

On the Dynamic of a Nonautonomous Difference Equation

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Received: 27 April 2008 / Revised: 12 January 2009 / Accepted: 26 January 2009

Abstract

Nonlinear difference equations of higher order are important in applications; such equations appear naturally as discrete analogues of differential and delay differential equations which model various diverse phenomena in biology, ecology, economics, physics and engineering. The study of dynamical properties of such equations is of great importance in many areas. The autonomous difference equations have been studied extensively. The situation is more complicated when the considered model is nonautonomous. In this work we study global attractivity and boundedness of solutions of the following nonautonomous difference equation of order $k + 1$, $x_{n+1} = \frac{a_n x_n}{1 + x_n + \dots + x_{n-k}}$, in which $\{a_n\}$ is a positive bounded sequence. This equation is nonautonomous form of the logistic type difference equation with several delays. We prove that if $\liminf_{n \rightarrow \infty} a_n > 1$, then every positive solution is bounded and persistence. Furthermore we prove that when we have a positive solution $\{\bar{x}_n\}$ such that $k(k+1) \limsup_{n \rightarrow \infty} \bar{x}_n \leq 1$, then for all positive solutions $\{x_n\}$, $\lim_{n \rightarrow \infty} \frac{x_n}{\bar{x}_n} = 1$.

Keywords: Difference equations; Boundedness; Attractivity

Introduction

Population models, as taught and practiced by ecologists over the past century, represent the attempts to understand the nature affecting species populations. Applications of difference equations (species with non overlapping generations) and differential equations

(species with overlapping generations) on modeling of population dynamics have been widely studied (see [1-5,8,10-12]).

The classical logistic equation

$$\frac{dx(t)}{dt} = rx(t)\left(1 - \frac{x(t)}{P}\right), \quad t > 0 \quad (1)$$

represents a simple model of a single species dynamics in which r and P denote the growth rate and the carrying capacity of the population, respectively.

Usually, bearing in mind the known properties of the continuous model, the analogous difference equations proposed providing a best discrete approximation. For example

$$x_{n+1} = \frac{\alpha x_n}{1 + \beta x_n} \quad (2)$$

is the discrete version of (1). The model (1) ignores many complicating factors such as those due to age structure, spatial distribution and migration, sexual categories and others. The delayed logistic equation with a discrete delay or Hutchinson's equation [3]

$$\frac{dx(t)}{dt} = rx(t) \left(1 - \frac{x(t-\tau)}{P}\right) \quad (3)$$

provides a simple modification of (1). Pielou [11] considered the delay difference equation

$$x_{n+1} = \frac{\alpha x_n}{1 + \beta x_{n-k}} \quad (4)$$

as a discrete analogue of the delay logistic equation (3). Also, Kocic et al. in [7] studied the discrete logistic nonautonomous equation

$$x_{n+1} = \frac{\alpha_n x_n}{1 + x_{n-k}} \quad (5)$$

In order to introduce several delays, equation (4) has been modified to the following:

$$x_{n+1} = \frac{\alpha x_n}{1 + \sum_{i=1}^m \beta_i x_{n-k_i}} \quad (6)$$

which can be considered as a discrete analogue of the delay logistic equation:

$$\frac{dx(t)}{dt} = rx(t) \left(a - \sum_{j=1}^m b_j x(t - \tau_j)\right) \quad (7)$$

with several delays τ_j ($j = 1, 2, \dots, m$). Eq.(6) studied extensively in [6].

We consider in this paper the nonautonomous difference equation

$$x_{n+1} = \frac{a_n x_n}{1 + x_n + \dots + x_{n-k}}, \quad n = 0, 1, \dots \quad (8)$$

with initial conditions $x_{-k}, \dots, x_{-1} \geq 0, x_0 > 0$ and

$$0 < C \leq a_n \leq D < \infty, \quad n = -k, -k+1, \dots$$

in which k is a nonnegative integer. If for some $\alpha \in (0, 1)$ and for all $n \in \mathbb{N}$, $a_n \leq \alpha$, then by [9, Th. 2] the trivial solution $x = 0$ is the global exponential attractor of all solutions of Eq.(8). Furthermore author in [10] proved that if $a_n < 1$ and $\lim_{n \rightarrow \infty} a_n = 1$, the zero solution is globally asymptotically stable.

In this work we obtain new sufficient conditions for global attractivity of the solutions of Eq.(8). Also we study the boundedness and persistence of the solutions.

Results

1. Boundedness

We use the following notation in the sequel:

$$a = \liminf_{n \rightarrow \infty} a_n, \quad b = \limsup_{n \rightarrow \infty} a_n \quad (9)$$

Lemma 1. Assume that one of the following two conditions hold:

- (1) $b < 1$
- (2) $b = 1$ and there exists an integer N_0 such that $a_n \leq 1$ for $n \geq N_0$.

Then every positive solution $\{x_n\}$ of (8) is eventually decreasing and $\lim_{n \rightarrow \infty} x_n = 0$.

Proof. For sufficiently large n we have $a_n \leq 1$ and

$$x_{n+1} = \frac{a_n x_n}{1 + x_n + \dots + x_{n-k}} < x_n$$

hence $\{x_n\}$ is eventually decreasing, and therefore convergent, that is $\lim_{n \rightarrow \infty} x_n = x \geq 0$. Now there exists a subsequence $\{a_{n_i}\}$ such that $\lim_{i \rightarrow \infty} a_{n_i} = \limsup_{n \rightarrow \infty} a_n = b$, and we have

$$x = \lim_{i \rightarrow \infty} x_{n_i} = \lim_{i \rightarrow \infty} \frac{a_{n_i} x_{n_i}}{1 + x_{n_i} + \dots + x_{n_i-k}} = \frac{bx}{1 + (k+1)x}$$

hence we have $x = 0$ or $x = \frac{b-1}{k+1} \leq 0$ which yield $x = 0$. ■

Lemma 2. The following statements are true:

- (i) If one of the following two conditions holds:
 - (a) $b > 1$;
 - (b) $b = 1$ and for every positive integer N_0 there exists $n > N_0$ such that $a_n > 1$;

Then there exists a positive integer N_0 such that for every $n > N_0$,

$$x_{n-k}, x_{n-k+1}, \dots, x_n \geq \bar{x} > b-1 \Rightarrow x_{n+1} < x_n.$$

(ii) If $a > 1$ then there exists a positive integer N_0 such that for every $n > N_0$

$$x_{n-k}, x_{n-k+1}, \dots, x_n \leq \bar{x} > a-1 \Rightarrow x_{n+1} > x_n.$$

Proof. We prove part (i). Let $0 < \varepsilon < (k+1)\bar{x} + 1 - b$, then there exists $N_0 > k$ such that $a_n < b + \varepsilon$ for $n > N_0$. Furthermore, if $n > N_0$ and $x_{n-k}, x_{n-k+1}, \dots, x_n \geq \bar{x} > b-1$ then we have

$$x_{n+1} = \frac{a_n x_n}{1 + x_n + \dots + x_{n-k}} \leq \frac{(b + \varepsilon)x_n}{1 + (k+1)\bar{x}} < x_n.$$

Part (ii) can be proved by a similar argument. ■

A sequence $\{x_n\}$ is said to oscillate if the terms x_n are neither eventually all positive nor eventually all negative. Otherwise the sequence is called nonoscillatory. A sequence $\{x_n\}$ is called strictly oscillatory if for every $n_0 \geq 0$, there exist $n_1 > n_2 \geq n_0$ such that $x_{n_1} x_{n_2} < 0$. A sequence $\{x_n\}$ is said to oscillate about the sequence $\{\bar{x}_n\}$ if the sequence $\{x_n - \bar{x}_n\}$ oscillates. The sequence $\{x_n\}$ is called strictly oscillatory about $\{\bar{x}_n\}$ if the sequence $\{x_n - \bar{x}_n\}$ is strictly oscillatory.

For positive sequences $\{x_n\}$ and $\{\bar{x}_n\}$, we define a positive semicycle of $\{x_n\}$ relative to the sequence $\{\bar{x}_n\}$ as a string of terms $C_+ = \{x_{l+1}, x_{l+2}, \dots, x_m\}$ such that $x_i \geq \bar{x}_i$ for $i = l+1, \dots, m$ with $l \geq -k$ and $m \leq \infty$ and such that either $l = -k$ or $l \geq 0$ and $x_l < \bar{x}_l$ and either $m = \infty$ or $m < \infty$ and $x_{m+1} < \bar{x}_{m+1}$. A term $x_p \in C_+$ is said to be a maximum of the positive semicycle C_+ relative to $\{\bar{x}_n\}$ if

$$\frac{x_p}{x_p} = \max \left\{ \frac{x_{l+1}}{x_{l+1}}, \dots, \frac{x_m}{x_m} \right\}.$$

A negative semicycle is defined similarly.

Lemma 3. Every positive solution of equation (8) is bounded from above by a positive constant.

Proof. We have the following possible cases:

(1) Either $b < 1$ or $b = 1$ and there exists an integer N_0 such that $a_n \leq 1$, for $n \geq N_0$. From lemma 1 it follows that every positive solution of (8) converges to 0 and therefore it is bounded above.

(2) $b > 1$ or $b = 1$ and for every positive integer N_0 there exists $n > N_0$ such that $a_n > 1$. In this case $B = \sup\{a_n\} > 1$. Let $\bar{x} = B - 1$ and $\{x_n\}$ be a nontrivial positive solution of (8). The following three cases are possible:

(a) $x_n \leq \bar{x}$ for $n \geq N_0$. In this case $\{x_n\}$ is bounded.

(b) $x_n > \bar{x}$ for $n \geq N_0$. By the lemma 2(i), $\{x_n\}$ is decreasing, and therefore convergent and bounded.

(c) $\{x_n\}$ strictly oscillates about $\bar{x}$. Let $\{x_{q_{i+1}}, \dots, x_{p_i}\}$ be the i -th negative semicycle relative to $\bar{x}$ followed by the i -th positive semicycle $\{x_{p_{i+1}}, \dots, x_{r_i}\}$. Let x_{M_i} be the maximum in the i -th positive semicycle. From lemma 2(i) it follows that the maximum in the positive semicycle occurs in the first $k+1$ terms. Now we have

$$\begin{aligned} x_{M_i} &= \frac{a_{M_i} x_{M_i}}{1 + x_{M_i-1} + \dots + x_{M_i-1-k}} \\ &= \dots = \frac{a_{M_i-1} \dots a_{p_i}}{(1 + \dots + x_{M_i-1-k}) \dots (1 + \dots + x_{p_i-k})} x_{p_i} \\ &< B^{M_i-p_i} \bar{x} < B^{k+1} \bar{x} \end{aligned}$$

hence

$$\limsup_{n \rightarrow \infty} x_n = \limsup_{i \rightarrow \infty} x_{M_i} \leq B^{k+1} \bar{x}$$

therefore $\{x_n\}$ is bounded. ■

Lemma 4. If $a > 1$ then every positive solution of (8) is bounded below by a positive constant.

The proof is omitted since it is similar to the proof of the above lemma. Now the following theorem is a direct consequence of lemmas 3 and 4.

Lemma 5. Assume that $a > 1$. Then every positive solution of (8) is bounded and persist, that is there exist positive constants A, B such that for every $n \geq -k$

$$0 < A < x_n < B < \infty$$

2. Attractivity

In this section we study attractivity of solutions of (8).

Lemma 6. Assume that $a > 1$ and let $\{\bar{x}_n\}$ be a positive solution of (8). Then the following statements are true:

- (i) If for some integer m , $x_{m-k} = \bar{x}_{m-k}$, ..., $x_m = \bar{x}_m$ then for every $n = 0, 1, \dots$ $x_n = \bar{x}_n$.
- (ii) The extremum in every semicycle relative to $\{\bar{x}_n\}$, except perhaps in the first one, occurs in the first $k+1$ terms.

Proof.

Part (i) is obvious. For the proof of (ii) we consider only positive semicycles. If the semicycle has no more than $k+1$ terms the extremum is one of them. Assume that it has at least $k+2$ terms, that is $x_{n-k} \geq \bar{x}_{n-k}$, ..., $x_{n+1} \geq \bar{x}_{n+1}$. then

$$\begin{aligned} x_{n+1} &= \frac{a_n x_n}{1 + x_n + \dots + x_{n-k}} \\ &\leq \frac{a_n x_n}{1 + \bar{x}_n + \dots + \bar{x}_{n-k}} = \bar{x}_{n+1} \frac{x_n}{\bar{x}_n} \\ \frac{x_{n+1}}{\bar{x}_{n+1}} &\leq \frac{x_n}{\bar{x}_n} \end{aligned}$$

and the result follows. ■

For two positive sequences $\{x_n\}$, $\{\bar{x}_n\}$ we use the notation $x_n \sim \bar{x}_n$, if $\lim_{n \rightarrow \infty} \frac{x_n}{\bar{x}_n} = 1$.

Lemma 7. Assume that $a > 1$. Then for every positive solution $\{x_n\}$ of (8) which is nonoscillatory relative to $\{\bar{x}_n\}$,

$$x_n \sim \bar{x}_n.$$

Proof. We consider only the case when $x_n \geq \bar{x}_n$ for $n \geq N_0$. From lemma 6

$$1 \leq \frac{x_{n+1}}{x_n} \leq \frac{\bar{x}_{n+1}}{\bar{x}_n} \text{ for } n > N_0.$$

Therefore

$$\lim_{n \rightarrow \infty} \frac{x_n}{\bar{x}_n} = \alpha \geq 1.$$

We prove that $\alpha = 1$. Since $\{x_n\}$, $\{\bar{x}_n\}$, $\{a_n\}$ are bounded and persistent sequences, there exists a subsequence $\{n_i\}$ of integers such that the subsequences $\{x_{n_i}\}$, ..., $\{x_{n_i-k}\}$, $\{\bar{x}_{n_i}\}$, ..., $\{\bar{x}_{n_i-k}\}$, $\{a_{n_i}\}$ are convergent.

Let

$$\lim_{i \rightarrow \infty} \bar{x}_{n_i} = x'_0 > 0, \dots, \lim_{i \rightarrow \infty} \bar{x}_{n_i-k} = x'_k > 0,$$

$$\lim_{i \rightarrow \infty} a_{n_i} = p > 0.$$

Now from the above relations $\lim_{i \rightarrow \infty} x_{n_i} = \alpha x'_0$, ..., $\lim_{i \rightarrow \infty} x_{n_i-k} = \alpha x'_k$.

Therefore we have

$$\begin{aligned} \alpha &= \lim_{i \rightarrow \infty} \frac{x_{n_i+1}}{x_{n_i+1}} \\ &= \lim_{i \rightarrow \infty} \frac{\frac{a_{n_i} x_{n_i}}{1 + x_{n_i} + \dots + x_{n_i-k}}}{\frac{a_{n_i} x_{n_i}}{1 + \bar{x}_{n_i} + \dots + \bar{x}_{n_i-k}}} = \frac{p \alpha x'_0}{1 + \alpha x'_0 + \dots + \alpha x'_k} \\ &= \frac{p x'_0}{1 + x'_0 + \dots + x'_k} \end{aligned}$$

which yields that $\alpha = 1$ and the proof is complete. ■

Theorem 8. Assume that $a > 1$. Let $\{\bar{x}_n\}$ be a positive bounded solution of (8) with

$$0 < r = \liminf_{n \rightarrow \infty} \bar{x}_n \leq \limsup_{n \rightarrow \infty} \bar{x}_n = s < \infty.$$

If $k(k+1)s \leq 1$, then for every positive solution $\{x_n\}$ of (8), $x_n \sim \bar{x}_n$.

Proof. For the nonoscillatory solution $\{x_n\}$ with respect to $\{\bar{x}_n\}$ the previous lemma proves that $x_n \sim \bar{x}_n$. Assume that $\{x_n\}$ is oscillatory with respect to $\{\bar{x}_n\}$. Let $\{x_{q_i+1}, \dots, x_{p_i}\}$ be the i -th negative semicycle followed by the i -th positive semicycle $\{x_{p_i+1}, \dots, x_{r_i}\}$. Denote by x_{M_i} the maximum in the i -th positive semicycle and x_{m_i} the minimum in the i -th negative semicycle. Let

$$\lambda = \liminf_{n \rightarrow \infty} \frac{x_n}{x_n} = \liminf_{i \rightarrow \infty} \frac{x_{m_i}}{x_{m_i}}$$

and

$$\mu = \limsup_{n \rightarrow \infty} \frac{x_n}{x_n} = \limsup_{i \rightarrow \infty} \frac{x_{m_i}}{x_{m_i}}.$$

Clearly $\lambda \leq 1 \leq \mu$ and we must show that $\lambda = \mu = 1$.

From lemma 6(ii) it follows that the maximum in the positive semicycle occurs in the first $k+1$ terms of the semicycle, and we have $M_i - p_i \leq k+1$. Clearly for

every $\varepsilon > 0$, and n sufficiently large $\lambda - \varepsilon < \frac{x_n}{x_n}$ and $\bar{x}_n < s + \varepsilon$ (11)

For i sufficiently large we have $\frac{x_{M_i}}{x_{M_i}} =$

$$\begin{aligned} & \frac{a_{M_i-1} x_{M_i-1}}{1 + x_{M_i-1} + \dots + x_{M_i-1-k}} \frac{1 + \bar{x}_{M_i-1} + \dots + \bar{x}_{M_i-1-k}}{a_{M_i-1} x_{M_i-1}} \\ & < \frac{1 + \bar{x}_{M_i-1} + \dots + \bar{x}_{M_i-1-k}}{1 + (\lambda - \varepsilon)(x_{M_i-1} + \dots + x_{M_i-1-k})} \frac{x_{M_i-1}}{x_{M_i-1}} \\ & < \dots < A \times \frac{x_{p_i}}{x_{p_i}} \end{aligned}$$

in which

$$\begin{aligned} A &= \frac{1 + \bar{x}_{M_i-1} + \dots + \bar{x}_{M_i-1-k}}{1 + (\lambda - \varepsilon)(x_{M_i-1} + \dots + x_{M_i-1-k})} \dots \\ & \quad \frac{1 + \bar{x}_{p_i} + \dots + \bar{x}_{p_i-k}}{1 + (\lambda - \varepsilon)(x_{p_i} + \dots + x_{p_i-k})}. \end{aligned}$$

Now we consider two cases. First, consider the case when the maximum occurs in the first k terms of the positive semicycle. Since $\lambda - \varepsilon < 1$ and $M_i - p_i \leq 1$ we have

$$\begin{aligned} \frac{x_{M_i}}{x_{M_i}} &< A \times \frac{1 + \bar{x}_{p_i-1} + \dots + \bar{x}_{p_i-1-k}}{1 + (\lambda - \varepsilon)(x_{p_i-1} + \dots + x_{p_i-1-k})} \\ &\times \dots \times \frac{1 + \bar{x}_{M_i-k-1} + \dots + \bar{x}_{M_i-2k}}{1 + (\lambda - \varepsilon)(x_{M_i-k-1} + \dots + x_{M_i-2k})} \frac{x_{p_i}}{x_{p_i}} \end{aligned}$$

Furthermore $\frac{x_{p_i}}{x_{p_i}} < 1$ and function

$\frac{1 + (k+1)x}{1 + (k+1)(\lambda - \varepsilon)x}$ is increasing for all x . Now from the above relation and (11) it follows

$$\frac{x_{M_i}}{x_{M_i}} < \left(\frac{1 + (s + \varepsilon)(k+1)}{1 + (k+1)(\lambda - \varepsilon)(s + \varepsilon)} \right)^{k+1}.$$

Now we consider the case when the maximum occurs in the $(k+1)$ -th term of the positive semicycle relative to $\{\bar{x}_n\}$. Then $M_i = p_i + k + 1$ and we have

$$\begin{aligned} \frac{x_{M_i}}{x_{M_i}} &= \frac{1 + \bar{x}_{M_i-1} + \dots + \bar{x}_{M_i-1-k}}{1 + x_{M_i-1} + \dots + x_{M_i-1-k}} \frac{x_{M_i-1}}{x_{M_i-1}} < \dots \\ &< \frac{1 + \bar{x}_{p_i+k} + \dots + \bar{x}_{p_i}}{1 + (\lambda - \varepsilon)(x_{p_i+k} + \dots + x_{p_i})} \dots \\ & \quad \frac{1 + \bar{x}_{p_i} + \dots + \bar{x}_{p_i-k}}{1 + (\lambda - \varepsilon)(x_{p_i} + \dots + x_{p_i-k})} \frac{x_{p_i}}{x_{p_i}} \end{aligned}$$

Since $\frac{x_{p_i}}{x_{p_i}} < 1$ we have

$$\frac{x_{M_i}}{x_{M_i}} < \left(\frac{1 + (s + \varepsilon)(k+1)}{1 + (k+1)(\lambda - \varepsilon)(s + \varepsilon)} \right)^{k+1}.$$

Now since $\varepsilon > 0$ is arbitrary, we obtain

$$\frac{x_{M_i}}{x_{M_i}} < \left(\frac{1 + (k+1)s}{1 + (k+1)\lambda s} \right)^{k+1} = G(\lambda).$$

Hence $\mu \leq G(\lambda)$. Similarly one can show that $\lambda \leq G(\mu)$. Therefore by [7, Lemma 2] it follows that $\lambda = \mu = 1$ and the proof is completed. ■

Remark: In the case that $a_n = a$ (autonomous case), if

$a > 1$, Eq.(8) has a positive equilibrium $\bar{x} = \frac{a-1}{k+1}$.

From theorem 8, it follows that the sufficient condition for global attractivity of $\bar{x}$ is $k(a-1) \leq 1$. Also if $0 < a < 1$ then the trivial solution 0 is the global attractor of all positive solutions of Eq.(8).

Acknowledgments

The author would like to thank the referees for their useful comments. This work was financially supported by the Semnan University.

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