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On Isomorphism Theorems of F^n -Polygroups

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Abstract

In this paper, the notion of fuzzy n-polygroups (F^n -polygroups) is introduced and some related properties are investigated. In this regards, the concepts of normal F-subpolygroups and homomorphisms of F^n-polygroups are adopted. Also, the quotient of F^n-polygroups by defining regular relations are studied. Finally, the classical isomorphism theorems of groups are generalized to F^n-polygroups provided that the F-subpolygroups considered in them are normal.

Keywords: Hyperstructure; Fuzzy set; F^n -Hyperoperation; F^n -Polygroup; Regular and strongly regular relations

Introduction

In this section, we describe the motivation and a survey of related works. Hyperstructure theory has been introduced by Marty in [15]. He defined hypergroups, investigated their properties and applied them to groups and rational algebraic functions. Later on these subjects has been studied by many mathematicians. Canonical hypergroups are a particular case of Marty's hypergroups. The notion of canonical hypergroups independent of other operations, was studied for the first time by J. Mittas in 1970. Some connected hyperstructures with canonical hypergroups were introduced and studied by Corsini, Bonansinga, Serafimidis, Kostantinidou, Mittas and De Salvo. Canonical hypergroups were used in the character theory of finite groups by Roth in 1975. Quasicanonical hypergroups which satisfy all conditions of canonical hypergroups except commutativity, were introduced by Corsini. This class of hypergroups were studied by Comer independently and he named them polygroups. There exists a rich bibliography: publications appeared within 2012 can be found in “Polygroup Theory and Related Systems" by B. Davvaz [2]. This book contains the principal definitions endowed with examples and the basic results of the theory.

n-ary generalizations of algebraic structures is another topic in hyperstructure theory. The concept of n-hypergroups which are a nice generalization of groups were introduced by Davvaz and Vougiouklis in [6], which is a generalization of the concept of Marty's hypergroup and n-group. Later on, n-polygroups which are a special case of n-hypergroups studied by Ghadiri and Waphare [11], Leoreanu-Fotea and Davvaz [5, 14] and others.

Following the introduction of fuzzy set by Zadeh in 1965 [18], fuzzy set theory has been developed by many others in mathematics and other branches of science. In 1971, the concept of a fuzzy subgroup has defined and studied by Rosenfeld [16]. He formulated the concept of a fuzzy subgroup of a group. The connections between the fuzzy sets and algebraic hyperstructures have been considered by Corsini, Davvaz, Leoreanu, Zahedi and others. Some applications of fuzzy algebra, such as in automata theory and coding theory can be found in [1]. Zahedi et al. introduced the notion of fuzzy subpolygroups of a polygroup, also see [3, 7]. Then, the notion of fuzzy n-ary subpolygroups is studied in [8, 9, 10, 12, 13]. The notion of fuzzy polygroup (F-polygroup), has been introduced and studied by Zahedi and Hasankhani [20, 21].

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In this paper by considering the notion of n-polygroups, first we introduce the notions of F^n-polygroup, F-subpolygroups, regular and strongly regular relations. Then, by using a normal F-subpolygroup we construct quotient F^n-polygroups and finally we state isomorphism theorems for F^n-polygroups.

Preliminaries

In this section, we recall some definitions and simple results of hyperstructures and fuzzy subsets we need for development of our paper. We shall use the notation x_i^j to denote the sequence $x_i, x_{i+1}, \ldots, x_j$. Also, $\overset{(i)}{a}$ denotes the sequence $\overbrace{a,\ldots,a}^{i}$. Let P be a non-empty set. A *hyperoperation* on P is a function $\circ: P\times P \to P^*(P)$, where $P^*(P)$ is the set of all the non-empty subsets of P. A couple $(P,\circ)$, endowed with a bijective function $^{-1}: P \to P$ (unitary operation), is called a *polygroup* if the following three conditions are satisfied: (i) $(x\circ y)\circ z = x\circ(y\circ z)$, for every x,y,z in P; (ii) there exists $e\in P$ such that $x\circ e = e\circ x = x$, for every $x\in P$; (iii) $z\in x\circ y \Rightarrow x\in z\circ y^{-1} \Rightarrow y\in x^{-1}\circ z$, for every x,y,z in P. The following elementary facts about polygroups follow easily from the axioms:

$$e\in x\circ x^{-1}\cap x^{-1}\circ x,\ e^{-1}=e,\ (x^{-1})^{-1}=x, \text{ and } (x\circ y)^{-1}=y^{-1}\circ x^{-1},$$

where $A^{-1}=\{a^{-1} \mid a\in A\}$. A *fuzzy subset* of P is a mapping $\mu: P\to I$, where I is the unit interval $[0,1]\subseteq \mathbb{R}$. The set of all fuzzy subsets of P will be denoted by I^P, that is

$$I^P=\{\mu \mid \mu: P\to I \text{ is a function}\}.$$

Let $\{\mu_\alpha : \alpha\in\Lambda\}$ be a collection of fuzzy subsets of P, where Λ is a non-empty index set. Then, we define the fuzzy subset $\bigcup_{\alpha\in\Lambda}\mu_\alpha$ as follows: for all $x\in P$, $(\bigcup_{\alpha\in\Lambda}\mu_\alpha)(x) = \bigvee_{\alpha\in\Lambda}\{\mu_\alpha(x)\}$. If $\mu\in I^P$, then the *support* of μ, is defined by $\operatorname{supp}(\mu)=\{x\in P \mid \mu(x)>0\}$. If $A\subseteq P$ and $t\in I$, then we define $A_t\in I^P$ as follows:

$$A_t(x)=\begin{cases} t & \text{if } x\in A,\\ 0 & \text{if } x\in P\setminus A.\end{cases}$$

Let $I_*^P = I^P\setminus\{0\}$. An F^n*-hyperoperation* on P is a function $f: P^n \to I_*^P$. In other words, for any $x_1^n\in P$, $f(x_1^n)$ is a non-zero fuzzy subset of P. If for all $x_1^n \in P$, $\operatorname{supp}(f(x_1^n))$ is a singleton set, then f is called an F^n*-operation*. An F^n-hyperoperation f on P is called *associative* if

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$$

for all $i,j\in\{1,\ldots,n\}$ and $x_1^{2n-1}\in P$.

If $\mu_1^n\in I_*^P$, then $f(\mu_1,\ldots,\mu_n)$ is defined by

$$f(\mu_1,\ldots,\mu_n)=\bigcup_{x_i\in\operatorname{supp}(\mu_i)} f(x_1,\ldots,x_n).$$

Let $\mu_1,\ldots,\mu_n,\mu\in I_*^P$ and $x_1^n\in P$. Then, for $i\in\{1,\ldots,n\}$ we define

(1) $f(x_1^{i-1},\mu,x_{i+1}^n) = f(\chi_{\{x_1\}},\ldots,\chi_{\{x_{i-1}\}},\mu,\chi_{\{x_{i+1}\}},\ldots,\chi_{\{x_n\}})$

(2) $f(x_1^{i-1},A,x_{i+1}^n) = f(\chi_{\{x_1\}},\ldots,\chi_{\{x_{i-1}\}},\chi_A,\chi_{\{x_{i+1}\}},\ldots,\chi_{\{x_n\}})$

(3) $f(\mu_1^{i-1},x,\mu_{i+1}^n) = f(\mu_1^{i-1},\chi_{\{x\}},\mu_{i+1}^n)$,

(4) $f(\mu_1^{i-1},A,\mu_{i+1}^n) = f(\mu_1^{i-1},\chi_A,\mu_{i+1}^n)$,

where A is a non-empty subset of P and χ_X is the characteristic function of set X.

Definition 2.1 *Let P be a non-empty set, f be an F^n-hyperoperation on P and let $^{-1}: P\to P$ be a unitary operation. Then, (P,f) is called an F^n-polygroup if the following axioms hold:*

(1) f is associative,

(2) there exists an element $e\in P$ such that $e^{-1}=e$ and $\operatorname{supp}(f(\overset{(i-1)}{e}, x, \overset{(n-i)}{e}))=\{x\}$, for all $x\in P$ and for all $i\in\{1,\ldots,n\}$

(in this case we say e is an F*-identity* element of P),

(3) $z\in\operatorname{supp}(f(x_1^n))$ implies $x_i\in\operatorname{supp}(f(x_{i-1}^{-1},\ldots,x_1^{-1},z,x_n^{-1},\ldots,x_{i+1}^{-1}))$ for all $z,x_1^n\in P$.

An F^n-polygroup (P,f) is said to be *canonical* if $f(x_1^n)=f(x_{\sigma(1)}^{\sigma(n)})$, *for all* $x_1^n\in P$ *and for every* $\sigma\in S_n$.

We can see easily that if (P,f) is an F^n-polygroup, $\mu_1,\dots,\mu_n,\mu \in I_*^P$ and x_1^n, x are arbitrary elements of P, then

(1) $(x^{-1})^{-1}=x$,

(2) $(\mathrm{supp}(f(x_1^n)))^{-1}=\mathrm{supp}(f(x_n^{-1},\dots,x_1^{-1}))$, where $A^{-1}=\{a^{-1}\mid a\in A\}$,

(3) $\mathrm{supp}(f(x_1^{i-1},\mu,x_{i+1}^n))=\bigcup_{t\in\mathrm{supp}(\mu)}\mathrm{supp}(f(x_1^{i-1},t,x_{i+1}^n))$, for all $i\in\{1,\dots,n\}$,

(4) $\mathrm{supp}(f(\mu_1,\dots,\mu_n))=\bigcup_{x_i\in\mathrm{supp}(\mu_i)}\mathrm{supp}(f(x_1,\dots,x_n))$.

Example 1: *Let* $t\in(0,1]$ *and let* P *be a polygroup. We define an* F^n*-hyperoperation* f *on* P *as follows:* $f(x_1,\dots,x_n)(x)=(x_1\circ\dots\circ x_n)_t(x)$, *for all* $x_1,\dots,x_n,x\in P$. Then, (P,f) is an F^n-polygroup.

Definition 2.2 *Let* (P,f) *be an* F^n*-polygroup and let* S *be a non-empty subset of* P*. Then,* S *is called an* F*-subpolygroup if*

(1) $e\in S$,

(2) $\mathrm{supp}(f(x_1^n))\subseteq S$, for all $x_1^n\in S$,

(3) $x\in S$ implies $x^{-1}\in S$.

Notice that condition (2) of Definition 2.2 is equivalent to $f(x_1^n)\le\chi_S$, for all $x_1^n\in S$.

Lemma 2.3 *Let* (P,f) *be a canonical* F^n*-polygroup and let* S_1^n *be* F*-subpolygroups of* P*. Then,*

(1) $\mathrm{supp}(f(S_1^n))$ is an F-subpolygroup of P,

(2) $\mathrm{supp}(f(S_1^{i-1},e,S_{i+1}^n))$ is an F-subpolygroup of $\mathrm{supp}(f(S_1^n))$, for all $i\in\{1,\dots,n\}$.

Quotient F^n-polygroups

The goal of this section is to introduce an equivalence relation S^* on an F^n-polygroup and to construct a quotient F^n-polygroup.

Let (P,f) be an F^n-polygroup and let θ be an equivalence relation on P. If A and B are non-empty subsets of P, then

(1) we write $A\overline{\theta}B$ if for every $a\in A$, there exists $b\in B$ such that $a\theta b$ and for every $b\in B$ there exists $a\in A$ such that $a\theta b$,

(2) $A\overline{\overline{\theta}}B$ means that for every $a\in A$ and for every $b\in B$, we have $a\theta b$.

An equivalence relation θ defined on an F^n-polygroup (P,f) is called *regular* if for every $x_1^n,y_1^n\in P$, $x_1\theta y_1,\dots,x_n\theta y_n$ implies that $\mathrm{supp}(f(x_1^n))\overline{\theta}\mathrm{supp}(f(y_1^n))$ and θ is called *strongly regular* if $x_1\theta y_1,\dots,x_n\theta y_n$ implies that $\mathrm{supp}(f(x_1^n))\overline{\overline{\theta}}\mathrm{supp}(f(y_1^n))$.

It is easy to verify that if θ is a regular relation on (P,f), then
$\{\theta[x]\mid x\in\mathrm{supp}(f(x_1,\dots,x_n))\}=\{\theta[x]\mid x\in\mathrm{supp}(f(\theta[x_1],\dots,\theta[x_n]))\}$, where $\theta[x]$ is the equivalence class of x. Also, whenever θ is a strongly regular relation, reflexivity of θ implies that for every $z_1,z_2\in\mathrm{supp}(f(x_1^n))$ we have $\theta[z_1]=\theta[z_2]$ and so $\{\theta[x]\mid x\in\mathrm{supp}(f(x_1^n))\}$ is singleton.

Throughout the paper, we use the notation $\theta_{[x_i]}^{[x_j]}$ to denote the sequence $\theta[x_i],\dots,\theta[x_j]$.

Theorem 3.1 *Let* (P,f) *be an* F^n*-polygroup and let* θ *be a regular relation on* P*. Then,* $P/\theta=\{\theta[x]\mid x\in P\}$ *is an* F^n*-polygroup with the* F^n*-hyperoperation* f_θ *and unitary operation* $^{-1}$ *on* P *defined as follows:*

$$f_\theta(\theta_{[x_1]}^{[x_n]})=\chi_{\{\theta[z]\mid z\in\mathrm{supp}(f(x_1^n))\}},\quad \forall x_1^n\in P,$$

$$(\theta[x])^{-1}=\theta[x^{-1}],\quad \forall x\in P.$$

Proof. Since θ is a regular relation, f_θ is well-defined. We show that f_θ is associative. For every $i,j\in\{1,\dots,n\}$ and for every $\theta_{[x_1]}^{[x_{2n-1}]}\in P/\theta$, we have:

$f_\theta(\theta_{[x_1]}^{[x_{i-1}]},f_\theta(\theta_{[x_i]}^{[x_{n+i-1}]}),\theta_{[x_{n+i}]}^{[x_{2n-1}]})=$

$$
\begin{aligned}
&= \bigcup_{z\in \operatorname{supp}(f(x_i^{n+i-1}))} f_\theta(\theta_{[x_1]}^{[x_{i-1}]}, \theta[z], \theta_{[x_{n+i}]}^{[x_{2n-1}]}) \\
&= \bigcup_{z\in \operatorname{supp}(f(x_i^{n+i-1}))} \chi_{\{\theta[t] \mid t\in \operatorname{supp}(f(x_1^{i-1}, z, x_{n+i}^{2n-1}))\}} \\
&= \chi_{\{\theta[t] \mid t\in \operatorname{supp}(f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}))\}} \\
&= \chi_{\{\theta[t] \mid t\in \operatorname{supp}(f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}))\}} \\
&= f_\theta(\theta_{[x_1]}^{[x_{j-1}]}, f_\theta(\theta_{[x_j]}^{[x_{n+j-1}]}), \theta_{[x_{n+j}]}^{[x_{2n-1}]}).
\end{aligned}
$$

Evidently, $(\theta[e])^{-1} = \theta[e]$ and $f_\theta(\theta[x], \overset{(n-1)}{\theta[e]}) = \{\theta[x]\}$ for every $x \in P$, where e is the F-identity element of (P, f). Now, if $\theta[z] \in \operatorname{supp}(f_\theta(\theta_{[x_1]}^{[x_n]}))$ then there exists $z' \in \operatorname{supp}(f(x_1^n))$ such that $\theta[z] = \theta[z']$. Since (P, f) is an F^n-polygroup, for every $i \in \{1, \ldots, n\}$ we have $x_i \in \operatorname{supp}(f(x_{i-1}^{-1}, \ldots, x_1^{-1}, z', x_n^{-1}, \ldots, x_{i+1}^{-1}))$ which implies that

$$
\begin{aligned}
\theta[x_i] &\in f_\theta((\theta[x_{i-1}])^{-1}, \ldots, (\theta[x_1])^{-1}, \theta[z'], (\theta[x_n])^{-1}, \ldots, (\theta[x_{i+1}])^{-1}) \\
&= f_\theta((\theta[x_{i-1}])^{-1}, \ldots, (\theta[x_1])^{-1}, \theta[z], (\theta[x_n])^{-1}, \ldots, (\theta[x_{i+1}])^{-1}).
\end{aligned}
$$

Thus, $(P/\theta, f_\theta)$ is an F^n-polygroup.

Let S be an F-subpolygroup of an F^n-polygroup (P, f). We define the relation S^* on P as follows:

$$xS^*y \text{ if and only if } x \in \operatorname{supp}(f(S, y, \overset{(n-2)}{e})).$$

Lemma 3.2 *The relation S^* is an equivalence relation.*

Proof. It is easy to see that S^* is reflexive and symmetric. Let x, y, z be elements of P such that xS^*y and yS^*z. Then, we have

$x \in \operatorname{supp}(f(S, y, \overset{(n-2)}{e}))$ and

$y \in \operatorname{supp}(f(S, z, \overset{(n-2)}{e}))$. Therefore,

$x \in \operatorname{supp}(f(a, y, \overset{(n-2)}{e}))$ and

$y \in \operatorname{supp}(f(b, z, \overset{(n-2)}{e}))$ for some $a, b \in S$. So,

$$
\begin{aligned}
x \in \operatorname{supp}(f(a, f(b, z, \overset{(n-2)}{e}), \overset{(n-2)}{e})) &= \operatorname{supp}(f(f(a, b, \overset{(n-2)}{e}), z, \overset{(n-2)}{e})) \\
&\subseteq \operatorname{supp}(f(S, z, \overset{(n-2)}{e})).
\end{aligned}
$$

Hence, xS^*z. Therefore, the relation S^* is transitive.

Definition 3.3 *An F-subpolygroup S of (P, f) is said to be normal in P if for every $x \in P$,*

$$\operatorname{supp}(f(x, f(S, x^{-1}, \overset{(n-2)}{e}), \overset{(n-2)}{e})) \subseteq S.$$

Lemma 3.4 *For F-subpolygroups S_1^n of a canonical F^n-polygroup, where S_j is normal, for some $1 \le j \le n$, we have*

(1) $\bigcap_{i=1}^{n} S_i$ is a normal F-subpolygroup of S_k, where $1 \le k \le n$,

(2) S_j is a normal F-subpolygroup of $\operatorname{supp}(f(S_1^n))$.

Lemma 3.5 *Let S be a normal F-subpolygroup of (P, f). Then,*

(1) $\operatorname{supp}(f(x, S, \overset{(n-2)}{e})) = \operatorname{supp}(f(S, x, \overset{(n-2)}{e}))$, for all $x \in P$,

(2) $\operatorname{supp}(f(x, f(S, x^{-1}, \overset{(n-2)}{e}), \overset{(n-2)}{e})) = S$, for all $x \in P$.

Proof. (1) For every $x \in P$ we have

$$
\begin{aligned}
\operatorname{supp}(f(x, S, \overset{(n-2)}{e})) &= \operatorname{supp}(f(f(x, S, \overset{(n-2)}{e}), \overset{(n-1)}{e})) \\
&\subseteq \operatorname{supp}(f(f(x, S, \overset{(n-2)}{e}), f(x^{-1}, \overset{(n-2)}{e}, x), \overset{(n-2)}{e})) \\
&= \operatorname{supp}(f(f(f(x, S, \overset{(n-2)}{e}), x^{-1}, \overset{(n-2)}{e}), x, \overset{(n-2)}{e})) \\
&\subseteq \operatorname{supp}(f(S, x, \overset{(n-2)}{e})).
\end{aligned}
$$

Similarly, we can prove that

$$\operatorname{supp}(f(S, x, \overset{(n-2)}{e})) \subseteq \operatorname{supp}(f(x, S, \overset{(n-2)}{e})).$$

(2) For every $x \in P$ we have

$$
\begin{aligned}
S = \operatorname{supp}(f(e, S, \overset{(n-2)}{e})) &\subseteq \operatorname{supp}(f(f(x, x^{-1}, \overset{(n-2)}{e}), S, \overset{(n-2)}{e})) \\
&= \operatorname{supp}(f(x, f(x^{-1}, S, \overset{(n-2)}{e}), \overset{(n-2)}{e})) \\
&= \operatorname{supp}(f(x, f(S, x^{-1}, \overset{(n-2)}{e}), \overset{(n-2)}{e})).
\end{aligned}
$$

Thus, for every $x \in S$, we have

$$\operatorname{supp}(f(x, f(S, x^{-1}, \overset{(n-2)}{e}), \overset{(n-2)}{e})) = S.$$

Proposition 3.6 *Let S be a normal F-*

subpolygroup of (P, f). *Then, for* $x, y \in P$, *the following assertions are equivalent:*

(1) $xS^* y$,

(2) $\operatorname{supp}(f(x, y^{-1}, \overset{(n-2)}{e})) \subseteq S$,

(3) $\operatorname{supp}(f(x, y^{-1}, \overset{(n-2)}{e})) \cap S \neq \varnothing$.

Proof. $1 \Rightarrow 2)$ Since $x \in \operatorname{supp}(f(S, y, \overset{(n-2)}{e}))$, we have

$$\operatorname{supp}(f(x, y^{-1}, \overset{(n-2)}{e})) \subseteq \operatorname{supp}(f(f(S, y, \overset{(n-2)}{e}), y^{-1}, \overset{(n-2)}{e}))$$
$$= \operatorname{supp}(f(y, f(S, y^{-1}, \overset{(n-2)}{e}), \overset{(n-2)}{e}))$$
$$= S.$$

$2 \Rightarrow 3)$ It is obvious.

$3 \Rightarrow 1)$ Let $z \in \operatorname{supp}(f(x, y^{-1}, \overset{(n-2)}{e})) \cap S$. Then, we have $x \in \operatorname{supp}(f(z, y, \overset{(n-2)}{e})) \subseteq \operatorname{supp}(f(S, y, \overset{(n-2)}{e}))$.

That is, $xS^* y$.

Proposition 3.7 *Let* S *be a normal* F-*subpolygroup of* (P, f). *Then, for every* $z \in \operatorname{supp}(f(x, y, \overset{(n-2)}{e}))$ *we have*

$$\operatorname{supp}(f(x, f(S, y, \overset{(n-2)}{e}), \overset{(n-2)}{e})) = \operatorname{supp}(f(S, z, \overset{(n-2)}{e})).$$

Proof. Let $z \in \operatorname{supp}(f(x, y, \overset{(n-2)}{e}))$ be an arbitrary element. Then,

$$\operatorname{supp}(f(S, z, \overset{(n-2)}{e})) \subseteq \operatorname{supp}(f(S, f(x, y, \overset{(n-2)}{e}), \overset{(n-2)}{e})) = \operatorname{supp}(f(x, f(S, y, \overset{(n-2)}{e}), \overset{(n-2)}{e})).$$

Now, we prove the converse inclusion. Let $a \in \operatorname{supp}(f(x, f(S, y, \overset{(n-2)}{e}), \overset{(n-2)}{e}))$ be an arbitrary element. Then, we have

$y \in \operatorname{supp}(f(f(S, x^{-1}, \overset{(n-2)}{e}), a, \overset{(n-2)}{e}))$. Therefore,

$$\operatorname{supp}(f(x, y, \overset{(n-2)}{e})) \subseteq \operatorname{supp}(f(x, f(f(S, x^{-1}, \overset{(n-2)}{e}), a, \overset{(n-2)}{e}), \overset{(n-2)}{e})$$
$$= \operatorname{supp}(f(f(x, f(S, x^{-1}, \overset{(n-2)}{e}), \overset{(n-2)}{e}), a, \overset{(n-2)}{e}))$$
$$= \operatorname{supp}(f(S, a, \overset{(n-2)}{e})).$$

This implies that $z \in \operatorname{supp}(f(S, a, \overset{(n-2)}{e}))$ and so we have $a \in \operatorname{supp}(f(S, z, \overset{(n-2)}{e}))$.

Lemma 3.8 *If* S *is a normal* F-*subpolygroup of* (P, f), *then*

(1) $S^*[x] = \operatorname{supp}(f(S, x, \overset{(n-2)}{e}))$,

(2) for every $a_2^n \in P$ we have $\operatorname{supp}(f(S, a_2^n)) = S^*[x]$, where $x \in \operatorname{supp}(f(S, a_2^n))$.

For a set A, we define $S^*[A] = \bigcup_{a \in A} S^*[a]$. Obviously $S^*[A] = \operatorname{supp}(f(S, A, \overset{(n-2)}{e}))$.

Lemma 3.9 *Let* S *be a normal* F-*subpolygroup of an* F^n-*polygroup* (P, f). *Then, for all* $x_1^n \in P$ *we have*

(1) $S^*[\operatorname{supp}(f(x_1^n))] = S^*[x]$, where $x \in \operatorname{supp}(f(x_1^n))$,

(2) $S^*[\operatorname{supp}(f(x_1^n))] = \operatorname{supp}(f(S^{*[x_n]}_{[x_1]}))$,

(3) $S^*[\operatorname{supp}(f(S^{*[x_n]}_{[x_1]}))] = \operatorname{supp}(f(S^{*[x_n]}_{[x_1]}))$.

Proof. (1) Assume that $x \in \operatorname{supp}(f(x_1^n))$. It is clear that $S^*[x] \subseteq S^*[\operatorname{supp}(f(x_1^n))]$. Since

$$x \in S^*[\operatorname{supp}(f(x_1^n))] = \operatorname{supp}(f(S, f(x_1^n), \overset{(n-2)}{e}))$$

we have $\operatorname{supp}(f(x_1^n)) \subseteq \operatorname{supp}(f(S, x, \overset{(n-2)}{e})) = S^*[x]$.

Therefore, we have $S^*[\operatorname{supp}(f(x_1^n))] \subseteq S^*[x]$ which completes the proof.

(2) By using Lemma 3.8, we have

$$\operatorname{supp}(f(S^{*[x_n]}_{[x_1]})) = \operatorname{supp}(f(f(S, x_1, \overset{(n-2)}{e}), \ldots, f(S, x_n, \overset{(n-2)}{e})))$$
$$= \operatorname{supp}(f(S, f(x_1^n), \overset{(n-2)}{e}))$$
$$= S^*[\operatorname{supp}(f(x_1^n))].$$

(3) By using (2), we have

$$S^*[\operatorname{supp}(f(S^{*[x_n]}_{[x_1]}))] = S^*[S^*[\operatorname{supp}(f(x_1^n))]] = S^*[\operatorname{supp}(f(x_1^n))].$$

Corollary 3.10 *The relation* S^* *is a strongly regular relation.*

Proof. Let $x_1 S^* y_1, \ldots, x_n S^* y_n$ and $x \in \operatorname{supp}(f(x_1^n))$ and $y \in \operatorname{supp}(f(y_1^n))$. Then,

$$x \in \operatorname{supp}(f(x_1^n)) \subseteq \operatorname{supp}(f(f(S, y_1, \overset{(n-2)}{e}), \ldots, f(S, y_n, \overset{(n-2)}{e})))$$

$$\begin{aligned} &= \mathrm{supp}(f(S, f(y_1^n), \overset{(n-2)}{e})) \\ &= S^*[\mathrm{supp}(f(y_1^n))] \\ &= S^*[y]. \end{aligned}$$

Therefore, we have xS^*y.

Let S be a normal F-subpolygroup of an F^n-polygroup (P, f). Since S^* is a strongly regular relation, by Theorem 3.1, P/S^* is an F^n-polygroup with F^n-operation f_{S^*} defined as follows:

$$f_{S^*}(S^{*[x_n]}_{[x_1]}) = \chi_{\{S^*[z]\}}, \quad \forall z \in \mathrm{supp}(f(x_1^n))$$

and so P/S^* is a quotient F^n-group.

Results and Discussion

In this section, with respect to the concepts of normal F-subpolygroups and strongly regular relations and homomorphisms, we state and prove isomorphism theorems for F^n-polygroups.

Let (P_1, f_1) and (P_2, f_2) be two F^n-polygroups. A *homomorphism* from P_1 to P_2 is a mapping $\varphi : P_1 \to P_2$ such that

(1) $\varphi(e_{P_1}) = e_{P_2}$, where e_{P_1} and e_{P_2} are F-identity elements,

(2) $\varphi(\mathrm{supp}(f_1(x_1^n))) = \mathrm{supp}(f_2(\varphi_{x_1}^{x_n}))$,

hold for all $x_1^n \in P_1$, where $\varphi_{a_i}^{a_j}$ denotes the sequence $\varphi(a_i), \ldots, \varphi(a_j)$.

An injective homomorphism is called a *monomorphism* and an onto homomorphism is called an *epimorphism*. An injective and onto homomorphism is called an *isomorphism*. We say that P_1 is *isomorphic* to P_2, denoted by $P_1 \cong P_2$, if there exists an isomorphism from P_1 to P_2.

The next lemma can be proved easily using previously defined notions and thus we omit its proof.

Lemma 4.1 *Let (P_1, f_1) and (P_2, f_2) be two F^n-polygroups and let $\varphi : P_1 \to P_2$ be a homomorphism. Then,*

(1) $\varphi(x^{-1}) = (\varphi(x))^{-1}, \quad \forall x \in P_1$,

(2) φ is injective if and only if $ker\varphi = \{e_{P_1}\}$, where $ker\varphi = \{x \in P_1 \mid \varphi(x) = e_{P_2}\}$,

(3) $ker\varphi$ is an F-subpolygroup of P_1,

(4) $Im\varphi$ is an F-subpolygroup of P_2.

Let θ be a regular relation on (P, f). Clearly, the natural map $\pi : P \to P/\theta$ by $\pi(x) = \theta[x]$ is an epimorphism. π is called the *canonical* homomorphism.

Lemma 4.2 *Let (P_1, f_1) and (P_2, f_2) be two F^n-polygroups and let $\varphi : P_1 \to P_2$ be a homomorphism. Then, there exists a monomorphism $\psi : P_1/\rho_\varphi \to P_2$ such that $\psi \circ \pi = \varphi$, where*

$\rho_\varphi = \{(x, y) \in P_1 \times P_1 \mid \varphi(x) = \varphi(y)\}$.

$$\begin{tikzcd} \mathcal{P}_1 \arrow[r, "\varphi"] \arrow[d, "\pi"'] & \mathcal{P}_2 \\ \mathcal{P}_1/\rho_\varphi \arrow[ur, dashed, "\psi"'] & \end{tikzcd}$$

Proof. First, we show that ρ_φ is a regular relation on P_1 and then P_1/ρ_φ is defined. For $x_1^n, y_1^n \in P_1$ if $x_1\rho_\varphi y_1, \ldots, x_n\rho_\varphi y_n$ and $a \in \mathrm{supp}(f_1(x_1^n))$, then we have

$$\begin{aligned} \varphi(a) \in \varphi(\mathrm{supp}(f_1(x_1^n))) &= \mathrm{supp}(f_2(\varphi_{x_1}^{x_n})) \\ &= \mathrm{supp}(f_2(\varphi_{y_1}^{y_n})) \\ &= \varphi(\mathrm{supp}(f_1(y_1^n))). \end{aligned}$$

Therefore, there exists $b \in \mathrm{supp}(f_1(y_1^n))$ such that $\varphi(a) = \varphi(b)$. This implies that $\mathrm{supp}(f_1(x_1^n))\overline{\rho_\varphi}\mathrm{supp}(f_1(y_1^n))$. Hence, ρ_φ is regular. Now, we define $\psi(\rho_\varphi[x]) = \varphi(x)$. It is easy to see that ψ is a monomorphism and $\psi \circ \pi = \varphi$.

Theorem 4.3 *Let γ and θ be regular relations on (P, f) such that $\gamma \subseteq \theta$. Then, there exists a regular* *relation μ on P/γ such that $(P/\gamma)/\mu$ is isomorphic to P/θ.*

Proof. We define the map $\varphi : P/\gamma \to P/\theta$ by $\varphi(\gamma[x]) = \theta[x]$. Since $\gamma \subseteq \theta$, φ is well-defined. For $\gamma[x_1],\ldots,\gamma[x_n] \in P/\gamma$ we have

$$\begin{aligned}\varphi(f_\gamma(\gamma_{[x_1]}^{[x_n]})) &= \{\varphi(\gamma[z]) \mid z \in \mathrm{supp}(f(x_1^n))\}\\ &= \{\theta[z] \mid z \in \mathrm{supp}(f(x_1^n))\}\\ &= f_\theta(\theta_{[x_1]}^{[x_n]})\\ &= f_\theta(\varphi(\gamma[x_1]),\ldots,\varphi(\gamma[x_n])).\end{aligned}$$

Therefore, φ is a homomorphism. Now, if

$$\mu = \{(\gamma[x],\gamma[y]) \in P/\gamma \times P/\gamma \mid \varphi(\gamma[x]) = \varphi(\gamma[y])\},$$

then by Lemma 4.2, there exists a monomorphism $\psi : (P/\gamma)/\mu \to P/\theta$ such that $\psi \circ \pi = \varphi$, and so ψ is an isomorphism.

Lemma 4.4 *Let (P, f), (P_1, f_1) and (P_2, f_2) be F^n-polygroups and let $\varphi_1 : P \to P_1$ and $\varphi_2 : P \to P_2$ be epimorphisms such that $\varphi_1^{-1} \circ \varphi_1 \subseteq \varphi_2^{-1} \circ \varphi_2$. Then, there exists a unique epimorphism $\psi : P_1 \to P_2$ such that $\psi \circ \varphi_1 = \varphi_2$.*

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \mathcal{P}_1 \arrow[dl, dashed, "\psi"] \\ \mathcal{P}_2 & \end{tikzcd}$$

Proof. Since φ_1 is onto, for every $z_1 \in P_1$ there exists $x \in P$ such that $\varphi_1(x) = z_1$. We define $\psi : P_1 \to P_2$ by $\psi(z_1) = \varphi_2(x)$. If $\varphi_1(y) = z_1$ $(y \in P)$, we have $(x, y) \in \varphi_1^{-1} \circ \varphi_1 \subseteq \varphi_2^{-1} \circ \varphi_2$, and so $\varphi_2(x) = \varphi_2(y)$ This proves that ψ is well-defined. We prove that ψ is an epimorphism. Clearly, we have $\psi(e_{P_1}) = e_{P_2}$. Now, if $x_1^n \in P_1$ are arbitrary elements, then there exist $y_1^n \in P$ such that $\varphi_1(y_i) = x_i$, $(1 \le i \le n)$ and we have $\mathrm{supp}(f_2(\psi_{x_1}^{x_n})) = \mathrm{supp}(f_2(\varphi_{2}{}_{y_1}^{y_n}))$

$$\begin{aligned}&= \varphi_2(\mathrm{supp}(f(y_1^n)))\\ &= \{\varphi_2(t) \mid t \in \mathrm{supp}(f(y_1^n))\}\\ &= \{\varphi_2(t) \mid \varphi_1(t) \in \mathrm{supp}(f_1(x_1^n))\}\\ &= \{\psi(\varphi_1(t)) \mid \varphi_1(t) \in \mathrm{supp}(f_1(x_1^n))\}\\ &= \psi(\mathrm{supp}(f_1(x_1^n))).\end{aligned}$$

It is routine to check that ψ is surjective and $\psi \circ \varphi_1 = \varphi_2$. The uniqueness is evident.

Theorem 4.5 *If γ and θ are regular relations on (P, f) such that $\gamma \subseteq \theta$, then there exists an epimorphism $P/\gamma \to P/\theta$.*

Proof. Let $\pi_1 : P \to P/\gamma$ and $\pi_2 : P \to P/\theta$ be canonical homomorphisms. Since $\gamma = \pi_1^{-1} \circ \pi_1$ and $\theta = \pi_2^{-1} \circ \pi_2$, by Lemma 4.4 the proof is completed.

Proposition 4.6 *Let (P_1, f_1) and (P_2, f_2) be two F^n-polygroups and $P = P_1 \times P_2 = \{(x, y) \mid x \in P_1, y \in P_2\}$ We define $f_\otimes : P^n \to I_*^P$ as follows:*

$$f_\otimes((x_1, y_1),\ldots,(x_n, y_n))(a,b) = \min\{f_1(x_1,\ldots,x_n)(a), f_2(y_1,\ldots,y_n)(b)\}.$$

for all $(a,b) \in P$. Then, $(P, f_\otimes)$ is an F^n-polygroup.

Recall that for relations ρ and σ on P the product relation is

$$\rho \circ \sigma = \{(x,y) \in P^2 \mid (x,u) \in \rho, (u,y) \in \sigma \text{ for some } u \in P\}.$$

The diagonal relation Δ on P is the set $\{(a,a) \mid a \in P\}$ and the full relation P^2 is denoted by ∇.

Theorem 4.7 *Let (P, f) be an F^n-polygroup and θ,θ^* be regular relations on P such that $\theta \cap \theta^* = \Delta$ and $\theta \circ \theta^* = \nabla$. Then,*

$$P \cong P/\theta \times P/\theta^*$$

under the map $\psi(x) = (\theta[x], \theta^*[x])$.

Proof. If $x, y \in P$ and $\psi(x) = \psi(y)$ then $\theta[x] = \theta[y]$ and $\theta^*[x] = \theta^*[y]$, so $(x, y) \in \theta \cap \theta^*$; hence $x = y$. This means that ψ is injective. Now, let $x, y \in P$ are given. Since $\theta \circ \theta^* = \nabla$, there exists z in P such that $x\theta z$ and $z\theta^* y$,

hence $\psi(z) = (\theta[z], \theta^*[z]) = (\theta[x], \theta^*[y])$, so ψ is onto. Now, for every $x_1,\ldots,x_n \in P$, we show that $\psi(\mathrm{supp}(f(x_1^n))) = \mathrm{supp}(f_{\otimes}(\psi_{x_1}^{x_n}))$.

We have

$$\psi(\mathrm{supp}(f(x_1^n))) = \{\psi(x) \mid x \in \mathrm{supp}(f(x_1^n))\}$$
$$= \{(\theta[x],\theta^*[x]) \mid x \in \mathrm{supp}(f(x_1^n))\}$$
$$\subseteq \{(\theta[x],\theta^*[y]) \mid x,y \in \mathrm{supp}(f(x_1^n))\}$$
$$= f_{\theta}(\theta_{[x_1]}^{[x_n]}) \times f_{\theta^*}(\theta^{*[x_n]}_{[x_1]})$$

$= \mathrm{supp}(f_{\otimes}(\psi_{x_1}^{x_n}))$, and so $\psi(\mathrm{supp}(f(x_1^n))) \subseteq \mathrm{supp}(f_{\otimes}(\psi_{x_1}^{x_n}))$.

Conversely, suppose that $(\theta[a],\theta^*[b]) \in \mathrm{supp}(f_{\otimes}(\psi_{x_1}^{x_n}))$.

Then,

$$(\theta[a],\theta^*[b]) \in \{(\theta[x],\theta^*[y]) \mid x,y \in \mathrm{supp}(f(x_1^n))\}.$$

Since $\theta \circ \theta^* = \nabla$, there exists c in P such that $a\theta c$ and $c\theta^* b$, and so $(\theta[a],\theta^*[b]) = (\theta[c],\theta^*[c])$, where $c \in \mathrm{supp}(f(x_1^n))$. Therefore, $(\theta[a],\theta^*[b]) \in \psi(\mathrm{supp}(f(x_1^n)))$. This completes the proof.

Theorem 4.8 *(First Isomorphism Theorem). Let* (P_1, f_1) *and* (P_2, f_2) *be two* F^n*-polygroups and let* $\varphi : P_1 \to P_2$ *be a homomorphism such that* $K = ker\varphi$ *is a normal* F*-subpolygroup of* (P_1, f_1). *Then,* $P_1/K^* \cong Im\varphi$.

Proof. We consider the map $\psi : P_1/K^* \to Im\varphi$ by $\psi(K^*[x]) = \varphi(x)$. By the following argument ψ is well-defined.

$$K^*[x] = K^*[y] \Leftrightarrow \mathrm{supp}(f_1(K, x, \overset{(n-2)}{e_{P_1}})) = \mathrm{supp}(f_1(K, y, \overset{(n-2)}{e_{P_1}}))$$
$$\Rightarrow \varphi(\mathrm{supp}(f_1(K, x, \overset{(n-2)}{e_{P_1}}))) = \varphi(\mathrm{supp}(f_1(K, y, \overset{(n-2)}{e_{P_1}})))$$
$$\Leftrightarrow \mathrm{supp}(f_2(\varphi(K), \varphi(x), \overset{(n-2)}{e_{P_2}})) = \mathrm{supp}(f_2(\varphi(K), \varphi(y), \overset{(n-2)}{e_{P_2}}))$$
$$\Leftrightarrow \mathrm{supp}(f_2(e_{P_2}, \varphi(x), \overset{(n-2)}{e_{P_2}})) = \mathrm{supp}(f_2(e_{P_2}, \varphi(y), \overset{(n-2)}{e_{P_2}}))$$
$$\Leftrightarrow \varphi(x) = \varphi(y).$$

Obviously, $\psi(K^*[e_{P_1}]) = \varphi(e_{P_1}) = e_{P_2}$ and for every $K^{*[x_n]}_{[x_1]} \in P_1/K^*$, we have

$$\psi(\mathrm{supp}(f_{1K^*}(K^{*[x_n]}_{[x_1]}))) = \psi(K^*[z]) = \varphi(\mathrm{supp}(f_1(x_1^n)))$$
$$= \mathrm{supp}(f_2(\varphi_{x_1}^{x_n}))$$
$$= \mathrm{supp}(f_2(\psi(K^*[x_1]),\ldots,\psi(K^*[x_n]))),$$

where z is an arbitrary element of $\mathrm{supp}(f_1(x_1^n))$. Therefore, ψ is a homomorphism.

If $y \in Im\varphi$ is an arbitrary element, then there exist $x \in P_1$ such that $y = \varphi(x) = \psi(K^*[x])$ which implies that ψ is onto. We have

$$ker\psi = \{K^*[x] \in P_1/K^* \mid \psi(K^*[x]) = e_{P_2}\}$$
$$= \{K^*[x] \in P_1/K^* \mid \varphi(x) = e_{P_2}\}$$
$$= \{K^*[x] \in P_1/K^* \mid x \in K\}$$
$$= \{K^*[e_{P_1}]\}.$$

Therefore, ψ is injective and so $P_1/K^* \cong Im\varphi$.

Theorem 4.9 *(Second Isomorphism Theorem). If* S_1^n *are* F*-subpolygroups of a canonical* F^n*-polygroup* (P, f) *such that* S_j *is normal for some* $j \in \{1,\ldots,n\}$ *then*

$$\mathrm{supp}(f(S_1^{j-1}, e, S_{j+1}^n))/(\mathrm{supp}(f(S_1^{j-1}, e, S_{j+1}^n)) \cap S_j)^* \cong \mathrm{supp}(f(S_1^n))/S_j^*.$$

Proof. By Lemma 2.3, $\mathrm{supp}(f(S_1^{j-1}, e, S_{j+1}^n))$ is an F-subpolygroup of $\mathrm{supp}(f(S_1^n))$. By Lemma 3.4, S_j is a normal F-subpolygroup of $\mathrm{supp}(f(S_1^n))$. Hence, by Lemma 3.4, $\mathrm{supp}(f(S_1^{j-1}, e, S_{j+1}^n)) \cap S_j$ is a normal F-subpolygroup of $\mathrm{supp}(f(S_1^{j-1}, e, S_{j+1}^n))$. So $\mathrm{supp}(f(S_1^n))/S_j^*$ and $\mathrm{supp}(f(S_1^{j-1}, e, S_{j+1}^n))/(\mathrm{supp}(f(S_1^{j-1}, e, S_{j+1}^n)) \cap S_j)^*$ are defined.

We consider the map $\psi : \mathrm{supp}(f(S_1^{j-1}, e, S_{j+1}^n)) \to \mathrm{supp}(f(S_1^n))/S_j^*$ by $\psi(x) = S_j^*[x]$. Clearly, $\psi(e) = S_j^*[e]$ and for every $x_1^n \in \mathrm{supp}(f(S_1^{j-1}, e, S_{j+1}^n))$ we have

$$\psi(\mathrm{supp}(f(x_1^n))) = \{\psi(x) \mid x \in \mathrm{supp}(f(x_1^n))\}$$
$$= \{S_j^*[x] \mid x \in \mathrm{supp}(f(x_1^n))\}$$
$$= \{S_j^*[z]\}$$
$$= \mathrm{supp}(f_{S_j^*}(S^{*[x_n]}_{j[x_1]}))$$
$$= \mathrm{supp}(f_{S_j^*}(\psi_{x_1}^{x_n})),$$

where z is an arbitrary element of $\mathrm{supp}(f(x_1^n))$. Therefore, ψ is a homomorphism. Let

$S_j^*[a] \in \mathrm{supp}(f(S_1^n))/S_j^*$ be an arbitrary element. Then, there exist $a_i \in S_i; 1 \le i \le n$ such that $a \in \mathrm{supp}(f(a_1^n))$. Now, for

$$x \in \mathrm{supp}(f(a_1^{j-1}, e, a_{j+1}^n)) \subseteq \mathrm{supp}(f(S_1^{j-1}, e, S_{j+1}^n))$$

we have

$$\begin{aligned}
\psi(x) = S_j^*[x] = S_j^*[\mathrm{supp}(f(a_1^{j-1}, e, a_{j+1}^n))] &= \mathrm{supp}(f(S_j, f(a_1^{j-1}, e, a_{j+1}^n), \overset{(n-2)}{e}))\\
&= \mathrm{supp}(f(a_1^{j-1}, f(S_j, \overset{(n-1)}{e}), a_{j+1}^n))\\
&= \mathrm{supp}(f(a_1^{j-1}, f(S_j, a_j, \overset{(n-2)}{e}), a_{j+1}^n))\\
&= \mathrm{supp}(f(S_j, f(a_1^n), \overset{(n-2)}{e}))\\
&= S_j^*[\mathrm{supp}(f(a_1^n))]\\
&= S_j^*[a].
\end{aligned}$$

Therefore, ψ is onto. We have

$$\begin{aligned}
ker\psi &= \{x \in \mathrm{supp}(f(S_1^{j-1}, e, S_{j+1}^n)) \mid S_j^*[x] = S_j^*[e]\}\\
&= \{x \in \mathrm{supp}(f(S_1^{j-1}, e, S_{j+1}^n)) \mid x \in S_j\}\\
&= \mathrm{supp}(f(S_1^{j-1}, e, S_{j+1}^n)) \cap S_j.
\end{aligned}$$

Hence, by the First Isomorphism Theorem the desired result holds.

Theorem 4.10 *(Third Isomorphism Theorem). If* S *and* J *are normal* F *-subpolygroups of* (P, f) *such that* $S \subseteq J$*, then* J/S^* *is a normal* F *-subpolygroup of* P/S^* *and* $(P/S^*)/(J/S^*)^* \cong P/J^*$.

Proof. First, we show that J/S^* is an F -subpolygroup of P/S^*. Since $e \in J$, we have $S^*[e] \in J/S^*$. If $S^*[x] \in J/S^*$ then $x \in J$ and so $x^{-1} \in J$ which implies that $(S^*[x])^{-1} = S^*[x^{-1}] \in J/S^*$. For every $S^{*[x_n]}_{[x_1]} \in J/S^*$ and for $z \in \mathrm{supp}(f(x_1^n)) \subseteq J$ we have $\mathrm{supp}(f_{S^*}(S^{*[x_n]}_{[x_1]})) = \{S^*[z]\} \subseteq J/S^*$. Thus, J/S^* is an F -subpolygroup of P/S^*. For all $S^*[x] \in P/S^*$ and for every $t \in J$ we have

$$\begin{aligned}
\mathrm{supp}(f_{S^*}(S^*[x], f_{S^*}(S^*[t], S^*[x^{-1}], \overset{(n-2)}{S^*[e]}), \overset{(n-2)}{S^*[e]})) = \mathrm{supp}(f_{S^*}(S^*[x], S^*[z], \overset{(n-2)}{S^*[e]})&\\
= \{S^*[w]\}&\\
\subseteq J/S^*,&
\end{aligned}$$

where $z \in \mathrm{supp}(f(t, x^{-1}, \overset{(n-2)}{e}))$ and

$w \in \mathrm{supp}(f(x, z, \overset{(n-2)}{e})) \subseteq \mathrm{supp}(f(x, f(J, x^{-1}, \overset{(n-2)}{e}), \overset{(n-2)}{e})) = J$.

So, J/S^* is normal.

If we define $\psi : P/S^* \to P/J^*$ by $\psi(S^*[x]) = J^*[x]$, then it is not difficult to see that ψ is an onto homomorphism and $ker\psi = J/S^*$. Therefore, by the First Isomorphism Theorem the desired result follows easily.

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