Journal of Sciences, Islamic Republic of Iran 24(4): 361 - 364 (2013) http://jsciences.ut.ac.ir
University of Tehran, ISSN 1016-1104

Some properties of n-capable and n-perfect groups

M. R. Rismanchian*

Department of Pure Mathematics, Faculty of Mathematical Sciences, Shahrekord University, Shahrekord, Islamic Republic of Iran.

Received: 7 July 2013 / Revised: 28 October 2013 / Accepted: 16 November 2013

Abstract

In this article we introduce the notion of n-capable groups. It is shown that every group G admits a uniquely determined subgroup ([Z^n)]^* (G) which is a characteristic subgroup and lies in the n-centre subgroup of the group G. This is the smallest subgroup of G whose factor group is n-capable. Moreover, some properties of n-central extension will be studied.

Keywords: n-central; n-capable; n-perfect; n-unicentral.

Introduction

In 1979 Fay and Waals [3] introduced the notion of the n-potent and the n-centre subgroups of a group G, for a positive integer n, respectivelyas follows:

$$G_n = <[x, y^n] |\ x, y \in G>,$$

$$Z^n(G) = \{x \in G |\ xy^n = y^n x, \forall y \in G\},$$

where $[x, y^n] = x^{-1}y^{-n}xy^n$. It is easy to see that G_n is a fully invariant subgroup and $Z^n(G)$ is a characteristic subgroup of group G. In the case $n = 1$, these subgroups will be G' and $Z(G)$, the drive and centre subgroups of G, respectively. If $G_n = G$, then G is said to be n-perfect. Let H be a subgroup of G, then $[H, G^n]$ is defined as follows:

$$[H, G^n] = <[h, g^n] |\ h \in H, g \in G>,$$

and in particular if $H = G$, we get G_n. The following lemma is similar to the Lemma 2.1 of [5].

Lemma 0.1. Let G and H be two groups and N be a normal subgroup of G. Then

(i) $G_n = \{1\} \Leftrightarrow Z^n(G) = G$,
(ii) $(G / N)_n = G_n N / N$,
(iii) $N \subseteq Z^n(G) \Leftrightarrow [N, G^n] = 1$,
(iv) $Z^n(G \times H) = Z^n(G) \times Z^n(H)$.

Materials and Methods

1. *n*-capability

Baer [1] initiated an investigation of the question "which conditions a group G must be fulfill in order to be isomorphic with the group of inner automorphisms of a group E? As $\mathrm{Inn}E \cong E/Z^n(E)$, it is equivalent to study when $G \cong E/Z(E)$. By Hall and Senior [4] such a group is called capable. Let n be a positive integer, this notion can be generalized as follows:

Definition 1.1. A group G is said to be n-capable if there exists a group E such that $G \cong E/Z(E)$. Consider the homomorphism $\psi : E \longrightarrow G$ such that $Z^n(E)$ includes the kernel of ψ. The intersection of all subgroups of G of the form $\psi(Z^n(E))$, for every such ψ, denoted by $(Z^n)^*(G)$.

The group G is said to be *n-unicentral* if $(Z^n)^*(G) = Z^n(G)$. It is easy to see that $(Z^n)^*(G)$ is a characteristic

* Corresponding author: Tel /Fax: +983814421622; Email: rismanchian@sci.sku.ac.ir

subgroup of G included in $Z^n(G)$, see [6].
The following theorem is useful in the sense that the quotient group G by $(Z^n)^*(G)$ is n-capable which is a generalized version of the work of Beyl, Feglner and Schmid in [2] and similar to the work of Mirebrahimi and Mashayekhy [7] in the case of varieties of groups, see also [8] for more investigations.

Theorem 1.2. Let H_i be normal subgroup of G and G/H_i be n-capable $(i \in I)$. If $N = \bigcap_{i\in I} H_i$, then G/N is n-capable.

Proof. By definition of n-capability, for any $i \in I$, there exists the following short exact sequence

$$1 \to z^n(E_i) \overset{\subset}{\to} E_i \overset{\Psi_i}{\to} G/H_i \to 1.$$

Let $B = \prod_{i\in I} Z^n(E_i)$, and

$A = \{(e_i) \in \prod_{i\in I} E_i \mid \exists\, g \in G \text{ s.t } \Psi_i(e_i) = gH_i, \forall i \in I\}$,

Where $\prod_{i\in I} X_i$ is the cartesian product of the groups $X_i's$. Clearly $B \subseteq A$. For any $g \in G$, we can choose the elements $e_{g,i}$ such that $\Psi_i(e_{g,i}) = gH$. Thus $e_g = (e_{g,i}) \in \prod_{i\in I} E_i$. Also it is clear that the map

$$G/N \longrightarrow A/B,$$
$$gN \longrightarrow e_g B$$

is an isomorphism. Now, as $B = Z^n(A)$, we conclude that G/N is n-capable.

Theorem 1.3. $(Z^n)^*(E)$ is the least subgroup lies in the n-centre of G such that $G/(Z^n)^*(G)$ is an n-capable group.

Proof. Let $1 \to K \to E \overset{\Psi}{\to} G \to 1$ be an n-central extension by G, i.e. $K \subseteq Z^n(E)$.
By isomorphism and Theorem 2.2z it is clear that $G/(Z^n)^*(G)$ is n-capable. Now let N be a normal subgroup of G, where G/N is n-capable. Therefore, there exists an n-central extension

$$1 \to Z^n(H) \longrightarrow H \overset{\varphi}{\to} G/N \to 1.$$

Let $E = \{(g,h) \in G \times H \mid gN = \varphi(h)\}$ and ϕ be the projection map $(g,h) \mapsto g$. Then

$$1 \to Ker\phi \to E \overset{\phi}{\to} G \to 1$$

is n-central extension, since $Z^n(G \times H) = Z^n(G) \times Z^n(H)$. Let $(g,h) \in Z^n(E), (g_1,h_1) \in G \times H$ such that $\varphi(h_1) = g_1 N$. Thus, we have

$$(1,1) = [(g,h),(g_1,h_1)^n] = ([g,g_1^n],[h,h_1^n]).$$

Therefore $[h,h_1^n] = 1, \forall h_1 \in H$ and then $h \in Z^n(H)$. Now we have $\phi(Z^n(E)) \subseteq N$. Thus by the definition $(Z^n)^*(G) \subseteq \phi(Z^n(E)) \subseteq N$, which completes the proof.

An immediate necessary and sufficient condition for a group G to be n-capable is,

Corollary 1.4. A group G is n-capable if and only if $(Z^n)^*(G) = 1$.

Now we have a sufficient condition for n-capability of a group.

Corollary 1.5. Let N be a normal subgroup of G, such that $N \cap (Z^n)^*(G) = 1$. If G/N is n-capable, then so is G.

The next theorem shows that the class of n-capable groups is closed under the direct product which generalizes Proposition 6.3 of [2]. A group G is said to be subdirect product of the groups $\{G_i\}_{i\in I}$, if G is a subgroup of the (unrestricted) direct product $\prod_{i\in I} G_i$ such that $p_i(G) = G_i, i \in I$, where p_i's are natural projections.

Theorem 1.6. Let G be a subdirect product of the n-capable groups $\{G_i\}_{i\in I}$. Then so is G.

Proof. Since G_i is n-capable, we have the following short exact sequences,

$$1 \to Z^n(E_i) \overset{\subset}{\to} E_i \overset{\psi_i}{\to} G_i \to 1, \quad i \in I.$$

Define

$$\psi = \{\psi_i\}_{i\in I}: \prod_{i\in I} E_i \xrightarrow[\{e_i\}\mapsto\{\psi_i(e_i)\}]{} \prod_{i\in I} G_i,$$

and let $E = \psi^{-1}(G)$, $A = \prod_{i\in I} Z^n(E_i)$. Then A is the n-central subgroup of $\prod_{i\in I} E_i$. Hence we obtain the following commutative diagram,

$$\begin{array}{ccccccccc} 1 & \to & A & \to & E & \overset{\psi|}{\to} & G & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \prod_{i\in I} Z^n(E_i) & \to & \prod_{i\in I} E_i & \overset{\psi}{\to} & \prod_{i\in I} G_i & \to & 1, \end{array}$$

where $\psi|$ is the restricted map of ψ and the vertical maps $E \to \prod_{i\in I} E_i$ and $G \to \prod_{i\in I} G_i$ are inclusions. Since G is a subdirect product and $\ker\psi \subseteq E$, the group E is a subdirect product of $\{E_i\}_{i\in I}$.
Now it is obvious that $A \subseteq Z^n(E)$. For the reveres inclusion, let $\{e_i\}_{i\in I} \in Z^n(E)$ and $t_i \in E_i$ for an arbitrary fixed group E_i. Denote also p'_i to be the natural projection for E.Therefore, there exists $\{t'_i\}_{i\in I} \in E$ such

that $p'_i\{t'_i\}_{i\in I} = t_i$. Thus

$$p'\left(\left[\{e_i\}_{i\in I}, \{t'_i\}_{i\in I}\right]\right) = p'_i([\{e_i, t'^n\}]_{i\in I}) = p'_i(\{1_i\}_{i\in I}) = 1.$$

On the other hand,

$$\begin{aligned} p'([\{e_i\}_{i\in I}, \{t'_i\}_{i\in I}]) &= [p'_i(\{e_i\}_{i\in I}), p'_i(\{t'_i\}^n{}_{i\in I})] \\ &= [p'_i(\{e_i\}_{i\in I}), p'_i(\{t'^n{}_i\}_{i\in I}] \\ &= [e_i, t^n{}_i]. \end{aligned}$$

Hence, $[e_i, t^n{}_i] = 1$ and so the reverse inclusion holds. By $A = Z^n(E)$, we get the n-capability of G, which completes the proof.

The following corollary is immediate.

Corollary 1.7. If $\prod^{(w)}_{i\in I_i} G_i$ is a weak direct product of the groups $\{G_i\}_{i\in I}$, then

$$(Z^n)^*(\prod^{(w)}_{i\in I} G_i) \subseteq \prod^{(w)}_{i\in I}(Z^n)^*(G_i).$$

2. Application of free presentation

The structure of $Z^*(G)$ by any free presentation for the group G is given in [2]. In this section in a similar way, we study the structure of $(Z^n)^*(G)$. First, we give the following useful lemma.

Lemma 2.1. Let $1 \to R \to F \xrightarrow{\pi} G \to 1$ be a free presentation of the group G, and $1 \to A \to B \to C \to 1$ be an n-central extension of a group C. If $\alpha: G \to C$ is a homomorphism, then there exists a homomorphism $\beta: F / [R, F^n] \to B$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 1 & \to & R/[R,F^n] & \to & F/[R,F^n] & \xrightarrow{\bar{\pi}} & G & \to & 1 \\ & & \downarrow \beta| & & \downarrow \beta & & \downarrow \alpha & & \\ 1 & \to & A & \to & B & \xrightarrow{\psi} & C & \to & 1 \end{array}$$

Where $\bar{\pi}$ is the natural homomorphism induced by π and $\beta|$ is the restriction of β.

Theorem 2.2. For any free presentation $1 \to R \to F \xrightarrow{\pi} G \to 1$, and every n-central extension $1 \to A \to E \xrightarrow{\varphi} G \to 1$, we have

$$\bar{\pi}\big((Z^n)^*(F/[R,F^n])\big) \subseteq \varphi\big((Z^n)^*(E)\big).$$

Proof. By Lemma 2.1 and putting $1 \to A \to E \xrightarrow{\varphi} G \to 1$ instead of the second row in the diagram, there exists a homomorphism $\beta: F/[R, F^n] \to E$ such that the corresponding diagram is commutative. It is easily to check that $E = A\,\beta(F/[R, F^n])$ and hence, $\beta\big((Z^n)^*(F/[R, F^n])\big) \subseteq (Z^n)^*(E)$. Therefore, we get $\varphi\big(\beta(Z^n)^*(F/[R, F^n])\big) \subseteq \varphi\big((Z^n)^*(E)\big)$, which completes the proof.

The following important result is immediate.

Corollary 2.3. For any free presentation $1 \to R \to F \xrightarrow{\pi} G \to 1$ of G, we have

$$(Z^n)^*(G) = \bar{\pi}(Z^n)^*(F/[R, F^n]).$$

3. n − perfect groups

The concept of covering of a central extension by another central extension has been studied in page 92 of [6]. Here we generalize this notion.

Let $e: 1 \to A \to H \xrightarrow{\phi} G \to 1$, be an n-central extension by the group G. Now we state the following definition.

Definition 3.1. We say that the n-central extension e (*uniquely*) *covers* n-central extension,

$$1 \to A_1 \to H_1 \xrightarrow{\psi} G \longrightarrow 1,$$

If there exists a (unique) homomorphism $\theta: H \to H_1$ such that the following diagram is commutative,

$$\begin{array}{ccccccccc} 1 & \to & A & \to & H & \xrightarrow{\emptyset} & G & \to & 1 \\ & & \downarrow \theta| & & \downarrow \theta & & \downarrow I_G & & \\ 1 & \to & A_1 & \to & H_1 & \to & G & \to & 1. \end{array}$$

The n-central extension e is said to be *universal*, if uniquely covers any other n-central extension by the group G.

The following useful lemma can be easily proved.

lemma 3.2. Let G be an n-perfect group. Then $1 \to 1 \to G \xrightarrow{I_G} G \to 1$, is a universal n-central extension if and only if any n-central extension by G splits.

Now, we present the following theorem which states some essential properties of universal n-central extension.

Theorem 3.3. Let $e_i: 1 \to A_i \to H_i \xrightarrow{\phi_i} G \to 1, i = 1,2,$ be n-central extensions by the group G. Then

(i) If e_1 and e_2 are universal n-central extensions, then there exists a homomorphism $H_1 \to H_2$ such that maps A_1 onto A_2,

(ii) If e_1 is universal n-central extension, then H_1 and G are n-perfect,

(iii) If $1 \to 1 \to H \xrightarrow{\phi} G \to 1$, is a universal n-central extension, then so is $1 \to 1 \to G \xrightarrow{I_G} G \to 1$.

Proof.

(i) The proof is easy, see also Lemma 2.10.1(i) of [6].

(ii) Consider the following n-central extension,

$$1 \to A_1 \times H_1 / H_{1n} \to H_1 \times H_1 / H_{1n} \overset{\psi}{\to} G \to 1,$$

where $\psi(a, bH_{1n}) = \phi_1(a)$, $a \in A_1, b \in H_1$. Now we define the following homomorphisms

$$\theta_i: H_1 \to H_1 \times H_1/H_{1n}, i = 1,2$$

$$\theta_1(h) = (h, 1),\ \theta_2(h) = (h, hH_{1n}), \qquad \forall h \in H_1.$$

Thus $\psi o\theta_i = \phi_1$, which implies that $\theta_1 = \theta_2$. Therefore $H_1 = H_{1n}$ and so $G = G_n$.

(iii) By the definition and part (ii), G and H are n-perfect. If $1 \to A \to G^* \overset{\psi}{\to} G \to 1$, is an n-central extension of A by G, then there exists a homomorphism $\alpha: H \to G^*$ such that $\phi = \psi o\alpha$. Also, $\alpha o\phi^{-1}$ is a homomorphism from G onto G^* such that $\psi o(\alpha o\phi^{-1}) = 1$. Thus, by Lemma 3.2 the extension splits.

Results

In this paper by means of n-centre of a group we generalize some properties of capablity. Furthermore we characterize a least normal subgroup which lies in the n-centre of a given group. We derive a necessary and sufficient condition for n-capability of a group, also a sufficient condition for a group to be n-capable. Moreover we prove that subdirect product of n-capable groups is n-capable. Further we present some properties of covering and uniquely covering of an n-central extension by another n-central extension.

Acknowledgements

The author is grateful to the referee for the useful comments which improved the presentation of manuscript. The author was partially supported by the Shahrekord University as well as by the Center of Excellence for Mathematics Shahrekord University.

References

1. Baer R., Groups with preassigned central and qoutient groups, *Trans. Amer. Math*. Soc. **44:**387-412 (1938).
2. Beyl F.R., Feglner V. and Schmid P., On groups occurring as centre factor groups, *J. Algebra* **61:**161-177 (1979).
3. Fay T.H. and Waals G.L., Some remarks on n-potent and n-abelian groups, *J. Indian Math. Soc.* **47:**217-222 (1983).
4. Hall M. and Senior J.K., The group of order $2^n(n \leq 6)$, Macmillan, New York, (1964).
5. Hekster H.N., On the structure of n-isoclinism classes of groups, *J. Pure and Applied Algebra* **40:**63-85 (1986).
6. Karpilovsky G., The Schur Multiplier, London Mathematical Society Monographs, New Series 2, Clarendon press, Oxford University press, Oxford, (1987).
7. Mirebrahimi H. and Mashayekhy B., On varietal capability of infinite direct products of groups, *International J. of Group Theory* **1:**33-37 (2012).
8. Rismanchian M.R. and Araskhan M., Generalized Baer-invariant of a pair of groups and marginal extension*, J. of Sciences, Islamic Republic of Iran*, **23:**251-256 (2012).