Derivations on Certain Semigroup Algebras

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Abstract

In the present paper we give a partially negative answer to a conjecture of
Ghahramani, Runde and Willis. We also discuss the derivation problem for both
foundation semigroup algebras and Clifford semigroup algebras. In particular, we
prove that if \( S \) is a topological Clifford semigroup for which \( E_s \) is finite, then
\( H^1(M(S),M(S))=\{0\} \).

Keywords: Foundation semigroup; Semigroup algebra; Derivation; First order cohomology; Clifford semigroup

1. Introduction

Let \( S \) be a locally compact topological semigroup, and let \( M(S) \) denote the space of all bounded complex
regular measures on \( S \). This space with the convolution
product and norm \( \|\mu\|=|\mu|(S) \) is a Banach algebra.
The space of all measures \( \mu \in M^*(S) \) for which the
mappings \( s \mapsto \delta_s^* \mu \) and \( s \mapsto |\mu| \delta_s \) from \( s \) into
\( M(S) \) are weakly continuous is denoted by \( M^*_w(S) \) (or \( L(S) \) as in [1]), where \( \delta_s \) denotes the Dirac
measure at \( s \). Note that the measure algebra \( M^*_w(S) \)
defines a two-sided closed \( L \)-ideal of \( M(S) \) (see [1]).

For a locally compact topological semigroup \( S \), let
\( M_0(S):=\{\mu \in M(S):\mu(S)=0\} \) and
\( I_0(S)=M_0(S) \cap M^*_w(S) \).

A semigroups \( S \) is called a foundation semigroup; if
\( \cup\{\text{supp}(\mu):\mu \in M_0(S)\} \) is dense in \( S \). Note that if
\( S \) is a foundation semigroup with an identity then
\( M^*_0(S) \) has a bounded approximate identity (c.f. [16]).

Let \( S \) be a foundation semigroup. Given any \( \mu \in M_0(S) \) and \( \phi \in M^*_0(S)' \), define the complex-valued function \( \phi \circ \mu \) and \( \mu \circ \phi \) on \( S \) by
\( (\phi \circ \mu)(s)=\phi(\delta_s^* \phi) \) and
\( (\mu \circ \phi)(s)=(\mu \delta_s \phi)(s) \).

It is clear that \( \phi \circ \mu \) and \( \mu \circ \phi \) are in \( C_b(S) \), where
\( C_b(S) \) denotes the space of all bounded continuous
complex-valued functions on \( S \). By Lemma 3.4 of
[16], for each \( \phi \in M^*_0(S)' \) and \( \mu,\nu \in M^*_0(S) \),
\( \phi(\mu \ast \nu)=\nu(\mu \ast \phi)=\mu(\phi \ast \nu) \).

Let \( S \) be a Banach algebra and \( X \) be a Banach \( A \)-
bimodule. A bounded linear map \( D:A \rightarrow X \) is called
an \( X \)-derivation, if
\( D(ab)=D(a)b+a.D(b) \) \( (a,b \in A) \).

For every \( x \in X \) we define \( \text{ad}_x \) by

\[ \text{ad}_x(\mu)(s)=\mu(x(s)) \]

2000 Mathematical Subject Classification: 43A20,46M20
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ad\_s (a) = a \cdot x - x \cdot a \quad (a \in A).

It is easily seen that \( \text{ad}\_s \) is a derivation. Derivations of this form are called inner derivations. The set of all derivations from \( A \) into \( X \) is denoted by \( Z^1(A, X) \), and the set of all inner \( X \)-derivations is denoted by \( B^1(A, X) \). Clearly, \( Z^1(A, X) \) is a linear subspace of the space of all bounded linear operators of \( A \) into \( X \) and \( B^1(A, X) \) is a linear subspace of \( Z^1(A, X) \). We denote by \( H^1(A, X) \) the difference space of \( Z^1(A, X) \) modulo \( B^1(A, X) \).

It is a conjecture raised by Ghahramani, Runde and Willis in [8]. This contradiction shows that \( M(S) \) is not semisimple. Also \( H^1(M(S), M(S)) \) is not zero (c.f. [8], Example on page 387). Thus the hypothesis of semisimplicity in Theorem 16.21 of [2] is necessary.

In the following we give examples of a semigroup \( S \) for which the first order cohomology \( H^1(M(S), M(S)) = \{0\} \), but \( S \) is neither left and nor right cancellative. This is a partially negative answer to the guess of Ghahramani, Runde and Willis in [8].

**Example 2.3.** Let \( A \) be a non-empty set, and let \( S = A \cup \{0\} \). With the multiplication defined by \( s^2 = s \) and \( st = 0 \) for all \( s, t \in S \) with \( s \neq t \), \( S \) is a commutative semigroup. Since for each \( t \in A \), the function \( \phi_t \) defined by \( \phi_t(s) = 0 \) for \( s \neq t \) and \( \phi_t(t) = 1 \) is a semicharacter on \( S \), the set of all semicharacters on \( S \), separates the points of \( S \). Hence by Proposition 4.1.4 of [6], \( \ell^i(S) \) is semisimple. From Theorem 16.21 of [2], it follows that \( H^1(\ell^i(S), \ell^i(S)) = \{0\} \), although \( S \) is not either left or right cancellative.

**Remark 2.4.** Let \( S \) be a compact, Hausdorff, cancellative right topological semigroup, then \( S \) is a compact topological group and so \( H^1(M(S), M(S)) = \{0\} \).

Before proving our next theorem we first need to prove two lemmas.

**Lemma 2.5.** Let \( S \) be a locally compact left zero semigroup with \( \text{Card}(S) > 2 \). Then \( S \) is a right cancellative semigroup for which \( H^1(M(S), M(S)) \neq \{0\} \).

Proof. Suppose first that \( S \) is a locally compact left zero semigroup, then it is clear that \( S \) is a right cancellative. Clearly for \( \mu, \nu \in M(S), \mu^*\nu = \nu(S)\mu \). Moreover we have

\[
Z^1(M(S), M(S)) = \{ L \in B(M(S), M(S)) : L(M(S)) \subseteq M_0(S) \}.
\]
To see this, take $D \in Z^1(M(S),M(S))$ and $\mu \in M(S)$, then

$$
\mu(S)D(\mu) = D(\mu(S)\mu) \\
= D(\mu^*\mu) \\
= D(\mu)^*\mu + \mu^*D(\mu) \\
= \mu(S)D(\mu) + D(\mu(S)\mu).
$$

Thus $D(\mu)(S) = 0$. This implies that $D(\mu) \in M_o(S)$. Conversely, if $D \in B(M(S),M(S))$, such that $D(\mu(S)) \subseteq M_o(S)$, then

$$
D(\mu^*\nu) = D(\nu(S)\mu) \\
= \nu(S)D(\mu) + D(\nu)(S)\mu \\
= D(\mu)^*\nu + \mu^*D(\nu).
$$

Now since Card$(S) \geq 3$, there exist $s_1, s_2, s_3 \in S$ such that $s_i \neq s_j$ for $i \neq j$. By the Hahn-Banach theorem there exists $D \in B\left(M(S),M_o(S)\right)$ such that $D(\delta_{s_1}) = 0$ and $D(\delta_{s_2}) = \delta_{s_1} - \delta_{s_1}$ (indeed, by the Hahn-Banach theorem there exists $\overline{D} \in B\left(M(S),\mathbb{C}\delta_{s_1} - \delta_{s_1}\right)$ that extends the following bounded linear map,

$$
\mathbb{C}\delta_{s_1} \oplus \mathbb{C}\delta_{s_2} \rightarrow \mathbb{C}(\delta_{s_1} - \delta_{s_1}) : \lambda_1\delta_{s_1} + \lambda_2\delta_{s_2} \mapsto \lambda_1(\delta_{s_1} - \delta_{s_1}),
$$

where $\lambda_1, \lambda_2 \in \mathbb{C}$. Now, define $D \in B\left(M(S),M_o(S)\right)$ by $D(\mu) = \overline{D}(\mu)$. By (1), $D$ is a derivation. If $D = ad_{\phi}$ for some $\nu \in M(S)$, then $D(\mu) = \nu(S)\mu - \mu\nu$. This implies that

$$
0 = D(\delta_{s_1}) = \nu(S)\delta_{s_1} - \delta_{s_1}(S)\nu \\
= \nu(S)\delta_{s_1} - \nu,
$$

and so $\nu = \nu(S)\delta_{s_1}$. Similarly $\delta_{s_1} - \delta_{s_1} = D(\delta_{s_2}) = \nu(S)\delta_{s_1} - \nu$. Therefore $\delta_{s_1} - \delta_{s_1} = \nu(S)(\delta_{s_2} - \delta_{s_1})$, and hence

$$
(\delta_{s_2} - \delta_{s_1})(\{s_1\}) = (\nu(S)(\delta_{s_2} - \delta_{s_1}))(\{s_1\}) = 0.
$$

This contradiction shows that $D \not\in B^1(M(S),M(S))$. Thus $H^1(M(S),M(S)) \neq \{0\}$. □

**Lemma 2.6.** Let $S$ be a left zero semigroup with Card$(S) = 2$, then $H^1(M(S),M(S)) = \{0\}$.

**Proof.** Let $S = \{s_1, s_2\}$ and $D \in Z^1(M(S),M(S))$. Then from (1) it follows that $D(M(S)) \subseteq M_o(S)$. Suppose that $D(\delta_{s_1}) = \alpha(\delta_{s_1} - \delta_{s_1})$ and $D(\delta_{s_2}) = \beta(\delta_{s_1} - \delta_{s_1})$. Set $\phi = \alpha\delta_{s_1} - \beta\delta_{s_1}$, then $ad_{\phi}(\delta_{s_1}) = \delta_{s_1}^*\phi - \phi^*\delta_{s_1} = \alpha(\delta_{s_1} - \delta_{s_1})$. Thus $ad_{\phi}(\delta_{s_1}) = D(\delta_{s_1})$. Similarly $ad_{\phi}(\delta_{s_2}) = D(\delta_{s_2})$. So $ad_{\phi} = D$. □

A combination of the above two lemmas yields the following result.

**Theorem 2.7.** Let $S$ be a left zero semigroup. Then $H^1(M(S),M(S)) = \{0\}$ if and only if Card$S \leq 2$.

**Remark 2.8.** Let $S$ be a left zero semigroup with two elements. Then by Lemma 2.6 we have $H^1(M(S),M(S)) = \{0\}$, but by Proposition 2.5 we have $H^1(M(S \times S),M(S \times S)) \neq \{0\}$.

### 3. Derivations on Foundation Semigroups

Our starting point of this section is the following definition.

**Definitions 3.1.** If a Banach algebra $A$ is contained in another Banach algebra $B$ as a closed ideal, then the strict topology or strong operator topology (so) on $B$ with respect to $A$ is defined through the family of seminorms $(p_x)_{x \in A}$, where

$$
p_x(b) := \|bx\| + \|xb\| \quad (b \in B).
$$

For a topological semigroup $S$ the strict topology on $M(S)$ with respect to $M(S)$ is simply called the so topology or the strict topology on $M(S)$. 

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Lemma 3.2. Let $B$ be a Banach algebra and $A$ be an ideal of $B$. Then $T \in (B, so)^*$ if and only if there exits subset $\{a_1, a_2, \ldots, a_n\}$ and $\{a'_1, \ldots, a'_n\}$ of $A$ and $\{\phi_1, \ldots, \phi_n\}$ and $\{\phi_1', \ldots, \phi_n'\}$ of $A^*$ such that for each $b \in B$

$$T(b) = \sum_{i=1}^{n} \phi_i(a, b) + \sum_{i=1}^{n} \phi'_i(ba_i)$$

Proof. Let $T \in (B, so)^*$. Then by Theorem 3.1 of [4] there exist $a_1, a_2, \ldots, a_n$ in $A$, such that

$$|T(b)| \leq \sum_{i=1}^{n} (\|b\| b_{ab} + b_{ba} 0) (b \in B).$$

Let $M = \{(a_1, b_1, a_2, b_2, \ldots, a_n, b_n) : b \in B\}$, and define the functional $F_0 : M \mapsto \mathbb{C}$ by

$$F_0(a, b, \ldots, a_n b, b_1, \ldots, b_n) = T(b).$$

Clearly $M \subseteq \Theta_{a_1}^{a_n} A$ and $F_0$ is well defined and bounded. By the Hahn-Banach theorem there is a bounded functional $F$ on $\Theta_{a_1}^{a_n} A$ such that $F|_u = F_0$.

For all $1 \leq i \leq n$ and $1 \leq j \leq 2$, define $\phi_0 \in A^*$ by

$$\phi_0(a) = F(0, \ldots, a_i, \ldots, 0) (a \in A).$$

Now for any $b \in B$ we have

$$T(b) = F_0(a_1 b_1, a_2 b_2, \ldots, a_n b_n) = F(a_1 b_1, a_2 b_2, \ldots, a_n b_n) = \sum_{i=1}^{n} \phi_i(a, b) + \sum_{i=1}^{n} \phi'_i(ba_i).$$

The other side is trivial. $\square$

The following result is a generalization of Proposition 3.41(i) of [5] from locally compact groups to the case of foundation semigroups with completely different technique of proof.

Theorem 3.3. Let $S$ be a foundation semigroup. Then $\ell^1(S)$ is so-dense in $M(S)$. Now let $\phi \in M_{\alpha}(S) \cap M_{\beta}(S)$, and $\nu \in M_{\gamma}(S)$, then

$$\phi(\mu_n \ast \nu) = \mu_n (\nu \phi) \to \mu(\nu \phi) = \phi(\mu \ast \nu).$$

Therefore by Lemma 3.2 for any $T \in (M(S), so)^*$ we have $T(\mu_n) \to T(\mu)$. So $\ell^1(S)$ is weakly dense in the locally convex space $(M(S), so)$. Since $\ell^1(S)$ is convex, by Theorem 3.12 of [15] we have $\mu \in \ell^1(S)^\ast$. $\square$

Proposition 3.4. Let $S$ be a foundation semigroup with identity. Then $D(M_{\alpha}(S)) \subseteq M_{\alpha}(S)$ for any $D \in Z^1(M_{\alpha}(S), M(S))$. Proof. Let $\{e_\alpha\}$ be a bounded approximate identity for $M_{\alpha}(S)$, then for each $\mu \in M_{\alpha}(S)$,

$$D(\mu) = \lim_{\alpha} D(\mu^* e_\alpha) = (\lim_{\alpha} D(\mu^*) e_\alpha + \mu^*D(e_\alpha)) \in M_{\alpha}(S).$$

Recall that $S$ is said to be left compactly cancellative if $C^{-1} S$ is a compact subset of $S$ for all compact subsets $C$ and $D$ of $S$, where

$$C^{-1} S = \{x \in S : cx \in D \text{ for some } c \in C\}.$$ Right compactly cancellative locally compact semigroups are defined similarly. A semigroup which is both left and right compactly cancellative is called compactly cancellative.

Let $A$ be a Banach algebra. A pair $(L, R)$ of operators $L$ and $R$ on $A$ is called a multiplier if for each $a, b \in A$, $L(ab) = L(a)b, R(ab) = aR(b)$ and $aL(b) = R(a)b$. The set of all multipliers on $A$, denoted by $M(A)$ with the multiplication defined by

$$(L_1, R_1)(L_2, R_2) = (L_1 \circ L_2, R_2 \circ R_1) \quad ((L_1, R_1), (L_2, R_2) \in M(A)),\)$$

is a Banach algebra that called the multiplier algebra of $A$.

In the proof of the following lemma we have been inspired by that of Theorem 3.340 of [5].

Lemma 3.5. Let $S$ be a compactly cancellative foundation semigroup with identity, Then the multiplier
algebra of \( M_\ast (S) \) is isomorphic with \( M(S) \).

**Proof.** For \( \mu \in M(S) \), define

\[
L_\mu (v) = \mu * v \quad \text{and} \quad R_\mu (v) = v * \mu \quad (v \in M_\ast (S)).
\]

Clearly \( (L_\mu , R_\mu ) \) is a multiplier of \( M_\ast (S) \). We show that the mapping \( \mu \mapsto (L_\mu , R_\mu ) \) is an isomorphism from \( M(S) \) onto the multiplier algebra of \( M_\ast (S) \). Let \( (e_\mu ) \) be a bounded approximate identity for \( M_\ast (S) \), and \( (L,R) \) be a multiplier of \( M_\ast (S) \), then \( \big( L(e_\mu ) \big) \) is a bounded net in \( M(S) \).

By Banach-Alaoglu’s Theorem, passing to a subnet if necessary, we can assume that there exists \( \mu \in M(S) \), such that \( L(e_\mu ) \to \mu \) in the weak* topology. Let \( v \in M_\ast (S) \) and \( \phi \in C_0(S) \). By Lemma 1 of [12], \( \phi \circ v \in C_0(S) \). So

\[
\lim_n \langle \phi , L\big( e_\mu \big)^n v \rangle = \langle \phi , L\big( e_\mu \big) v \rangle = \langle \phi , \mu \rangle = \langle \phi \circ v , \mu \rangle = \langle \phi , \mu * v \rangle = \langle \phi , L_\mu (v) \rangle,
\]

and hence \( L\big( e_\mu \big)^n v \to L_\mu (v) \) in the weak* topology.

Now, since \( L\big( e_\mu \big)^n v \to L(v) \) in the norm topology, we have \( L = L_\mu \). Similarly \( R = R_\mu \). The remainder of proof is trivial. \( \square \)

**Proposition 3.6.** Let \( S \) be a compactly cancellative foundation semigroup with identity, Then \( H^1(M(S), M(S)) = H^1\big( M_\ast (S) , M(S) \big) \).

Furthermore each \( D \in \mathbb{Z}^1\big( M_\ast (S) , M(S) \big) \) has a unique so-weak* continuous extension \( \overline{D} \in \mathbb{Z}^1\big( M(S), M(S) \big) \).

**Proof.** From Lemma 3.5 the set of all multipliers on \( M_\ast (S) \) is equal with \( M(S) \). On the other hand, by Lemma 1 of [12] we have \( M_\ast (S) * C_0(S) \subseteq C_0(S) \).

Also, let \( (e_\mu ) \) be a bounded approximate identity for \( M_\ast (S) \). As in Lemma 2.1 from [12],

\[
\| e_\mu \cdot f - f \|_\infty \to 0 \quad (f \in C_0(S)).
\]

Thus \( M_\ast (S) * C_0(S) = C_0(S) \) by Cohen factorization theorem. Similarly, \( C_0(S) * M_\ast (S) = C_0(S) \). Therefore \( C_0(S) \) is a neo-unital \( M_\ast (S) \)-module. By Propositions 1.9 and 1.11 from [10] the proof is complete. \( \square \)

### 4. Derivations on Clifford Semigroups

An element \( e \) of a semigroup \( S \) is called an idempotent if \( e^2 = e \). We denote be \( E(E_\ast) \) the set of idempotents in \( S \). Recall that a semigroup \( S \) is a *Clifford semigroup* if it is an inverse semigroup for which each idempotent is central (cf. [9], 4.2). By Theorem 4.2.1 of [9], \( S \) is a semilattice of groups and if \( S = \bigcup \{ G_e : e \in E(S) \} \), then for \( e, f \in E, e \leq f \) if and only if \( e f = f \), and moreover for every \( e, f \in E, G_e G_f \subseteq G_{ef} \).

**Lemma 4.1.** Let \( S \) be a topological Clifford semigroup, and \( D \in \mathbb{Z}^1\big( M(S), M(S) \big) \), then \( D\big( \ell^1(S) \big) \subseteq M_0(S) \).

**Proof.** Suppose that \( S = \bigcup_{r \in E} G_r \). Let \( x \in S \), then there exists \( e \in E \) such that \( x \in G_e \). If \( H \) is a subgroup of \( G_e \), \( S_e \subseteq S \) generated by \( x \) and \( e \), then \( H \) is abelian and therefore \( \ell^1(H) \) is amenable. We note that \( M(S) \) is a \( \ell^1(H) \)-bimodule and the restriction of \( D \) on \( \ell^1(H) \) denoted by \( D_x \) is a derivation. Thus \( D_x S \) is inner. That is there is \( \mu_x \in M(S) \) such that \( D_x = \text{id}_{\mu_x} \). Therefore for any \( x \in H \), we have \( D_x (\delta_{s}) = \delta_{s} * \mu_x - \mu_x * \delta_{s} \) and so that \( D(\delta_{s}) = \delta_{s} * \mu_x - \mu_x * \delta_{s} \). Thus \( D(\delta_{s})(S) = 0 \). This implies that \( D\big( \ell^1(S) \big) \subseteq M_0(S) \). \( \square \)

The following theorem is a generalization of Proposition 7.1 of [8].

**Theorem 4.2.** Let \( S \) be a compactly cancellative foundation Clifford semigroup with identity and \( D \in \mathbb{Z}^1\big( M_\ast (S), M(S) \big) \), then \( D\big( M_\ast (S) \big) \subseteq I_0(S) \).

**Proof.** By Proposition 3.6, \( D \) has a unique extension \( \overline{D} \in \mathbb{Z}^1\big( M(S), M(S) \big) \). Using Theorem 3.3 and Lemmas 3.6 and 4.1 we obtain
\[ D(\mu S) \subseteq B(M(S)) = B(\ell'(S)^*_{\text{weak}}) \]

\[ = B(\ell'(S)^*_{\text{weak}}) = M_0(S). \]

On the other hand by Proposition 3.4 \( D(M_\mu(S)) \subseteq M_\mu(S), \) thus \( D(M_\mu(S)) \subseteq I_\mu(S). \)

**Remark 4.3.** (a) Let \( T \) be a compact foundation semilattice with identity, for example \( T = \{1, 2, \ldots, n\}, \) where \( n \in \mathbb{N} \) with the \( k \cdot l = \max \{k, l\} \) \((k, l \in T)\). Let SGS be any locally compact group. Then \( S = T \times G \) with the product topology and coordinatise multiplication defines a foundation semigroup (see [7], page 43) with identity that is compactly cancellatively.

Let \( G_t = \{t\} \times G \) for \( t \in T \). It is clear that \( G_t \) is a group with the identity \((t, e_G)\). Clearly \( S = \bigcup_{e \in G} \)

\[ S \text{ is a Clifford semigroup. Furthermore } E_S = \{(t, e_G): t \in T\}. \] By Theorem 4.2, if \( D \in Z_1(M_\mu(S), M(S)) \), then \( D(M_\mu(S)) \subseteq I_\mu(S). \)

(b) The proof of the Theorem 4.2 shows that if \( S \) is a compactly cancellative foundation semigroup with identity such that \( S \) is a union of groups, then \( D(M_\mu(S)) \subseteq I_\mu(S). \)

**Lemma 4.4.** Let \( S = \bigcup\{G_e: e \in E_S\} \) be a topological Clifford semigroup and \( D \in Z_1(M(S), M(S)) \). If \( e \in E_S \) and \( \text{supp} (\mu) \subseteq G_e, \) then \( \text{supp}(D(\mu)) \subseteq \bigcup_{j \in S} G_j. \)

**Proof.** Since \( e \) is central, so

\[ D(\delta_e) = D(\delta_e \cdot \delta_e) = 2\delta_e \cdot D(\delta_e) \]

and hence

\[ \delta_e \cdot D(\delta_e) = \delta_e \cdot (2D(\delta_e)) = 2\delta_e \cdot D(\delta_e). \]

Since \( \text{supp}(\mu) \subseteq G_e \), we have

\[ D(\mu) = D(\mu \cdot \delta_e) = D(\mu \cdot \delta_e) + \mu \cdot D(\delta_e) = D(\mu) \cdot \delta_e. \]

Thus

\[ \text{supp}(D(\mu)) = \text{supp}(D(\mu) \cdot \delta_e) \subseteq e \cup \bigcup_{j \in S} G_j. \]

The following theorem is indeed the main result of this paper.

**Theorem 4.5.** Let \( S = \bigcup\{G_e: e \in E_S\} \) be a topological Clifford semigroup such that \( E_S \) is finite and each \( G_e \) is closed. Then \( H^1(M(S), M(S)) = \{0\}. \)

**Proof.** Let \( D \in Z_1(M(S), M(S)) \). Each \( e \in E_S \) defines a bounded derivation \( D_e: M(G_e) \rightarrow M(S) \) by \( D_e(\mu) = D(\mu_e) \), where \( \mu_e \in M(S) \) is given by

\[ \int_{G} f d \mu_e = \int_{G_e} f \mid_{G_e} d \mu_e \quad (f \in C_0(S)). \]

By Lemma 4.4, \( D_e(M(G_e)) \subseteq M(\bigcup_{j \in S} G_j). \) Since each \( G_j \) is closed and \( E_S \) is finite, so each \( G_j \) is also open and hence \( M(\bigcup_{j \in S} G_j) = \bigoplus_{j \in S} M(G_j). \)

Thus we have

\[ D_e(M(G_e)) \subseteq M(\bigcup_{j \in S} G_j) = \bigoplus_{j \in S} M(G_j). \]

Therefore we can decompose \( D_e \) across \( \bigoplus_{j \in S} M(G_j) \) as \( D_e(\mu) = \sum_{j \in S} D_j(\mu) \), where \( D_j(\mu) \) denotes the \( j \) th projection of \( D_e(\mu) \) on \( M(G_j) \).

Since \( j \leq e \), so \( j = j \), and hence \( D_j \) is a derivation from \( M(G_j) \) into \( M(G_j) \). We call each associated derivation from \( M(G_e) \) to \( M(G_j) \) the *principle component* of \( D \) on \( G_j \). By [13], if \( G \) is a locally compact group, then

\[ H^1(M(G_j), M(G_j)) = 0. \]

By using the method of Theorem 3.2 of [3], we get a bounded derivation \( D^{\times} = D - ad_\xi \), where \( \xi \in M(S) \) and \( D^{\times} \) has zero component on each \( G_e \) \((e \in E_S)\). If \( e \leq u \) and \( \mu_u \in M(G_u) \), then \( D^{\times}(\delta_e \cdot \mu_u) = \delta_e \cdot D^{\times}(\mu_u) \) and \( \text{supp}(\delta_e \cdot \mu_u) \subseteq G_e \cdot G_u \subseteq G_u = G_j \). So we can apply the argument of Theorem 3.2 of [3] to obtain \( D^{\times} = 0. \)

Hence \( D \) is inner. \( \square \)

**Example 4.6.** Let \( n \in \mathbb{N} \) and \( T = \{1, 2, \ldots, n\} \) with the \( k \cdot l = \max \{k, l\} \) \((k, l \in T)\). Suppose \( G \) is a locally compact group. Then \( S = T \times G \) with the product topology and coordinatise multiplication defines a Clifford semigroup that satisfies the hypothesis of Theorem 4.5. Therefore \( H^1(M(S), M(S)) = \{0\}. \)

**Remark 4.7.** Let \( S \) be a left zero semigroup with at least three elements. Then \( S = \bigcup_{s \in S} \{s\}, \) but
$H^1(\ell^1(S),\ell^1(S)) \neq \{0\}$ by Lemma 2.5. Therefore Theorem 4.5 is not valid in general for every semigroup $S$ which is a union of groups.

Acknowledgements

The authors would like to express their deep gratitude to the referees for their careful reading of the earlier version of the manuscript and several insightful comments. The first author also wishes to thank both The Center of Excellence for Mathematics and The Research Affairs (Research Project No. 850709) of the University of Isfahan for their financial supports. The second author wishes to thank the University of Bu-Ali Sina for moral support.

References