{\mathrm{Im}v}} x$ be a map which is such that $f\left(l + \Gamma_{xR}(L)\right) = \frac{l}{x} + \mathrm{Im}v$ for all $l \in L$. It is easy to see that f is an R-epimorphism and that $\ker f = x\left(\frac{L}{\Gamma_{xR}(L)}\right)$. Moreover $0:_{\frac{L_x}{\mathrm{Im}v}} x$ is a homomorphic image of the Artinian R-module $\frac{L}{xL}$ and the proof is complete. □

The next theorem shows that if the ideal a is principal, then the assertion of the above theorem holds without any restriction on $\dim\left(\frac{R_0}{a \cap R_0}\right)$. The following lemma is needed in the proof of theorem 2.5.

Lemma 2.4. Suppose that the ideal a is principal. Then $\frac{R_0}{a_0} \otimes_{R_0} H^i_a(M)$ and $H^j_{a_0R}\left(H^i_a(M)\right)$ are Artinian $R-$modules for all non negative integers i and j.

Proof. Let $y \in a$ be such that $a = Ry$. Note that if $i \geq 2$, then $H^i_a(M) = 0$. So we assume that $i \leq 1$. The case $i = 0$ is obvious, since $\frac{H^0_a(M)}{a_0 H^0_a(M)}$ is a finitely generated R-module which is annihilated by some power of $a_0R + a$. Let $i = 1$. In view of [4, 2.1.7], we can replace M by $\frac{M}{\Gamma_a(M)}$; and hence we may assume that y is a non-zero divisor on M.

Now, the short exact sequence $0 \longrightarrow M \xrightarrow{y} M \longrightarrow \frac{M}{yM} \longrightarrow 0$ provides an exact sequence

$$0 \longrightarrow \frac{M}{yM} \longrightarrow H^1_a(M) \xrightarrow{y} H^1_a(M) \longrightarrow 0,$$

which in turn, yields the exact sequences

$$\frac{R_0}{a_0} \otimes_{R_0} \frac{M}{yM} \longrightarrow \frac{R_0}{a_0} \otimes_{R_0} H^1_a(M)$$

$$\xrightarrow{y} \frac{R_0}{a_0} \otimes_{R_0} H^1_a(M) \longrightarrow 0$$

and

$$H^j_{a_0R+a}\left(\frac{M}{yM}\right) \longrightarrow H^j_{a_0R}\left(H^1_a(M)\right)$$

$$\xrightarrow{y} H^j_{a_0R}\left(H^1_a(M)\right).$$

Since the modules $\frac{R_0}{a_0} \otimes_{R_0} \frac{M}{yM}$ and $H^j_{a_0R+a}\left(\frac{M}{yM}\right)$ are Artinian, we can use the above exact sequences and [4, 7.1.2], to see that the R-modules $\frac{R_0}{a_0} \otimes_{R_0} H^1_a(M)$ and $H^j_{a_0R}\left(H^1_a(M)\right)$ are Artinian for all $j \geq 0$. □

The next theorem improves [11, 2.6].

Theorem 2.5. Suppose that the ideal a is principal. Let i, j be non-negative integers. Then the R-modules $\frac{R_0}{a_0} \otimes_{R_0} H^i_a(M,N)$ and $H^i_{a_0R}\left(H^i_a(M,N)\right)$ are Artinian.

Proof. Let $y \in a$ be such that $a = Ry$. The case $i = 0$ is trivial. Let $i = 1$ and $T := \frac{Hom_R(M,N)}{H^0_a\left(Hom_R(M,N)\right)}$. In view of [4, 2.2.17], we have an isomorphism $\frac{Hom_R(M,N)_y}{T} \cong H^1_a\left(Hom_R(M,N)\right)$; so that, by 2.4, the R-modules $\frac{R_0}{a_0} \otimes_{R_0} \frac{Hom_R(M,N)_y}{T}$ and $H^j_{a_0R}\left(\frac{\mathrm{Hom}_R(M,N)_y}{T}\right)$ are Artinian for all $j \geq 0$. On the other hand in view of [5, 2.2 and 2.5(i)], we have an exact sequence

$$0 \longrightarrow \frac{Hom_R(M,N)_y}{T} \longrightarrow H^1_a(M,N)$$

$$\xrightarrow{\alpha} Ext^1_R(M,N),$$

which in turn yields the exact sequences

$$\frac{R_0}{a_0} \otimes_{R_0} \frac{Hom_R(M,N)_y}{T} \longrightarrow \frac{R_0}{a_0} \otimes_{R_0} H^1_a(M,N)$$

$$\longrightarrow \frac{R_0}{a_0} \otimes_{R_0} \mathrm{Im}\,\alpha$$

and

$$H^j_{a_0R}\left(\frac{Hom_R(M,N)y}{T}\right)\longrightarrow H^j_{a_0R}\left(H^1_a(M,N)\right)$$

$$\longrightarrow H^j_{a_0R}(\operatorname{Im}\alpha)$$

for all $j\geq 0$. Since $\operatorname{Im}\alpha$ is a finitely generated and a-torsion R-module, we can deduce from the above exact sequences that the R-modules $\frac{R_0}{a_0}\otimes_{R_0}H^1_a(M,N)$ and $H^j_{a_0R}\left(H^1_a(M,N)\right)$ are Artinian for all $j\geq 0$. Now, let $j\geq 2$. The module M can be included in an exact sequence $0\longrightarrow K\longrightarrow F\longrightarrow M\longrightarrow 0$ of finitely generated R-modules, in which F is free. Note that $H^i_a(F,N)=0$ for all $i\geq 2$. So, application of the functor $H^i_a(-,N)$ to this short sequence induces the isomorphism $H^{i-1}_a(K,N)\cong H^i_a(M,N)$ for all $i\geq 3$. Therefore, in order to complete the proof, it is enough for us to show that $\frac{R_0}{a_0}\otimes_{R_0}H^2_a(M,N)$ and $H^j_{a_0R}\left(H^2_a(M,N)\right)$ are Artinian for all $j\geq 0$. By repeating the above argument, we obtain the exact sequence

$$H^0_a(K,N)\longrightarrow H^1_a(M,N)\xrightarrow{\beta}H^1_a(F,N)$$

$$\longrightarrow H^1_a(K,N)\longrightarrow H^2_a(M,N)\longrightarrow 0.$$

If we break this sequence in two exact sequences

$$0\longrightarrow\ker\beta\longrightarrow H^1_a(M,N)\longrightarrow\operatorname{Im}\beta\longrightarrow 0$$

and

$$0\longrightarrow\operatorname{Im}\beta\longrightarrow H^1_a(F,N)\longrightarrow co\ker\beta\longrightarrow 0.$$

then we get the following exact sequences

$$H^{j+2}_{a_0R}\left(H^1_a(M,N)\right)\longrightarrow H^{j+2}_{a_0R}(\operatorname{Im}\beta)$$

$$\longrightarrow H^{j+3}_{a_0R}(\ker\beta)$$

and

$$H^{j+1}_{a_0R}\left(H^1_a(F,N)\right)\longrightarrow H^{j+1}_{a_0R}(co\ker\beta)$$

$$\longrightarrow H^{j+2}_{a_0R}(\operatorname{Im}\beta).$$

As $\ker\beta$ is finitely generated and a-torsion, it follows, from 2.1 (i), that the R-module $H^j_{a_0R}(\ker\beta)$ is Artinian for all $j\geq 0$. On the other hand, $H^{j+2}_{a_0R}\left(H^1_a(M,N)\right)$ and $H^{j+1}_{a_0R}\left(H^1_a(F,N)\right)$ are Artinian; so that $H^{j+1}_{a_0R}(co\ker\beta)$ is too. Finally, we use the exact sequence $0\longrightarrow co\ker\beta\longrightarrow H^1_a(K,N)\longrightarrow H^2_a(M,N)\longrightarrow 0$ to obtain the exact sequences

$$\frac{R_0}{a_0}\otimes_{R_0}co\ker\beta\longrightarrow\frac{R_0}{a_0}\otimes_{R_0}H^1_a(K,N)$$

$$\longrightarrow\frac{R_0}{a_0}\otimes_{R_0}H^2_a(M,N)\longrightarrow 0$$

and

$$H^j_{a_0R}\left(H^1_a(K,N)\right)\longrightarrow H^j_{a_0R}\left(H^2_a(M,N)\right)$$

$$\longrightarrow H^{j+1}_{a_0R}(co\ker\beta).$$

Now, since the assertion is true for $i=1$, it follows that the R-modules $\frac{R_0}{a_0}\otimes_{R_0}H^2_a(M,N)$ and $H^j_{a_0R}\left(H^2_a(M,N)\right)$ are Artinian for all $j\geq 0$, as required. □

The following example, which has already been presented in [3], shows that the R-modules $\frac{R_0}{a_0}\otimes_{R_0}H^1_a(M,N)$, $H^0_{a_0R}\left(H^2_a(M,N)\right)$ and $H^2_{a_0R}\left(H^1_a(M,N)\right)$ are not Artinian in general.

Example 2.6. Let K be a field and let x,y,t be indeterminate. Let $R_0=K[x,y]_{(x,y)}$ $m_0=(x,y)R_0$ and $R=R_0[m_0t]$ be the Rees ring of m_0. Set $a:=(xt,yt)R$. In view of [3, 4.1 and 4.2] the R-modules $\frac{R_0}{a_0}\otimes_{R_0}H^1_a(R)$ and $H^0_{a_0R}\left(H^2_a(R)\right)$ are not Artinian. Also, the example 2.9 of [12] shows that $H^2_{m_0R}\left(H^1_a(R)\right)$ is not Artinian, in this specific case.

The next theorem is an improvement of [9, 2.4].

Theorem 2.7. Let $\dim R_0=d$ and $t\geq 0$ be an integer such that the R-module $H^i_a(M,N)$ is Artinian for all $i\geq t+1$. Then the R-modules $H^p_{a_0R}\left(H^q_a(M,N)\right)$ and $\frac{R_0}{a_0}\otimes_{R_0}H^q_a(M,N)$ are Artinian for all $q\geq t$.

Proof. First, consider the spectral sequence

$$E^{p,q}_2:=H^p_{a_0R}\left(H^q_a(M,N)\right)\underset{p}{\Rightarrow}H^{p+q}_{a_0R+a}(M,N).$$

Whenever $d-1 \le p \le d$ and $q \ge t$, it is easy to see that $E_{\infty}^{p,q} \cong \frac{E_2^{p,q}}{L}$ for some Artinian submodule L of $E_2^{p,q}$. Also $E_{\infty}^{p,q}$ is a sub-quotient of the Artinian R-module $H_{a_0R+a}^{p+q}(M,N)$. Therefore, $E_2^{p,q}$ = $H_{a_0R}^{p}\left(H_a^q(M,N)\right)$ is Artinian.

Next we prove that $\frac{R_0}{a_0} \otimes_{R_0} H_a^q(M,N)$ is Artinian for all integer $q \ge t$. To this end, it should be noted that the set $Supp_R\left(\frac{R_0}{a_0} \otimes_{R_0} H_a^q(M,N)\right)$ consist entirely of finitely many maximal ideals of R. Therefore, in order to prove the assertion, it is enough for us to show that $\left(\frac{R_0}{a_0} \otimes_{R_0} H_a^q(M,N)\right)_m$ is an Artinian R_m-module for all $m \in Supp_R\left(\frac{R_0}{a_0} \otimes_{R_0} H_a^q(M,N)\right)$. On the other hand, by the flat base change theorem, it is easy to see that, without lose of generality, we can replace R, M and N, respectively, by R_m, M_m and N_m and hence we assume that R is local. Now, we proceed by induction on $n := \dim N$. If $n=0$, then $\Gamma_a(N)=N$. So for each $l \ge 0$, $H_a^l(M,N) \cong Ext_R^l(M,N)$ is a finitely generated R-module which is annihilated by some power of a. Therefore, there is nothing to prove in this case. Now assume that $\dim N = n > 0$ and that the result has been proved for any finitely generated R-module of dimension $n-1$.

By Lemma 2.2, we can assume that $\Gamma_{a_0R}(N)=0$; so that there exists an element $x \in a_0$ such that $x \notin \bigcup_{q \in Ass_R(N)}(q \cap R_0) \bigcup \ker\varphi$. Hence x is a non-zero divisor on N and $\dim\frac{N}{xN} = n-1$. Also, we can use the exact sequence $0 \longrightarrow N \xrightarrow{x} N \longrightarrow \frac{N}{xN} \longrightarrow 0$ to see that $a_a\left(M, \frac{N}{xN}\right) \le t$. Now, from the exact sequence $H_a^q(M,N) \xrightarrow{x} H_a^q(M,N) \longrightarrow H_a^q\left(M, \frac{N}{xN}\right) \longrightarrow H_a^{q+1}(M,N)$ we deduce the short exact sequence

$$0 \longrightarrow \frac{H_a^q(M,N)}{xH_a^q(M,N)} \longrightarrow H_a^q\left(M, \frac{N}{xN}\right) \longrightarrow 0 :_{H_a^{q+1}(M,N)} x \longrightarrow 0$$

which yields the following exact sequence

$$Tor_1^{R_0}\left(\frac{R_0}{a_0}, 0 :_{H_a^{q+1}(M,N)} x\right) \longrightarrow \frac{R_0}{a_0} \otimes_{R_0} \frac{H_a^q(M,N)}{xH_a^q(M,N)} \longrightarrow \frac{R_0}{a_0} \otimes_{R_0} H_a^q\left(M, \frac{N}{xN}\right).$$

Let $q \ge t$. Then as a sub-module of $H_a^{q+1}(M,N)$, the R-module $0 :_{H_a^{q+1}(M,N)} x$ is Artinian and so is $Tor_1^{R_0}\left(\frac{R_0}{a_0}, 0 :_{H_a^{q+1}(M,N)} x\right)$ by 2.1(ii). Also using the inductive hypothesis, the R-module $H_a^q\left(M, \frac{N}{xN}\right)$ is Artinian. Now, it follows from the previous exact sequence that $\frac{R_0}{a_0} \otimes_{R_0} \frac{H_a^q(M,N)}{xH_a^q(M,N)} \cong \frac{R_0}{a_0} \otimes_{R_0} H_a^q(M,N)$ is Artinian. □

The cohomological dimension $cd_a(M,N)$ of M and N with respect to a is defined as $cd_a(M,N) = \sup\{i \ge 0 : H_a^i(M,N) \ne 0\}$ with the convention that sup is taken in $N_0 \bigcup \{+\infty\}$ and $\sup\varphi = -\infty$. Note that if $pd_R(M)$ is finite, then, by an easy induction, we can show that $cd_a(M,N) < \infty$. The example 1.1 shows that if we delete the assumption $pd_R(M) < \infty$, then it may happen that $cd_a(M,N) = +\infty$. Next, we prove that $H_{a_0R}^{d-1}\left(H_a^c(M,N)\right)$ is an Artinian R-module. Also our next corollary recovers [11, 2.8] and [2, 2.3(b)].

Corollary 2.8. Suppose that $c = cd_a(M,N)$ is finite and $\dim R_0 = d$. Then the R-modules $\frac{R_0}{a_0} \otimes_{R_0} H_a^c(M,N)$ and $H_{a_0R}^j\left(H_a^c(M,N)\right)$ are Artinian for $j = d, d-1$.

Proof. It is clear that, for all $i \ge c+1$, the R-module $H_a^i(M,N)$ is Artinian. Therefore the result follows immediately from theorem 2.7. □

References

1. M. H. Bijan-Zadeh, A common generalization of local cohomology theories Glasgow Math., J. 21 (1980), 173-181.
2. M. Brodmann, Asymptotic behaviour of cohomology: tameness, supports and associated primes, in: commutative Algebra and Algebraic Geometry, contemp. Math. 390 (2005), 31-61.
3. M. Brodmann, S. Fumasoli and R. Tajarod, Local cohomology over homogenous rings with one-dimensional local base ring, proceedings of AMS. 131 (2003), 2977-2985.
4. M. Brodmann, R. Y. Sharp, Local cohomology: An Algebraic Introduction with Geometric Applications, Cambridge University Press (1998).
5. K. Divaani-Azar, R, Sazeedeh, Cofiniteness of generalized local cohomology modules, Colloq. Math. 99 (2004), 283-290.
6. J. Herzog, Komplex Auflosungen und Dualitat in der lobalen algebra, preprint, Universitiit Regensurg, 1974.
7. M. Jahangiri, H. Zakeri, Local cohomology modules with respect to an ideal containing the irrelevant ideal, to appear in J.Pure Appl.Alg.
8. K. Khashayarmanesh, Associated primes of graded components of generalized local cohomology modules, Comm. Algebra. 33(9) (2005), 3081-3090.
9. C.Rotthaus, L.M.Sega, Some properties of graded local cohomology modules,Journal of Algebra 283(2005),232-247.
10. R. Sazeedeh, Artinianess of Graded local cohomology modules, Proceedings of the AMS. 135 (2007), 2339-2345.
11. R. Sazeedeh, Finiteness of graded local cohomology modules, J. Pure Appl. Alg. 212(1) (2008), 275-280.
12. N. Zamani, On graded generalized local cohomology, Arch. Math. 86(2006), 321-330.