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Automorphism Group of a Possible 2-(121, 16, 2) Symmetric Design

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Abstract

Let D be a symmetric 2-(121, 16, 2) design with the automorphism group of Aut(D). In this paper the order of automorphism of prime order of Aut(D) is studied, and some results are obtained about the number of fixed points of these automorphisms. Also we will show that $|Aut(D)|=2^p\, 3^q\, 5^r\, 7^s\, 11^t\, 13^u$, where p, q, r, s, t and u are non-negative integers such that r, s, t, u $\le$ 1. In addition we present some general results on the automorphisms with prime order of a symmetric design and some general results on the automorphism groups of a symmetric design are given and in Section 3, we prove a series of Lemma. Based on them we can prove main Theorem. One of the reasons for the emergence and growth of block designs is the combined irrigation of fields having a lot of patches, at the end of the paper, there is offered an application of block design in modern irrigation.

Keywords: Automorphism group of a design; Block design; Fixed blocks; Irrigation; Symmetric design

Introduction

The terminology and notations in this paper for designs and automorphism groups are as in [3,6,8]. In [3], some incidence structures have been discussed and block design, as a special case, has been examined. In this paper, we consider a block design, although the existence or non-existence of such design is not known with known parameters.

Let $X=\{x_1, x_2, ..., x_v\}$ denote a v-set, and $P_i(X)$ the set of *i* -subset (a subset with *i* elements) of X, where $1 \le i \le v$. The ordered pair D=(X, **B**) in which **B** is a collection of the elements of $P_k(X)$ usually called blocks) is called an incident structure denoted by 2-(v, k, λ) design if every element of $P_2(X)$ appears on B, λ times. D is called a symmetric design if |B|= v. A symmetric design with λ =2 is called a biplane. In a symmetric design, the appearance of each element, r, in the blocks of design is equal to k.

A mapping ψ between two designs D=(X, **B**), and D'=(X', **B** ') is an isomorphism if $\psi: X \to X'$ is a 1-1 correspondence and ψ (**B**)= **B** '. Every isomorphism of a design D to itself is called an automorphism and the set of all automorphisms of the same design with the natural composition rule among mappings form the automorphism group of the design, denoted by Aut (D). For $f \in$ Aut(D) and $x \in X$ we denote by fix(f)={$x \in X|$

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x^f =x}, and $\overline{x}$={x^f | f $\in$ Aut(D)}, the set of fixed points of f and the orbit of x, respectively.

Material and Methods

First we have the following lemma, which we shall refer to as Burnside's Lemma.

Lemma 1.1. [7, Theorem 3. 26] Let G be a permutation group on a finite set X. If r is the number of distinct orbits in X, then $r = \frac{1}{|G|}\sum_{\alpha\in G} |\text{fix}\ (\alpha)|$.

Let D be a 2-(v, k, λ) design and $\alpha \in$ Aut (D). Then α induces an automorphism α_B on the blocks of D. If D is a symmetric 2-(v, k, λ) design, then there is a close connection between α and α_B which has been proved as the following lemmas in [6].

Lemma 1.2. Let α be an automorphism of a non-trivial symmetric 2-(v, k, λ) design D. Then |fix(α)|=|fix (α_B)|.

Lemma 1.3. Let α be an automorphism of a non-trivial symmetric 2-(v, k, λ) design D. Then the number of orbits of G on points equals that of on blocks.

The following lemma is resulted from Bowler. [4, Theorem 3.1. and 3.3.]

Lemma 1.4. For a non-identity automorphism α of a symmetric 2-(v, k, λ) design fixing m points, we have

$$f \le \frac{1}{4}\ (v+3k-6) \quad \text{if } o(\alpha) \ge 3,$$

$$f \le \frac{1}{3}\ (v+2k-4) \ \text{ if } o(\alpha)=2.$$

Results and Discussion

The purpose of this paper is to study the automorphism group of a possible symmetric 2-(121, 16, 2) design. Precisely, we prove the following main theorem.

Theorem 1.5 Let G be an automorphism group of a possible 2-(121, 16, 2) design and $\alpha \in$ G, then $|G| = 2^p\ 3^q\ 5^r\ 7^s\ 11^t\ 13^u$, where r, s, t,u $\in$ {0, 1} and p, q are non-negative integers. In addition,

(i) If o(α)=3, then |fix(α)|=1 or 7. Also, if o(α)=7, then |fix(α)|=2 and also there exists a 2-(7, 4, 2) symmetric design in the structure of this design.

(ii) If o(α)= 5, 11 or 13, then |fix(α)|= 1, 0 or 4, respectively.

Some Results on Automorphism Group of Symmetric Design

In this section, we present some general results on the automorphisms with prime order of a symmetric design. The following lemmas are basic.

Lemma 1.6. [1, Lemma 2.3.] Let D=(X, **B**) be a 2-(v, k, λ) symmetric design. Assume that α is an automorphism of prime order p, such that (λ, p)=1. If a block B $\in$ **B** contains at least two fixed points, then the number of fixed points in B is congruent to k mod p. Also the number of the fixed blocks of B which containing the fixed points x is congruent to k mod p.

Lemma 1.7. [1, Lemma 2.4.] Let D=(X, **B**) be a 2-(v, k, λ) symmetric design. Suppose that B_1 and B_2 are two fixed blocks on the action of the automorphism α of prime order p of D, where λ<p. If $x \in B_1 \cap B_2$, then x $\in$ fix (α).

Lemma 1.8. If x, y $\in$ fix (α) and λ<p, then all blocks of B containing x and y are fixed blocks.

Proof. Since p<λ, (λ,p)=1 and by Lemma 1.6, proof is complete. □

Lemma 1.9. [1, Lemma 2.6.] Suppose that D=(X, **B**) is a 2-(v, k, λ) symmetric design, and p is a prime number such that p>λ-1>1. Take $\alpha \in$ Aut (D) with order p. Then B $\not\subset$ fix (α) for any block B. The following lemmas play vital role in proving the main theorem of this paper.

Lemma 1.10. Let D be a 2-(v, k, 2) symmetric design, and $\alpha \in$ Aut(D). Suppose that B is a block of D containing m($\ge$ 3) fixed points. Then there exist at least $\binom{m}{2}+1$ fixed block. Also

$$|\text{fix}(\alpha)| \ge \binom{m}{2}+1.$$

Proof. Let B $\in$ **B**, and x, y $\in$ B such that x, y $\in$ fix(α). Then α maps B to a block which contains x, y. So if B is a fixed block, since λ=2, every other block containing x,y must be fixed. Now since B contains m fixed points and $m \geq 3$, B is a fixed block and hence every pair of this m fixed points must be in the distinct fixed blocks. So we have at least $\binom{m}{2}$ fixed blocks. In addition, B is a fixed block, hence we have at least $\binom{m}{2}+1$ fixed blocks. By Lemma 1.2, $|\text{fix}(\alpha)| \geq \binom{m}{2}+1$. □

Lemma 1.11. Let D be a 2-(v, k, 2) symmetric design, $\alpha \in$ Aut (D), and o(α)=p is a prime. If p> λ , then

$$\{\sum_{B \in fix(\alpha_B)} |B - fix(\alpha)|\} + |fix(\alpha)| \leq v.$$

Proof. By Lemma 1.7, (B-fix(α)) $\cap$ (B'-fix(α))= ϕ , for any fixed blocks B and B'. Therefore, the number of elements out of intersection is equal to $\sum_{B \in fix(\alpha_B)} |B - fix(\alpha)|$ and if we add it to |fix(α)| then sum is less than v. □

Lemma 1.12. Let D be the 2-(v, k, 2) symmetric design, $\alpha \in$ Aut (D), and o(α)=p is a prime. If p> λ , and every block contains at most m fixed points, then $|\text{fix}(\alpha)|-1 \leq \frac{1}{2}(\text{m}^2 - m)$ and if equality holds there is a 2-(|fix(α)|, m, 2) symmetric design in the structure of this design.

Proof. By Lemma 1.6, if a block B contains two fixed points, then it is a fixed block. Hence every pair of fixed points is contained in a fixed block. Now we have $\binom{|\text{fix}(\alpha)|}{2} \times 2$ pairs of fixed points and since the fixed blocks contain at most $\binom{m}{2}|\text{fix}(\alpha)|$ pairs of fixed points, we have

$$\binom{|\text{fix}(\alpha)|}{2} \times 2 \leq \binom{m}{2} |\text{fix}(\alpha)|.$$

Therefore $|\text{fix}(\alpha)|-1 \leq \frac{1}{2}(\text{m}^2 - m)$. If the equality holds, then every fixed block contains exactly m fixed points and every pair of fixed points is in two fixed blocks. take $\mathbf{B} = \{B \cap \text{fix}(\alpha)\}$ where B is a fixed block and X=fix(α). Then D=(X, **B**) is a 2-(|fix(α)|, m, 2) symmetric design. The following lemma is useful to find the real path of knowing the problem. Its proof can be found in [5].

Lemma 1.13. If $\alpha \in$ Aut (D) and D is a 2-(v, k, λ) symmetric design, and o(α)=2, then

$$1+\frac{\text{k-1}}{\lambda} \leq |\text{fix}(\alpha)| \leq \text{k}+\sqrt{\text{k-}\lambda}$$

and α induces the odd number of orbits on points.

Automorphism of Prime Order

From Ashbacher's theorem [2, Theorem 2.7. p. 274], if a prime p divides the order of automorphism group of a 2-(v, k, λ) symmetric design, then p divides v or $p \leq k$. Therefore, if f is an automorphism of a 2-(121, 16, 2) symmetric design of prime order, then o(f) $\in$ {2, 3, 5, 7, 11, 13}.

Automorphism of Order 3

Let α be an automorphism of order 3. Then by Lemmas 1.4 and 1.6 we have |fix(α)| $\in$ {1, 4, 7, …, 49}. As Lemma 1.6, every block which contains a pair of fixed points is a fixed block, so from the Lemma 1.10, every fixed block contains at most ten fixed points. Otherwise, if the number of fixed points of fixed block is greater than ten then we have $|\text{fix}(\alpha)| \geq \binom{13}{2} + 1 = 79$ which is contradiction. If there exists a block with 10 fixed points, we claim that |fix(α)|= 46.

Now suppose that B is a fixed block with 10 fixed points. Then by Lemma 1.12, $|\text{fix}(\alpha)| \leq \frac{1}{2}(\text{m}^2 - m)+1=46$ and $|\text{fix}(\alpha)| \geq \binom{10}{2}+1 = 46$ and so |fix(α)|= 46. Now |fix(α)|=46, then every fixed block contains 10 fixed points, and |B|-|fix(α)|=6 for any fixed block. Hence every fixed block contains at most 7 fixed points. Now suppose that there exists a block with 7 fixed points. Hence |fix (α)| $\geq$ 22. By Lemma 1.13, fix(α) $\leq$ 22, and therefore |fix(α)|=22. In this case every fixed block contains 7 fixed points, and there exists a 2-(22, 7, 2) symmetric design in the structure of this design and this is a contradiction, since there exists no design with this parameter [4]. Therefore, every block of this design contains at most 4 fixed points.

Now if a block of design contains a block with 4 fixed points, then by Lemma 1.13, |fix(α)|=7 and fixed block contains 4 fixed points, so there exists a 2-(7, 4, 2) symmetric design in the structure of this design (which is a complement of a Fano plane). We summarize our argument about the automorphism of order 3 in the following lemma.

Lemma 1.14. If α is an automorphism of a 2-(121,16,2) symmetric design of order 3, then $|fix(\alpha)|=1$ or 7, and if $|fix(\alpha)|=7$, then there exists a complement of Fano plane in the structure of design.

Automorphism of Order 11

Let α be an automorphism of 2-(121, 16, 2) design of order 11. Then we have the following lemma.

Lemma 1.15. $|fix(\alpha)|=0$.

Proof. If $|fix(\alpha)|\geq 11$, then there exist at least 11 fixed blocks. By Lemma 1.6 every fixed block contains exactly 5 fixed points. Hence $|fix(\alpha)|\neq 22$ or 33 or 44. If $|fix(\alpha)|=11$, then there exists a 2-(11, 5, 2) symmetric design in the structure of this design. This is a contradiction to Lemma 1.11. □

Lemma 1.16. If $11^q \mid |Aut(D)|$, then $q\leq 1$.

Proof. If $11^2 \mid |Aut(D)|$, then for every non-identity element we have $|fix(\alpha)|=0$. By Burnside's Lemma, the number of orbits r is equal to $\frac{1}{121}\sum|fix(\alpha)|=1$. Hence D has a transitive automorphism group. Therefore, D is generated by a (121,16,2) difference set and hence by [6] this is impossible.

Automorphism of Order 5

Suppose $\alpha\in$ Aut(D), and $o(\alpha)=5$. By Lemma 1.11, every fixed block contains at most 6 fixed points. Hence by Lemma 1.13, $|fix(\alpha)|\leq 16$. Now if a fixed block in design contains 6 fixed points, then $|fix(\alpha)|\leq$ 16. So in this case, there exists a 2-(16,6,2) symmetric design in the structure of this design. But by Lemma 1.12 this is impossible. Therefore, we have the following lemma.

Lemma 1.17. (i) If α is an automorphism of a 2-(121, 16, 2) symmetric design of order 5, then $|fix(\alpha)|=1$.

(ii) If $5^q \mid |Aut(D)|$, then $q\leq 1$.

Proof. Suppose that $5^2 \mid |Aut(D)|$, and G is an automorphism group of design of order 25. So by Burnside's lemma, the number of orbits $r = \frac{1}{25}\sum_{\alpha\in G}|fix(\alpha)|$, where r is not an integer number, and so this is a contradiction. □

Automorphism of Order 7 and 13

Suppose that $\alpha\in$ Aut (D) and $o(\alpha)=7$. Then by Lemma 1.6, every fixed block contains at most 9 fixed points. If a block contains 9 fixed points, by Lemmas 1.10 and 1.12, $|fix(\alpha)|=37$, and hence there is a 2-(37, 9, 2) symmetric design in the structure of this design. But this is a contradiction to Lemma 1.11. So every fixed block contains 2 fixed points and we have

Lemma 1.18. (i) $|fix(\alpha)|=2$.

(ii) If $5^q \mid |Aut(D)|$, then $q\leq 1$.

Proof. By the same argument as Lemma 4. 2. □

Now with the same argument, if $\alpha\in$ Aut(D) and $o(\alpha)=13$, then $|fix(\alpha)|=4$, and if $13^u \mid |Aut(D)|$ then $u\leq 1$. All the above arguments complete the proof of Theorem 1.5.

An Application of Theorem 1.5 in Advanced Irrigation

The concept of block designs was first introduced when the farm in Netherlands were to be irrigate a land then was referred to mathematicians. If a farmland has v land patches and the water is pumped so rapidly that k land patches are irrigated simultaneously, agricultural products need r times of irrigation, a set of land patches are irrigated twice simultaneously the possibility or lack of possibility of such as irrigation leads to existence or nonexistence of a 2-(v, k, λ) design. The automorphism group of a design is away to find a design through which several researchers have found the design the existence or non-existence of which hasn't been known. What was proved in this article shown that if there exists a method to irrigate a field with 121 land patch with input water needed to irrigated 16 land patches while every two land patches while are irrigated twice simultaneously and each land patches is irrigated 16 time, there will be $2^p\ 3^q\ 5^r\ 7^s\ 11^t\ 13^u$, methods to irrigated where r, s, t, u ≤ 1 and p, q, r, s, t, u are non-negative integers.

Finally, one may ask whether there exists the 2-(121, 16, 2) design, which demands further investigation.

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