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Modified Progressive Type-II Censoring Procedure in Life-Testing under the Weibull Model

M. Rezaei[1] and A. Khodadadi[2,*]

[1]*Department of Statistics, School of Sciences, Birjand University, Birjand, Islamic Republic of Iran*
[2]*Department of Statistics, Faculty of Mathematical Sciences, Shahid Beheshti University, Tehran, Islamic Republic of Iran*

Abstract

In this paper we introduce a new scheme of censoring and study it under the Weibull distribution. This scheme is a mixture of progressive Type II censoring and self relocating design which was first introduced by Srivastava [8]. We show the superiority of this censoring scheme (PSRD) relative to the classical schemes with respect to "asymptotic variance". Comparisons are also made with respect to the total expected time under experiment (TETUE) as an important feature of time and cost saving. These comparisons show that $\text{TETUE}_{(SRD)} < \text{TETUE}_{(PSRD)} < \text{TETUE}_{(PC2)}$ if $0 < \beta < 1$, $\text{TETUE}_{(PSRD)} = \text{TETUE}_{(SRD)} < \text{TETUE}_{(PC2)}$ if $\beta = 1$ and $\text{TETUE}_{(PSRD)}$ is the best among all the designs if $\beta = 2$ (Rayleigh distribution case).

Keywords: Asymptotic variance; Fisher information matrix; Maximum likelihood; Self relocating design (SRD); Total expected time under experiment

1. Introduction

In life testing, if an early decision is of more importance, we may plan a censored life testing instead of doing complete one. Although censored life testing loses efficiency compared to complete life testing of size n, the feature of censored life testing is time saving. For a review, see [12,13].

The two most common censoring schemes are termed as Type-I (C1) and Type-II (C2) censoring schemes. Briefly, they can be described as follows: Consider n items under observation in a particular experiment. In the conventional Type-I censoring scheme, the experiment continues up to a pre-specified time T. On the other hand, the conventional Type-II censoring scheme requires the experiment to continue until a pre-specified number of failures $m \le n$ occurs. The mixture of Type-I and Type-II censoring scheme is known as a hybrid censoring scheme, which was first introduced by Epstein [4,5].

One of the drawbacks of the conventional Type-I, Type-II or hybrid censoring schemes is that they do not allow for removal of units at points other than the terminal point of the experiment. One censoring scheme known as the Type-II progressive censoring (PC2) scheme, which has this advantage, has become very popular in the last few years. It can be described as follows: consider n units in a study and suppose that $m < n$ is fixed before the experiment. Moreover, m other integers, $R_1, \ldots, R_m$, are also fixed so that $R_1 +, \ldots, + R_m + m = n$. At the time of the i^{th} failure,

* Corresponding author, Tel.: +98(21)29903011, Fax: +98(21)22431649, E-mail: a_khodadadi@sbu.ac.ir

R_i of the remaining units are randomly removed. For further details on Type-II progressive censoring the readers may refer to the recent excellent monograph of Balakrishnan and Aggarwala [1].

We now describe the modified schemes, which are called Self Relocating Designs (SRD). Basically, the idea behind an SRD is as follows. As under the classical case, we start with the same number u of machines of each brand. However, here, at all times during the experiment we maintain the same number of competing machines of each brand. This is done as follows. If we have u machines of each brand, we have a total of um machines. This set of um machines is divided into u subsets, such that in each subset, there is exactly one machine of each brand. The subsets are made at the beginning of the experiment, we wait until the first failure occurs. Instantaneously after the first failure, the $(m-1)$ machines which are in the subset to which the failed machines belong, are removed from the experiment. Thus, after the first failure, there are exactly $(u-1)$ machines of each brand still continuing in the experiment. The experiment is continued until the second failure, instantaneously after the second failure, the $(m-1)$ machines which are in the subset to which the second failed machine belongs are censored. Thus, after the second failure, we are left with $(u-2)$ machines of each brand remaining in the experiment. This process is continued until a specified number G of failures has occurred (corresponding to Type-II censoring) or until a specified time period T has passed (corresponding to Type-I censoring). These classes of designs were first introduced in [8] and particular cases of these were studied in some details under exponential distribution. These studies established the superiority of SRD over individual censoring of Type-II in the sense that the former gave rise to a smaller value for the trace of the asymptotic variance (in a certain sense). You can find some more results related to these types of censoring in [6,9,10].

In this paper, we introduce a Type-II SRD progressive censoring (PSRD) scheme, and study the properties of this scheme under Weibull distribution which is widely used as a failure model, particularly for mechanical components. As the name suggests, it is a mixture of Type-II progressive and SRD censoring schemes. The paper is organized as follows: in section 2 we discuss progressive Type-II SRD censoring, Weibull model and likelihood function under this new design. In section 3, we focus on Information matrices under competing designs and compare the asymptotic variances. In section 4, and 5 we compare the Total Expected Time Under Experiment (TETUE) with respect to the different designs and numerical results for Weibull distribution are presented in 4 Tables.

2. New Class of SRD, Model Description and MLE

2.1. Weibull Models

The generalized Weibull distribution is given by the survival function

$$\psi(t) = \exp[-\lambda\theta(t)], \quad t \geq 0 \tag{1}$$

Where the λ is a positive real number, and $\theta(t)$ is a nondecreasing function of t, such that $\theta(0)=0$, $\theta(\infty)=\infty$.

It is clear that the density function is given by

$$f(t) = \lambda\theta'(t)e^{-\lambda\theta(t)}, \quad t > 0$$

where prime(\`) denotes differential coefficient. Also then, $\Lambda(t)$ the hazard rate is given by

$$\Lambda(t) = \lambda\theta'(t).$$

and, $\Pr(t > x \mid t > y) = e^{-\lambda[\theta(x)-\theta(y)]}$, where $x \geq y \geq 0$,

Also, the density function of the life time of such a machine is $f_t(x \mid y)$, where

$$f_t(x \mid y) = \lambda\theta'(x)e^{-\lambda[\theta(x)-\theta(y)]}, \quad x \geq y \geq 0. \tag{2}$$

Many inference-oriented studies have been made under this distribution, for example [2,3]. Note that the two parameter standard Weibull distribution with scale parameter $\alpha(>0)$ and shape parameter β (>0) is a special case of (1) with $\theta(t) = t^{\beta}, \lambda = (\frac{1}{\alpha})^{\beta}$ so that

$$F(t;\theta) = 1-\exp[-(\frac{t}{\alpha})^{\beta}], \quad t > 0.$$

The model can be written in alternate parametric forms as indicated below:

$$F(t;\theta) = 1-\exp[-(\lambda' t)^{\beta}], t \geq 0 \text{ with } \lambda' = \frac{1}{\alpha}$$

$$F(t;\theta) = 1-\exp(-\frac{t^{\beta}}{\alpha'}), t \geq 0 \text{ with } \alpha' = \alpha^{\beta} \tag{3}$$

$$F(t;\theta) = 1-\exp(-\lambda t^{\beta}), t \geq 0 \text{ with } \lambda = (\frac{1}{\alpha})^{\beta}$$

Although they are all equivalent, depending on the context a particular parametric representation might be more appropriate. A variety of models have evolved by transformation (linear or nonlinear), functional relationship, from this standard model. The parameters of the standard Weibull model are constant. For some models this is not the case. As a result, they are either a function of the variable t or some other variable (such as stems level) or are random variables. Besides there are stochastic point process models with links to the standard Weibull model. For further details on Weibull models the readers may refer to the recent monograph of Murthy, D. N. P., Xie, Min and Jiang, Renyan [7].

2.2. Progressive Type-II SRD Censoring (PSRD)

Suppose we have um machines as those in SRD case. We start the experiment at time $t_0\,(=0)$ and as the first failure occurs, we remove R_1+1 subsets, one of which contains the failed machine and choose the other R_1 ones randomly from the survived subsets. Thus after the first failure, there are exactly $u-R_1-1$ machines of each brand still functioning in the experiment. After the second failure, we remove R_2+1 subsets so that we are left with $(u-R_1-R_2-2)$ machines of each brand still continuing in the experiment. This process is continued until a specified number G of failures had occurred (corresponding to progressive Type-II censoring).

2.3. Likelihood Function under PSRD

For ease of discussion, we shall consider the cases $G=1$ and 2 first, and then the case of general G. Let L_g denote the likelihood for $G=g$. Clearly, L_1 is the probability that $um-1$ machines survive time t_1, and one machine (of type i_1) fails at time t_1. Thus, using (1) and (2), we have

$$L_1=\binom{u}{1}\lambda_{i_1}\theta'(t)e^{-u\sum_{i=1}^{m}\lambda_i\theta(t)} \tag{4}$$

Next, L_2 is the probability that $um-1$ machines survive time t_1, one machine(of type i_1) fails at time t_1, and furthermore that out of the $(u-R_1-1)m$ machines working at time (t_1+0), $(u-R_1-1)m-1$ machines survive time t_2,and one machine (of type i_2) fails at time t_2. Now, obviously, the life times of the machines working at any particular time are distributed independently of each other (and of all other machines which may have failed previously). Thus, effectively, out of the um machines 'launched' at time $0, (m-1)(u-R_1-1)-1$ machines survived time t_2, one machine (of type i_2) (which could be anyone out of the $(u-R_1-1)$ machines of type i_2. which were 'alive' at time (t_1+0)) failed at time t_2, one machine(of type i_1) (which could be anyone out of u machines of type i_1, alive at time $t_0=0$) failed at time t_1. Since, as mentioned above, the life times of the individual machines are independent random variables, we obtain

$$\begin{aligned}L_2&=\binom{u-R_1-1}{1}\lambda_{i_2}\theta'(t_2)e^{-(u-R_1-1)\sum_{i=1}^{m}\lambda_i\theta(t_2)}\\&\times\binom{u}{1}\lambda_{i_1}\theta'(t_1)e^{-u\sum_{i=1}^{m}\lambda_i\theta(t_i)}\end{aligned} \tag{5}$$

It is convenient to derive the expression for L_2 by the following 'conditional' argument. The events up to time t_2 can be divided into two stages, the first being the events till time t_1, and the other events in the time $(t_1,t_2]$. Notice that the events inside the two stages are independent of each other, except that in the second stage we have at time (t_1+0) exactly $(u-R_1-1)m$ machines, each of which is t_1 time units old. The likelihood of the events in the first stage is $L_1\,(=\xi_1, say)$. Let ξ_2 be the likelihood of the events in the second stage (given t_1). Clearly, from (1) and (2), we get

$$\begin{aligned}\xi_2&=\binom{u-R_1-1}{1}\lambda_{i_2}\theta(t_2)\\&\times e^{-(u-R_1-1)\sum_{i=1}^{m}\lambda_i[\theta(t_2)-\theta(t_1)]}\end{aligned} \tag{6}$$

since we must have $L_2=\xi_1\xi_2$, (6) and (4) lead to (5).

Now, consider the whole number of experiments divided into G stages, the time period for the g^{th} $(g=1,\ldots,G)$ stage being $(t_{g-1},t_g]$. Let ξ_g be the 'conditional ' likelihood for the g^{th} stage, given t_{g-1}. Since at time $(t_{g-1}+0)$, exactly $u-\sum_{i=1}^{g-1}R_i-g$ machines of each type were alive, we obtain

$$\xi_g = \begin{pmatrix} u - \sum_{i=1}^{g-1}(R_i - g - 1) \\ 1 \end{pmatrix} \lambda_{i_g} \theta'(t_g) \tag{7}$$
$$\times e^{-(u - \sum_{i=1}^{g-1} R_i - g - 1) \sum_{i=1}^{m} \lambda_i [\theta(t_g) - \theta(t_{g-1})]}$$

If L_G is the likelihood for the whole experiment, we must have $L_G = \xi_1 . \xi_2 \ldots . \xi_G$, and hence after some simplification, we obtain

$$L_G = C \prod_{g=1}^{G} \lambda_{i_g} \theta'(t_g) e^{-\sum_{i=1}^{m} \lambda_i \Theta(t)}$$

where, for any function $\theta(.)$, we define $\Theta(t) = \sum_{g=1}^{G}(R_g + 1)\theta(t_g)$ and $C = u(u - R_1 - 1)(u - R_1 - R_2 - 2)\ldots(u - R_1 - \ldots - R_G - G)$.

We define

$$b_{ig} = \begin{cases} 1, \text{if at time } t_g \text{, the machine that fails is of type i,} \\ 0, \text{ otherwise,} \end{cases}$$

and $b_i = \sum_{g=1}^{G} b_{ig}$ so that b_i is the number of times a machine of type i failed in the whole experiment. Clearly, we have $\prod_{g=1}^{G} \lambda_{i_g} = \prod_{i=1}^{m} \lambda_i^{b_i}$. So we have established:

Theorem 2.1. Under PSRD, the likelihood is given by

$$L_G = C \prod_{i=1}^{m} \lambda_i^{b_i} \prod_{g=1}^{G} \theta'(t_g) e^{-\sum_{i=1}^{m} \lambda_i \Theta(t)}$$

where $C = u(u - R_1 - 1)(u - R_1 - R_2 - 2) \ldots . (u - R_1 - \ldots - R_G - G)$.

3. Information Matrices under Competing Designs

We now consider the performance of designs under various situations. First we consider the PSRD under the assumption that $\theta(t)$ is a known function of t, We assume that the λ_i's are unknown parameters, and consider $\hat{\lambda}_i$'s, the respective maximum likelihood estimators. We have

$$l_G = \log L_G = \log C + \sum_{i=1}^{m} b_i \log \lambda_i$$
$$+ \sum_{g=1}^{G} \log \theta(t_g) - \Theta(t) \sum_{i=1}^{m} \lambda_i$$

then

$$\frac{\partial l_G}{\partial \lambda_i} = \frac{b_i}{\lambda_i} - \Theta(t), \quad i = 1, \ldots, m,$$

Hence, for all i,

$$\hat{\lambda}_i = \frac{b_i}{\Theta(t)}, \quad i = 1, \ldots, m.$$

Furthermore,

$$\frac{\partial^2 l_G}{\partial \lambda_i^2} = -\frac{b_i}{\lambda_i^2}, \qquad i = 1, \ldots, m.$$
$$\frac{\partial^2 l_G}{\partial \lambda_i \partial \lambda_j} = 0, \qquad i \neq j \; ; i, j = 1, \ldots, m. \tag{8}$$

Define

$$q_{ij} = -E\left(\frac{\partial^2 l_G}{\partial \lambda_i \partial \lambda_j}\right), \quad i \neq j = 1, \ldots, m, \tag{9}$$

then, from (8) $q_{ij} = 0$ if $i \neq j$. Also, we get $q_{ii} = \frac{1}{\lambda_i^2} E(b_i)$. But $E(b_i) = \sum_{g=1}^{G} E(b_{ig})$. Since b_{ig} takes values 0 and 1 only, for all i and g, we have $E(b_{ig}) = Prob(b_{ig} = 1)$.

Theorem 3.1. let $Y_i, (i = 1, \ldots, m)$ be independent random variables having a Weibull distribution with survival function $\psi(t)$ then:

$$\pi_i = Prob(Y_i = Min\{Y_1, \ldots, Y_m\}) = \frac{\lambda_i}{\sum_{i=1}^{m} \lambda_j}$$

Proof.

$$\pi_i = \int_0^\infty Prob(Y_i > y_j ; \forall j \neq i \,|Y_j = y) f_{y_j}(y) dy$$
$$= \int_0^\infty \prod_{\substack{i=1 \\ i \neq j}}^{m} Prob(Y_i > y) f_{y_j}(y) dy$$
$$= \int_0^\infty \lambda_j \theta'(y) e^{-\sum_{i=1}^{m} \lambda_i \theta(y)} dy = \frac{\lambda_i}{\sum_{i=1}^{m} \lambda_i} \qquad \square$$

From Theorem (3.1), it follows that $Prob(b_{ig} = 1) = \pi_i$, Since b_i is a Bin(G , π_i) then

$$E(b_i) = \frac{G\lambda_i}{\sum_{i=1}^{m}\lambda_i} \quad , \quad i = 1,\ldots,m .$$

which is independent of $\theta(.)$.

We now study the (asymptotic Fisher) information matrix (say, Q), Clearly $Q = (q_{ij})$, where $i, j = 1,\ldots,m$. we have

$$Q = \left(\frac{G}{\sum_{i=1}^{m}\lambda_i}\right) diag\,(\frac{1}{\lambda_1},\ldots,\frac{1}{\lambda_m}).$$

In [11], a comparison of the (asymptotic Fisher) information matrices (or, more precisely, the asymptotic variance (ASVAR) matrices) under SRD2 and C2 was made. In this paper, we shall consider PSRD against PC2. To clarify, PC2 means that we shall consider m separate experiments, wherein the ith experiment ($i = 1,\ldots,m$), we test u machines under progressive Type-II censoring, observing the first $G_0 = \frac{G}{m}$ failures, where the life time of a machine is assumed to obey the Weibull distribution. Notice that the two designs under comparison are on the same footing in the sense that the total number of failures is the same in both cases, and is equal to G . All the theory developed above for the PSRD holds for all m , including $m = 1$. Hence the results for the i^{th} ($i = 1,\ldots,m$) experiment can be derived from the corresponding results for the PSRD by taking $m = 1$, and then replacing G_0 by G . Now, for each i , suppose $\tilde{\lambda}_i$ is the maximum likelihood estimate of λ_i and $\tilde{\upsilon}_{ii}$ the asymptotic variance of $\tilde{\lambda}_i$. Then, (8) gives

$$\tilde{\upsilon}_{ii}^{-1} = \frac{G}{m}\frac{1}{\lambda_i^2}, \; i = 1,\ldots,m .$$

If $V^*(m \times m)$ denotes the ASVAR matrix of $(\tilde{\lambda}_1,...,\tilde{\lambda}_m)$, then clearly,

$$V^* = [\frac{m}{G}]diag\,(\lambda_1^2,\ldots,\lambda_m^2).$$

On the other hand, if V is the ASVAR matrix corresponding to PSRD, then $V = Q^{-1}$, and hence

$$V = [\frac{\sum_{i=1}^{m}\lambda_i}{G}]diag\,(\lambda_1,\ldots,\lambda_m).$$

We compare the traces of the ASVAR matrices. It is well known that these are proportional to average variance of estimates of all normalized linear combinations of parameters. We have $trV = \frac{(\sum_{i=1}^{m}\lambda_i)^2}{G}$ and $trV^* = \frac{m}{G}\sum_{i=1}^{m}\lambda_i^2$. Because of the Cauchy-Schwartz inequality, and since variance is always nonnegative, we have $\sum_{i=1}^{m}\lambda_i^2 \geq \frac{(\sum_{i=1}^{m}\lambda_i)^2}{m}$.

This leads to the following important result.

Theorem 3.2. For the ASVAR matrices of the estimators of λ_i under two approaches, we have $trV \leq trV^*$.

The above result assures the superiority of the PSRD procedures compared to the classical ones with respect to the trace of ASVAR matrices.

4. Total Expected Time under Experiment

We first define the 'total time under experiment'(TTUE) as follows. Suppose, in an experiment, n machines are used, and suppose that the j^{th} machine ($j = 1,\ldots,n$) was under observation for a time period τ_j . Then we say that for this experiment, the TTUE equals $\tau = (\sum_{j=1}^{n}\tau_j)$. Generally, the τ 's are random variables. In any case, we define the 'total expected time under experiment'(TETUE)' to be $E(\tau)$, where the expectation is taken over the events occurring in the whole experiment. The following result is obvious.

Theorem 4.1. The cumulative distribution function of t_r $1 \leq r \leq G$ is given by

$$F_{t_{r:G}}(t) = 1 - c_{r-1}\sum_{i=1}^{r}\frac{a_{i,r}}{\gamma_i}\psi(t)^{\gamma_i} \;, \quad t \geq 0$$

Where

$$\gamma_r = G - r + 1 + \sum_{i=r}^{m}R_i \quad and \quad c_{r-1} = \prod_{i=1}^{r}\gamma_i$$

and $a_{i,r} = \prod_{\substack{i=1\\ i\neq j}}^{r}\frac{1}{\gamma_j - \gamma_i}, \quad 1 \leq i \leq r \leq G .$

Table 1. Values of TETUE for $m=3$, $\lambda_1=1$, $\lambda_2=4$, $\lambda_3=7$ for certain values of (U,G) and β

(U,G)	Scheme	β = 2.00	β = 1.50	β = 1.25	β = 1.00	β = 0.75	β = 0.50
(24,24)	PSRD	18.420	12.400	9.186	6.000	3.120	1.000
	PC2	24.950	18.605	14.987	11.143	7.244	3.631
	PSRD	13.025	7.812	5.276	3.000	1.238	0.250
(24,12)	SRD2	13.924	8.276	5.494	3.000	1.116	0.164
	PC2	16.307	10.959	8.214	5.571	3.229	1.444
	C2	17.643	11.709	8.598	5.571	2.893	0.944

Table 2. Values of TETUE for $m=3$, $\lambda_1=0.5$, $\lambda_2=2.5$, $\lambda_3=4.5$ for certain values of (U,G) and β

(U,G)	Scheme	β = 2.00	β = 1.50	β = 1.25	β = 1.00	β = 0.75	β = 0.50
(24,24)	PSRD	23.300	16.964	13.379	9.600	5.840	2.560
	PC2	33.454	27.818	24.530	20.978	17.338	14.113
	PSRD	16.475	10.687	7.684	4.800	2.317	0.640
(24,12)	SRD2	17.612	11.321	8.002	4.800	2.089	0.419
	PC2	21.865	16.385	13.444	10.489	7.730	5.613
	C2	23.656	17.508	14.074	10.489	6.925	3.668

Table 3. Values of TETUE for $m=4$, $\lambda_1=0.25$, $\lambda_2=0.5$, $\lambda_3=0.75$, $\lambda=1.00$ for certain values of (U,G) and β

(U,G)	Scheme	β = 2.00	β = 1.50	β = 1.25	β = 1.00	β = 0.75	β = 0.50
(24,24)	PSRD	53.808	47.050	42.958	38.400	33.688	30.720
	PC2	64.252	58.216	54.397	50.000	45.368	43.036
	PSRD	38.048	29.639	24.673	19.200	13.369	7.680
(24,12)	SRD2	40.674	31.397	25.695	19.200	12.053	5.027
	PC2	41.877	34.225	29.779	25.000	20.247	17.083
	C2	45.010	36.353	31.041	25.000	18.350	11.727

Table 4. Values of TETUE for $m=6$, $\lambda_1=0.03$, $\lambda_2=0.06$, $\lambda_3=0.09$, $\lambda_4=0.12$, $\lambda_5=0.15$, $\lambda_6=0.18$ for certain values of (U,G) and β

(U,G)	Scheme	β = 2.00	β = 1.50	β = 1.25	β = 1.00	β = 0.75	β = 0.50
(24,24)	PSRD	160.782	176.889	194.097	228.571	317.460	725.624
	PC2	197.429	228.982	261.498	326.667	502.331	1443.76
	PSRD	113.690	111.433	111.479	114.286	125.984	181.406
(24,12)	SRD2	121.537	118.046	116.097	114.286	115.585	118.731
	PC2	129.032	134.964	143.445	163.333	222.507	552.366
	C2	136.374	141.300	148.015	163.333	206.883	420.279

Proof. See [1]. □

Lemma 4.1. Under PSRD the TETUE equals,

$$\tau_0 = m\sum_{r=1}^{G}(R_r+1)E(t_{r:G})$$

Where

$$E(t_{r:G}) = \Gamma(\frac{1}{\beta})\frac{c_{r-1}}{\beta}\sum_{i=1}^{r}a_{i,r}\,(\frac{1}{\gamma_i\sum_{j=1}^{m}\lambda_j})^{\frac{1}{\beta}}. \;\square \qquad (10)$$

We now consider PC2 with β_i equal to β. There are m separate experiments, and each experiment lasts until we observe G_0 failures. Let $t_{r:G_0}$ $(r=1,\dots,m)$ be the time for which the r^{th} experiment lasts. Clearly, $E(t_{r:G_0})$ is obtained from (10) by replacing G by G_0 and $m=1$. We obtain:

Corollary 4.1. Under PC2, for the i^{th} experiment $(i=1,\dots,m)$, we have

$$E(t_{r:G_0}) = \Gamma(\frac{1}{\beta})\frac{c_{r-1}}{\beta}\sum_{i=1}^{r}a_{i,r}\,(\frac{1}{\gamma_i\lambda_r})^{\frac{1}{\beta}}. \;\square \qquad (11)$$

To compare the TETUE under PSRD and PC2, we use (10) and (11). We have:

Corollary 4.2. Consider the PSRD and PC2, with β_i 's equal to β. Then the total expected time under the two designs are respectively T_1 and T_2 with

$$T_1 = m\sum_{i=0}^{G}(R_i+1)E(t_{i:G}),$$

$$T_2 = \sum_{i=1}^{m}[\sum_{j=1}^{G_0}(R_{ij}+1)E(t_{ij:G_0})],$$

Where R_{ij} denotes the number of censored machines of type i at time t_{ij}.

Theorem 4.2.

i) When $U=G$, the following result holds:

$$T_1 = mG\int_0^{\infty}\sum_{i=1}^{m}\lambda_i\theta'(x)e^{-\sum_{i=1}^{m}\lambda_i\theta(x)}dx$$

$$= mG\Gamma(1+\frac{1}{\beta})(\sum_{i=1}^{m}\lambda_i)^{-\frac{1}{\beta}}\; for\ Weibull$$

ii) For the case of exponential distribution, *i.e.* when $\beta=1$ and for all U $\geq$ G, we have

$$T_1 = \frac{mG}{\sum_{i=1}^{m}\lambda_i} = \frac{G}{\bar{\lambda}}, \quad T_2 = \frac{G}{m\sum_{i=1}^{m}\frac{1}{\lambda_i}} = \frac{G}{\tilde{\lambda}},$$

where $\bar{\lambda}$ and $\tilde{\lambda}$ are respectively the Arithmetic and the Harmonic mean of λ_i. Since $\bar{\lambda}\geq\tilde{\lambda}$, then $T_1\leq T_2$.

5. Discussion of Tables

Total expected time under experiment (TETUE) is a very important factor in the composition of the total cost of a test procedure. We will analyze and compare the values of each one of TETUE's associated with the testing procedure and the value of β which has the most effect on the behavior of the hazard function. The following tables involve five kinds of parameters, $\beta, \lambda's, m, u$ and G. The range of β considered is between 0.5 and 2. Values of $\lambda's$ are selected in a rather large range (viz. 0.03 to 7.00) so as to cover almost all practical situations. Four sets of values of $\lambda's$ are selected in this range. These four sets respectively correspond to the four tables (1 to 4), the four tables correspond respectively to m=3, 3, 4 and 6, the case m=3 being repeated twice so as to offer a comparison between two sets of values of λ with the same m. Also, in each table, two sets of values of U and G are considered; in one of these sets, we have U=G, and in the second U=2G, so as to provide a common value for all tables, and since G must be divisible by m, we take G=12. In [9], a numerical study of the TETUE was made for SRD2 and C2. For this purpose four sets of values of m, λ's and β's were selected. These values are displayed below:

(I) m = 3 λ = (1, 4, 7)'

(II) m = 3 λ = (0.5, 2.5, 4.5)'

(III) m = 4 λ = (0.25, 0.5, 0.75, 1.00)'

(IV) m = 6 λ = (0.03, 0.06, 0.09, 0.12, 0.15, 0.18)'

For each of these sets of values of m and λ, the following values of U, G and β's were tried:

β = 2.00, 1.75, 1.50, 1.00, 0.75, 050

(U,G) = (24,24),(24,12)

We are using the same sets of values of the above parameters for the study of PSRD and PC2.

Consider now T_1 and T_2. It is seen that the values of T_1 and T_2 (for fixed U,G, and β) increase as we go from Table 1 to 2 to 3 to 4. This fact is a reflection of larger values of m and of smaller values of λ. For large value of β, this increase is respectively small, but for small β, it is very dramatic. To compare two designs we use the most common classification of Weibull distribution which is based on the effect of the value β on the hazard function behavior.

The Weibull failure rate for $0 < \beta < 1$ is unbounded at $t = 0$. The failure rate $\Lambda(t)$, decreases , thereafter monotonically and is convex, approaching the value of zero as $t \to \infty$, or $\Lambda(t) = 0$. This behavior makes it suitable for representing the failure rate of units exhibiting early-type failures, for which the failure rate decreases with age. When such behavior is seen, the following reasons can be considered.

- Burn-in testing and/or environmental stress screening are not well implemented.
- There are problems in the production line.
- Inadequate quality control.
- Packaging and transit problems.

It is clear that when $0 < \beta < 1$, the design SRD is better than PSRD and PSRD is better than PC2, in terms of TETUE.

For $\beta = 1$, $\Lambda(t)$ yields a constant value. This makes it suitable for representing the failure rate of chance-type failure and the useful life period failure rate of machines. In this case PSRD is the same as SRD but it is better than PC2 in terms of TETUE.

For $\beta > 1, \Lambda(t)$ increases as t increases and becomes suitable for representing the failure rate of machines exhibiting wear-out type failures. For $1 < \beta < 2$ the $\Lambda(t)$ curve, is concave, consequently, the failure rate increases at a decreasing rate as t increases . For $\beta = 2$, or the Rayleigh distribution case, there is a straight line relationship between $\Lambda(t)$ and t , which goes through the origin with a slop of 2. $\Lambda(t)$ increases at a constant rate as t increases.

In this case we have that PSRD is the best among all four designs (PSRD, SRD, PC2, C2) in terms of TETUE.

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