

## **Determination of Stability Domains for Nonlinear Dynamical Systems Using the Weighted Residuals Method**

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**ABSTRACT:** Finding a suitable estimation of stability domain around stable equilibrium points is an important issue in the study of nonlinear dynamical systems. This paper intends to apply a set of analytical-numerical methods to estimate the region of attraction for autonomous nonlinear systems. In mechanical and structural engineering, autonomous systems could be found in large deformation problems or control of structures. In order to have an appropriate estimation of stability domain, some suitable Lyapunov functions are calculated by satisfying the modified Zubov's partial differential equation in a finite area around the asymptotically stable equilibrium point. To achieve this, the techniques of Collocation, Galerkin, Least squares, Moments and Sub-domain are applied. Furthermore, a number of numerical examples are solved by the suggested techniques and Zubov's construction procedure. In most cases, the proposed approaches compared with Zubov's scheme give a better estimation stability domain.

**Keywords:** Asymptotically Stable, Autonomous Systems, Lyapunov Function, Method of Weighted Residuals, Modified Zubov's PDE, Stability Domain.

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### **INTRODUCTION**

The investigation of stable equilibrium points could be helpful to interpret the characteristics of nonlinear dynamical systems. These points play an important role in many scientific and engineering problems. There are countless numerical methods which can analyze a system in the time domain. These techniques reveal the system behavior for a particular initial condition. In other words, they do not draw a general picture of properties for a nonlinear dynamical system with initial independent

parameters. Therefore, an analytical method is needed to study the characteristics of stability problems. To do this, Lyapunov (1892) established a powerful concept of stability for ordinary differential equations (Khalil, 2002; Wiggins, 2003). Testifying the stability of an equilibrium point without solving the equations of motion is an advantage of Lyapunov's theorem. In this context, many efforts have been made by researchers in aerospace, mechanical and structural engineering areas (see for example, Lewis, 2002 and 2009; Tylikowski, 2005 and Pavlović et al., 2007).

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Having a suitable estimation of stability domain for asymptotically stable equilibrium points is a substantial issue in engineering problems. To have such estimations, several approaches have been proposed (Genesio et al., 1985). A subclass of these analytical techniques is known as Lyapunov methods. In these methods, an optimal Lyapunov function is computed to find a conservative region of attraction in some neighborhood of asymptotically stable equilibrium points (Tan, 2006; Tan and Packard, 2008; Chesi et al., 2005). Lyapunov methods are well addressed in the literature (see for instance, Chesi, 2007; Johansen, 2000; Kaslik et al., 2005b and Giesl, 2007).

Zubov's studies showed that the Lyapunov function satisfying a certain partial differential equation obtains the entire stability domain (Kormanik and Li, 1972; Camilli et al., 2008). In most cases, it is hard, if not impossible; to find a closed form solution for Zubov's Partial Differential Equation (PDE) (or modified Zubov's PDE). Hence, a number of numerical methods can be applied to approximate the region of attraction. Using power series (Margolis and Vogt, 1963; Dubljević and Kazantzis, 2002; Fermín Guerrero-Sánchez et al., 2009), Lie series (Kormanik and Li, 1972), rational solution (Vannelli and Vidyasagar, 1985; Hachicho, 2007), sum-of-squares method (Peet, 2009) and other numerical techniques (Kaslik et al., 2005a; O'Shea, 1964; Rezaiee-Pajand and Moghaddasie, 2012; Giesl, 2008; Giesl and Wendland, 2011) could be helpful to achieve a conservative stability domain. Although Zubov's PDE is formulated for some particular non-autonomous systems (Aulbach, 1983a and 1983b), most of the numerical methods are applicable to autonomous systems. It is noteworthy that in some engineering problems, the averaging technique can transform the non-autonomous system into the autonomous system with an acceptable

level of accuracy (see, for example, Gilsinn, 1975; Yang et al., 2010 and Hetzler et al., 2007).

In this paper, the modified Zubov's partial differential equation is approximately solved using the weighted residuals method for autonomous systems including an asymptotically stable equilibrium point in the origin. As such, we introduce a class of Lyapunov functions, which is a linear combination of polynomial basis functions. The capability that the Lyapunov functions can be simulated in n-dimensional spaces is an advantage of the suggested basis functions. By using the techniques of Collocation, Galerkin, Least squares, Moments and Sub-domain, an error function is minimized in the vicinity of the equilibrium point to obtain the supposed Lyapunov function. Afterwards, a global optimization procedure is applied to estimate the boundary of stability domain. Since the proposed Lyapunov function is polynomial, the theory of moments is used to solve the optimization problem. Larger conservative estimation of stability domains with less polynomial terms is the superiority of the proposed method in comparison with Zubov's construction procedure.

This paper is presented in nine sections. Section 2 presents some required definitions and reviews the Lyapunov stability theorem. In addition, a powerful procedure is described to obtain a conservative region of attraction. This consequently leads to introduce a useful global optimization method for polynomial nonlinear systems with a polynomial Lyapunov function in Section 3. Section 4 addresses the modified Zubov's PDE for autonomous systems. This partial differential equation can lead to the exact stability domain in some particular cases. Furthermore, Zubov's construction procedure, which is only applicable to polynomial nonlinear systems, is explained. Section 5 introduces the method of weighted

residuals and addresses the five most common subclasses of this technique. In Section 6, a suitable basis function is proposed for multiple scalar differential equations. Section 7 presents the numerical implementation of the suggested method. Some numerical examples are provided to prove the qualification of the proposed technique in Section 8. Although, the suggested method is applicable to any autonomous system with asymptotically stable equilibrium points, the most of examples are taken from large deformation problems in this research. In order to compare the results of the Zubov's construction procedure and the proposed technique, all the examples are polynomial nonlinear. Finally, concluding remarks are given in Section 9.

### LYAPUNOV STABILITY THEOREM

In this section, the stability of a nonlinear dynamical system is investigated by the stability theorem of Lyapunov (Khalil, 2002; Wiggins, 2003). To do this, the governing equations of a nonlinear system should be transformed into a finite number of coupled first-order ordinary differential equations:

$$\dot{x} = f_{(x,t)}, x \in \mathbb{R}^n, t \in \mathbb{R} \quad (1)$$

where  $\dot{x}$  represents the time derivative of independent variables  $x$ . Since  $f$  is a function of  $t$ , Eq. (1) is a non-autonomous system. Inhere, a subclass of Eq. (1) called an autonomous (or time invariant) system is studied:

$$\dot{x} = f_{(x)}, x \in \mathbb{R}^n, t \in \mathbb{R} \quad (2)$$

A noteworthy concept in stability theories is the equilibrium point. If the initial state,  $x_{(t_0)}$ , (or  $x_0$ ) starts at  $x_e$  and stays at this

point for all future time,  $x_e$  is an equilibrium point of that dynamical system. According to this definition, the real roots of the following equations are the equilibrium points of the autonomous system (Eq.(2)):

$$f_{(x)} = 0 \quad (3)$$

The equilibrium point  $x_e$  is stable, if  $\forall \varepsilon > 0$ , there is a  $\delta > 0$  such that:

$$\|x_0 - x_e\| < \delta \Rightarrow \|x_{(t,x_0)} - x_e\| < \varepsilon, \forall t \geq t_0 \quad (4)$$

Similarly, the equilibrium point,  $x_e$ , is called asymptotically stable, if it is stable and there exists a  $\delta > 0$  such that:

$$\|x_0 - x_e\| < \delta \Rightarrow \lim_{t \rightarrow \infty} x_{(t,x_0)} = x_e \quad (5)$$

Another beneficial concept in stability theories is the region of attraction or stability domain ( $S$ ) which is defined as follows:

$$S = \left\{ x_0 \in \mathbb{R}^n \mid \lim_{t \rightarrow \infty} x_{(t,x_0)} = x_e \right\} \quad (6)$$

It is important to highlight that by a simple change in variables, the supposed equilibrium point can be shifted to the origin. This paper studies autonomous systems including an asymptotically stable equilibrium point at the origin ( $x_e = 0$ ) without loss of generality. In order to investigate the stability of the origin, a continuously differentiable function  $V : D \rightarrow \mathbb{R}$  is defined (Hachicho, 2007); where  $D \subset \mathbb{R}^n$ .  $V_{(x)}$  is positive definite (or positive semi-definite) in  $D$ , if the condition (Eq. (7)) is held:

$$V_{(0)} = 0 \text{ and } V_{(x)} > 0 \text{ (or } V_{(x)} \geq 0), \forall x \in D - \{0\} \quad (7)$$

In the same way, if  $-V_{(x)}$  is positive definite (or positive semi-definite),  $V_{(x)}$  is known as a negative definite (or negative semi-definite) function. The time derivative of this energy like function along the trajectories of Eq. (2) is as follows:

$$\dot{V}_{(x)} = \frac{\partial V_{(x)}}{\partial x} \cdot \dot{x} = \nabla V_{(x)} \cdot f_{(x)} \quad (8)$$

According to Lyapunov's stability theorem,  $x = 0$  is a stable equilibrium point, if there exists a function  $V_{(x)}$  with the following conditions:

- 1 -  $V_{(x)}$  is a positive definite function in  $D$ .
- 2 -  $\dot{V}_{(x)}$  is negative semi-definite in  $D$ .

In addition, if  $\dot{V}_{(x)}$  is negative definite in  $D - \{0\}$ , the origin is asymptotically stable. The continuously differentiable function,  $V_{(x)}$ , which is satisfying the mentioned conditions, is called a Lyapunov function for the nonlinear dynamical system.

A very beneficial theorem, which is immediately obtained from Lyapunov's stability theorem, is that if all the eigenvalues of matrix  $\partial f_{(x)} / \partial x$  at  $x = 0$  have negative real parts, the origin is asymptotically stable. On the other hand, if one or more real parts of the eigenvalues are positive, the origin is unstable (Khalil, 2002). Needless to say, for any dynamical system including a stable (or asymptotically stable) equilibrium point at  $x = 0$ , a countless number of Lyapunov functions could be found. Each Lyapunov function and its time derivative can be helpful to find a conservative estimation of stability domain (Vannelli and Vidyasagar, 1985). To achieve this, domain  $\Omega$  is defined:

$$\Omega = \left\{ x \in \mathbb{R}^n \mid \dot{V}_{(x)} \leq 0 \right\} \quad (9)$$

Therefore, the estimation of a guaranteed region of stability is as follows:

$$S = \left\{ x \in \mathbb{R}^n \mid V_{(x)} < c^* \right\} \quad (10)$$

where  $c^*$  is the largest positive value keeping  $S$  in the interior of  $\Omega$ . As a result, finding  $c^*$  is an optimization problem and can be rewritten as a global constrained optimization problem (Hachicho, 2007):

$$\begin{cases} c^* = \min V_{(x)} \\ \dot{V}_{(x)} = 0 \\ x \neq 0 \end{cases} \quad (11)$$

Furthermore, the following equation displays the boundary of the attraction region:

$$V_{(x)} = c^* \quad (12)$$

Obviously, a suitable Lyapunov function could give a less conservative estimation of stability domain.

## GLOBAL OPTIMIZATION OF POLYNOMIALS

Finding the exact solution of the global constrained optimization problem, Eq. (11), would be impossible in most nonlinear problems. However, in the case of polynomial systems with a polynomial Lyapunov function the application of the theory of moments can be used to transform this global optimization into a sequence of convex linear matrix inequality (LMI) problems (Hachicho, 2007; Lasserre, 2001). Hence, the following optimization problem is considered:

$$\begin{cases} \min p_{(x)} \\ g_{i(x)} \geq 0, i = 1, \dots, r \end{cases} \quad (13)$$

where  $p_{(x)} : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $g_{i(x)} : \mathbb{R}^n \rightarrow \mathbb{R}$  are real-valued polynomials of degrees at most  $m$  and  $w_i$ , respectively. For more simplification, the following notation is used for polynomials:

$$p_{(x)} = \sum_{\alpha} p_{\alpha} x^{\alpha}, x^{\alpha} = \prod_{j=1}^n x_j^{\alpha_j}, \sum_{j=1}^n \alpha_j \leq m \quad (14)$$

In a similar way,  $g_{i(x)}$  can be written as follows:

$$g_{i(x)} = \sum_{\beta} g_{\beta} x^{\beta}, x^{\beta} = \prod_{j=1}^n x_j^{\beta_j}, \sum_{j=1}^n \beta_j \leq w_i, i = 1, \dots, r \quad (15)$$

Now, vector  $y = \{y_{\alpha}\}$ , where  $y_{\alpha}$  is the  $\alpha$ -order moment for some probability measure,  $\mu$ , is defined. Moreover, its first element ( $y_{0, \dots, 0}$ ) is equal to 1. For instance, Eq. (16) illustrates  $y_{\alpha}$  for a two-dimensional ( $n = 2$ ) problem:

$$y_{i,j} = \int x_1^i x_2^j \mu(d(x_1, x_2)) \quad (16)$$

After all elements of vector  $y$  are computed, they establish the corresponding moment matrix  $M_{m(y)}$ . In the case of a two-dimensional problem,  $M_{m(y)}$  is a block matrix:

$$M_{m(y)} = \left\{ M_{i,j(y)} \right\}_{0 \leq i, j \leq 2m} \quad (17)$$

where each block is a  $(i+1) \times (j+1)$  matrix:

$$M_{i,j(y)} = \begin{bmatrix} y_{i+j,0} & y_{i+j-1,1} & \cdots & y_{i,j} \\ y_{i+j-1,1} & y_{i+j-2,2} & \cdots & y_{i-1,j+1} \\ \vdots & \vdots & \ddots & \vdots \\ y_{j,i} & y_{j-1,i+1} & \cdots & y_{0,i+j} \end{bmatrix} \quad (18)$$

On the other hand, if the element  $(i, j)$  of the matrix  $M_{m(y)}$  ( $M_{m(y)i,j}$ ) equals to  $y_S$ , where subscript  $S$  is a function of  $i$  and  $j$ , and the polynomial  $q_{(x)} = \sum_{\alpha} q_{\alpha} x^{\alpha}$  is given, then the elements of moment matrix  $M_{m(q,y)}$  are defined as follows:

$$M_{m(q,y)i,j} = \sum_{\alpha} q_{\alpha} y_{S+\alpha} \quad (19)$$

Afterwards, the LMI optimization problem, Eq. (20), is considered:

$$\begin{cases} \inf_y \sum_{\alpha} p_{\alpha} y_{\alpha} \\ M_{N(y)} \geq 0 \\ M_{N-\bar{w}_i(g_i,y)} \geq 0, i = 1, \dots, r \end{cases} \quad (20)$$

where  $\bar{w}_i = \lceil w_i / 2 \rceil$  is the smallest integer larger than  $w_i / 2$ , and  $N$  should satisfy the following conditions:

$$\begin{cases} N \geq \left\lceil \frac{m}{2} \right\rceil \\ N \geq \bar{w}_i, i = 1, \dots, r \end{cases} \quad (21)$$

Lasserre's work shows that the infimum value of  $\sum_{\alpha} p_{\alpha} y_{\alpha}$  in the LMI problem, Eq. (20), converges to the minimum value of  $p_{(x)}$  in the global constrained optimization problem, Eq. (13), by increasing the order of  $N$  (Lasserre, 2001). Consequently, the

optimization problem, Eq. (11), can be transformed into a simple LMI optimization problem for polynomial systems with the polynomial Lyapunov function using the theory of moments (Hachicho, 2007).

## ZUBOV'S METHOD

In section 2, the importance of Lyapunov functions in the analysis of the stability domain is described. Here, Zubov's partial differential equation (PDE) for autonomous systems is introduced (Kormanik and Li, 1972; Margolis and Vogt, 1963). Furthermore, a construction procedure for polynomial dynamical systems is presented. Zubov's method looks for the functions  $u_{(x)}$  and  $\varphi_{(x)}$  such that satisfy the following equations (Camilli et al., 2008; Vannelli and Vidyasagar, 1985):

$$u_{(0)} = 0 \quad (22)$$

$$\dot{u}_{(x)} = \nabla u_{(x)} \cdot f_{(x)} = -\varphi_{(x)}(1-u_{(x)})\sqrt{1+f_{(x)} \cdot f_{(x)}} \quad (23)$$

where  $\varphi_{(x)}$  is a positive definite function. The exact solution of partial differential Eq. (23) with condition of Eq. (22) for autonomous systems obtains the whole of stability domain:

$$S = \left\{ x \in \mathbb{R}^n \mid 0 \leq u_{(x)} < 1 \right\} \quad (24)$$

Consequently, the boundary of stability domain has the following form:

$$u_{(x)} = 1 \quad (25)$$

It should be emphasized that Eq. (23) is complicated to solve. Hence, when  $f_{(x)}$  is continuously differentiable in the

neighborhood of the origin, the following PDE can be used instead of Eq. (23):

$$\dot{u}_{(x)} = \nabla u_{(x)} \cdot f_{(x)} = -\varphi_{(x)}(1-u_{(x)}) \quad (26)$$

In a similar way, the stability domain and its boundary are derived from Eqs. (24) and (25), respectively (Margolis and Vogt, 1963; Vannelli and Vidyasagar, 1985). The transformation  $V_{(x)} = -\ln(1-u_{(x)})$  can change Eq. (26) into a simpler form:

$$\dot{V}_{(x)} = \nabla V_{(x)} \cdot f_{(x)} = -\varphi_{(x)} \quad (27)$$

Subsequently, the region of attraction and the equation of its boundary are as follows:

$$S = \left\{ x \in \mathbb{R}^n \mid 0 \leq V_{(x)} < +\infty \right\} \quad (28)$$

$$V_{(x)} = +\infty \quad (29)$$

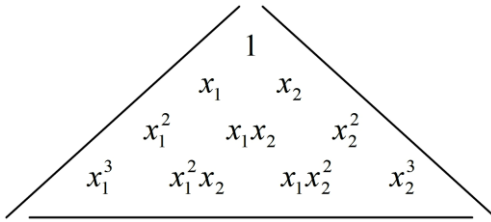
Eq. (27) is called the modified Zubov's PDE. Although Eq. (27) is quite simpler than Eq. (23), in most cases, it is hard or impossible to find the exact solution of  $V_{(x)}$ . Consequently, the approximate methods can be utilized to convince the modified Zubov's PDE in some neighborhood of the origin. It should be noted that the approximated Lyapunov function,  $V_{(x)}$ , is applicable when  $x$  satisfies the conditions given in Eqs. (9) and (10).

In the case of polynomial nonlinear systems, the power series approximation of  $V_{(x)}$  could be useful to conservatively estimate the stability domain (Margolis and Vogt, 1963):

$$V_{(x)} = V_2 + V_3 + \dots \quad (30)$$

where  $V_n$  is a homogenous polynomial relative to  $x$  of the  $n^{\text{th}}$  power with unknown coefficients. After substituting Eq. (30) into

Eq. (27) and equating the coefficients of similar terms, one can obtain a set of linear equations, which can be solved successively. Therefore, the series approximation of Lyapunov function,  $V_{(x)}$ , up to the  $n^{\text{th}}$  power will be available. This technique is called Zubov's construction procedure. Figure 1 displays the required terms for a Lyapunov function with two independent variables  $x_1$  and  $x_2$  up to the 3<sup>rd</sup> power.



**Fig. 1.** The required terms for approximate  $V$  with two independent variables  $x_1$  and  $x_2$ .

As previously mentioned, Zubov's construction procedure can be utilized for polynomial nonlinear systems. This method basically satisfies Eq. (27) up to the  $n^{\text{th}}$  power and neglects the higher terms. Hence, the approximation error will increase when  $x$  goes away from the origin. In the next section, a number of homogenous techniques, which are able to approximately satisfy the modified Zubov's PDE (27) in some neighborhood of the stable equilibrium points for non-polynomial dynamical systems, are proposed.

### METHOD OF WEIGHTED RESIDUALS

In the previous section, it is demonstrated that if there is a Lyapunov function  $V_{(x)}$  satisfying the modified Zubov's PDE for an autonomous system, the exact solution of stability domain will be achieved when  $0 \leq V_{(x)} < +\infty$ . On the other hand, it is impossible to find the exact solution of stability domain in most cases.

Consequently, the approximate techniques can be utilized to have a conservative estimation of stability domain. To do this, the method of weighted residuals is introduced.

Unlike Zubov's construction procedure, the weighted residuals approach minimizes the approximation error all over the domain of integration. This means that the approximation error is uniformly distributed in some neighborhood of the origin. Furthermore, the weighted residual technique is applicable to *non-polynomial* systems. Inhere; the five most common sub-methods of this technique are reviewed. The sub-methods dealt with in this research include Collocation, Galerkin, Least Squares, Moments and Sub-domain. In addition, a suitable basis function is also introduced in order to solve the partial differential equations with several independent variables.

The method of weighted residuals assumes that the Lyapunov function  $V_{(x)}$  is a linear combination of basis functions  $N_{i(x)}$  which are linearly independent (Akin, 2005). This can be written as follows:

$$V_{(x)} = \sum_{i=1}^m N_{i(x)} V_i \tag{31}$$

where  $V_i$  represents the value of  $V_{(x)}$  at the particular point  $x_i$ . Eqs. (32) and (33) show the conditions that basis functions should satisfy:

$$\begin{cases} N_{i(x_j)} = 1, x_i = x_j \\ N_{i(x_j)} = 0, x_i \neq x_j \end{cases} \tag{32}$$

$$\sum_{i=1}^m N_{i(x)} = 1 \tag{33}$$

By substituting the approximate Lyapunov function, Eq. (31), into the modified Zubov's PDE, Eq. (27), a residual function  $R_{(x)}$  is generated:

$$\left( \sum_{j=1}^m \nabla N_{j(x)} V_j \right) \cdot f_{(x)} + \varphi_{(x)} = R_{(x)} \quad (34)$$

In order to minimize the residual function in a supposed domain  $\Gamma$ , a set of weighted integrals of  $R_{(x)}$  is presumed to be equal to zero:

$$\int_{\Gamma} w_{i(x)} R_{(x)} d\Gamma = 0, i = 1, \dots, m \quad (35)$$

where  $w_{i(x)}$  denotes the  $i^{\text{th}}$  weighting function. Since there are  $m$  unknown values of  $V_j$  in  $R_{(x)}$ ,  $m$  weighted integrals are needed to have  $m$  equations. The matrix form of Eq. (35) is derived by substituting Eq. (34) into Eq. (35):

$$[A]_{m \times m} \{V\}_{m \times 1} = \{B\}_{m \times 1} \quad (36)$$

where

$$a_{ij} = \int_{\Gamma} w_{i(x)} (\nabla N_{j(x)} \cdot f_{(x)}) d\Gamma, i, j = 1, \dots, m \quad (37)$$

$$b_i = - \int_{\Gamma} w_{i(x)} \varphi_{(x)} d\Gamma, i = 1, \dots, m \quad (38)$$

It is noteworthy that the integration domain  $\Gamma$  should be contained in the stability domain and the stable equilibrium point is required to be included. Another point is that the linear Eq. (36) is not independent. Therefore, the boundary condition, Eq. (7), at the origin should be added to the Eq. (36):

$$V_{(0)} = \sum_{i=1}^m N_{i(0)} V_i = 0 \quad (39)$$

Eq. (39) and condition of Eq. (32) demonstrate that if one node is placed at the origin (for example, the  $o^{\text{th}}$  node),  $V_o$  is equal to zero. When the values of the computed Lyapunov function at  $x_i$  are achieved  $V_{(x)}$  will be obtained using Eq. (31). Subsequently, the stability domain will be estimated by Eqs. (9) and (10).

The choice of weighting function is the main difference between the sub-methods of the weighted residuals.

### Collocation Method

In this technique, the weights are assumed to be a Dirac Delta function:

$$w_{i(x)} = \delta_{(x-x_i)}, i = 1, \dots, m \quad (40)$$

The properties of the Dirac Delta function are as follows:

$$\begin{cases} \delta_{(x-x_i)} = +\infty, x = x_i \\ \delta_{(x-x_i)} = 0, x \neq x_i \end{cases} \quad (41)$$

$$\int_{\Gamma} \delta_{(x-x_i)} Q_{(x)} d\Gamma = Q_{(x_i)}, x_i \in \Gamma \quad (42)$$

It can be concluded from Eqs. (35) and (42) that the residual error vanishes at the particular points  $x_i$  in the Collocation method:

$$R_{(x_i)} = 0, i = 1, \dots, m \quad (43)$$

### Galerkin Method

The weighting functions are presumed to be the basis functions,  $N_{i(x)}$ , in this approach. More clearly, the residual error is forced to be orthogonal to the basis functions



in the Galerkin technique. As a result, the following condition is held:

$$\int_{\Gamma} N_{i(x)} R_{(x)} d\Gamma = 0, i = 1, \dots, m \quad (44)$$

### Least Squares Method

The residual error,  $R_{(x)}$ , exists throughout the integration domain,  $\Gamma$ . A criterion that could denote the total error,  $e_{\text{total}}$ , is the sum of  $R_{(x)}^2$  all over  $\Gamma$ :

$$e_{\text{total}} = \int_{\Gamma} R_{(x)}^2 d\Gamma \quad (45)$$

In order to find the minimum value of the total error, the derivative of  $e_{\text{total}}$  with respect to  $V_i$  should be equal to zero. This leads to the following relation:

$$\frac{\partial e_{\text{total}}}{\partial V_i} = 2 \int_{\Gamma} \frac{\partial R_{(x)}}{\partial V_i} R_{(x)} d\Gamma = 0, i = 1, \dots, m \quad (46)$$

The comparison between Eqs. (46) and (35) reveals the value of the weights in the Least Squares method:

$$w_{i(x)} = 2 \frac{\partial R_{(x)}}{\partial V_i}, i = 1, \dots, m \quad (47)$$

It should be noted that the coefficient matrix  $[A]_{m \times m}$  in Eq. (36) is *symmetric* in the Least Squares technique:

$$a_{ij} = 2 \int_{\Gamma} (\nabla N_{i(x)} \cdot f_{(x)}) (\nabla N_{j(x)} \cdot f_{(x)}) d\Gamma, i, j = 1, \dots, m \quad (48)$$

### Method of Moments

In this approach, the weighting functions are chosen from the terms of the polynomials:

$$w_{i(x)} = \prod_{q=1}^n x_q^{\alpha_q}, i = 1, \dots, m \quad (49)$$

where  $x_q$  represents the  $q^{\text{th}}$  component of variable vector  $x$  and  $\alpha_q$  is a non-negative scalar number. The maximum value of  $\alpha_q$  depends on the number of nodes laying on the direction of the  $q^{\text{th}}$  coordinate.

### Sub-Domain Method

The weights in this technique are defined as follows:

$$w_{i(x)} = \begin{cases} 1, & x \in \Gamma_i \\ 0, & x \notin \Gamma_i \end{cases}, i = 1, \dots, m \quad (50)$$

The sub-domains  $\Gamma_i$  are non-overlapping and completely fill the integration domain  $\Gamma$ :

$$\Gamma_i \cap \Gamma_j = \emptyset, i \neq j, i, j = 1, \dots, m \quad (51)$$

$$\Gamma = \bigcup_{i=1}^m \Gamma_i \quad (52)$$

Consequently, Eq. (35) can be rewritten in the following form:

$$\int_{\Gamma_i} R_{(x)} d\Gamma = 0, i = 1, \dots, m \quad (53)$$

The most important point is that the method of weighted residuals works properly when a set of basis functions compatible with the nature of the dynamical system is applied. A suitable basis function which can be used for multiple scalar differential equations is introduced in the next section.

## BASIS FUNCTIONS IN N-DIMENSIONAL PROBLEMS

As previously mentioned, the method of weighted residuals minimizes the residual

error over the integration domain,  $\Gamma$ . Depending on the number of independent variables  $n$ , the integration domain is defined in an  $n$ -dimensional space including the origin. An  $n$ -dimensional cuboid could be an appropriate suggestion for  $\Gamma$  for solving the modified Zubov's PDE with  $n$  independent variables:

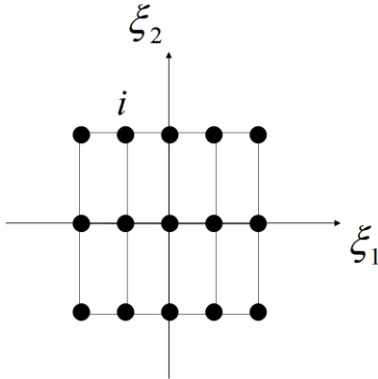
$$\Gamma = \left\{ x \in \mathbb{R}^n \mid |x_q - x_{q0}| \leq L_q, q = 1, \dots, n \right\} \quad (54)$$

where  $x_{q0}$  is the  $q^{\text{th}}$  component of the cuboid center  $x_0$  and  $2L_q$  represents the length of the side dealing with the direction  $x_q$ . The following transformation changes the integration domain into a simpler form:

$$\xi_q = \frac{x_q - x_{q0}}{L_q}, q = 1, \dots, n \quad (55)$$

$$\Gamma = \left\{ \xi \in \mathbb{R}^n \mid |\xi_q| \leq 1, q = 1, \dots, n \right\} \quad (56)$$

Eq. (56) represents an  $n$ -dimensional cube in space  $\xi$  with the center at the origin. If this cube is divided into  $n$ -dimensional cubic subspaces using a regular grid, the vertices of these subspaces could be a set of suitable locations for internal and external nodes. An example of node arrangement is given in Figure 2 for the two-dimensional space  $\xi$ .



**Fig. 2.** An example of node arrangement for 2D space.

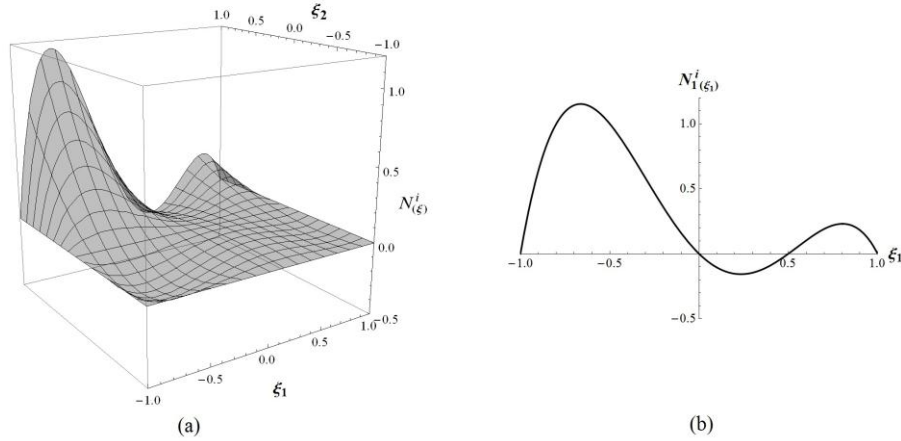
If the number of nodes in each principal direction is an odd number, there exists a node in the origin. Consequently, the boundary condition, Eq. (39), becomes simpler. Furthermore, the total number of nodes,  $m$ , is a product of the number of nodes in each axis. In order to generate the basis functions in  $n$ -dimensional spaces, the following product of polynomials in principal directions could be helpful to find a systematic method (Zienkiewicz and Taylor, 2000):

$$N_{(\xi)}^i = \prod_{q=1}^n N_{q(\xi_q)}^i, i = 1, \dots, m \quad (57)$$

where the superscript  $i$  indicates that the supposed basis function belongs to the  $i^{\text{th}}$  node. If each  $N_{q(\xi_q)}^i$  satisfies the conditions of Eqs. (32) and (33) in axis  $\xi_q$ , the obtained basis functions  $N_{(\xi)}^i$  in Eq. (57) also satisfies these conditions in  $n$ -dimensional spaces  $\xi$ . For this purpose, the Lagrange polynomials for  $N_{q(\xi_q)}^i$  are suggested (Ralston and Rabinowitz, 1978):

$$N_{q(\xi_q)}^i = \prod_k \frac{\xi_q - \xi_q^k}{\xi_q^i - \xi_q^k}, \begin{cases} i = 1, \dots, m \\ q = 1, \dots, n \end{cases} \quad (58)$$

where  $k$  represents a set of points in the direction of axis  $\xi_q$ , except point  $i$ . In addition,  $\xi_q^k$  denotes the value of  $\xi_q$  at the  $k^{\text{th}}$  point. Figure 3 displays the basis function of point  $i$  shown in Figure 2 and its first component:



**Fig. 3.** (a) The basis function of the point  $i$  (b) The first component of the basis function.

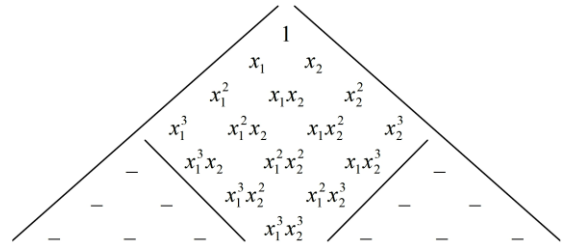
It should be noted that the terms used in Zubov's construction procedure are different from the terms applied by the proposed basis function. The maximum power of  $\xi_q$  in  $N_{q(\xi_q)}^i$  is equal to the number of nodes laying on the  $q^{\text{th}}$  axis minus one. On the other hand, the product of these components generates some higher terms. Figure 4 shows the required terms for a 2D problem with  $4 \times 4$  nodes.

The comparison between Figure 1 and Figure 4 reveals that some additional terms are needed to build the basis functions. The integration process in  $n$ -dimensional space  $\xi$  for polynomial dynamical systems is quite simple:

$$\int_{-1}^{+1} \xi^\beta d\xi = \begin{cases} 0 & , \text{if } \beta \text{ is odd} \\ \frac{2}{1+\beta} & , \text{if } \beta \text{ is even} \end{cases} \quad (59)$$

Another important issue regarding the suggested method is that the linear change of variables causing shift, rotation, scaling and so on in the integration domain,  $\Gamma$ , can affect the final result. Therefore, the union of the estimated stability domains is the largest conservative region of attraction.

Conversely, the linear change of variables has *no* effect on the final result of Zubov's construction procedure.



**Fig. 4.** The required terms for a 2D problem with  $4 \times 4$  nodes.

## COMPUTATIONAL STEPS

As suggested in this paper, the modified Zubov's partial differential equation is approximately solved using the method of weighted residuals. The same strategy is applied for all sub-methods. Therefore, the same basis functions are proposed for all weighted residuals methods. The only difference is in the selection of weighting functions. The steps needed to be taken for the application of the suggested method are described below.

### Step 1

The estimation of integration domain,  $\Gamma$ , is the first step to be taken in the proposed technique. Since the modified Zubov's PDE is defined in the region of attraction, the integration domain should be contained in it. In this way, an initial  $\Gamma$  is assumed. If it is located in the unknown exact stability domain, the estimated region of attraction could help choosing a new and, of course, a larger integration domain. Otherwise, we think that the computed stability domain will be quite smaller than  $\Gamma$ . To explain this, it should be noted that the method of weighted residuals attempts to find a suitable Lyapunov function,  $V_{(x)}$ , which its time derivative with respect to the time ( $\dot{V}$ ) is approximately equal to the negative definite function  $-\varphi_{(x)}$  over  $\Gamma$ . On the other hand, according to Eqs. (9) and (10), there is at least one point in the boundary of the computed stability domain at which  $\dot{V}$  is equal to zero. Consequently, there should be a huge jump in the time derivative of Lyapunov function around this point. This phenomenon can cause an inappropriate impression on the weighted residuals method.

### Step 2

As previously mentioned, the Lyapunov function,  $V_{(x)}$ , is replaced by a linear combination of the basis function,  $N_{i(x)}$ . Depending on the nature of the dynamical system, a set of suitable basis functions is chosen. These functions should convince the conditions of Eqs. (32) and (33). In addition, according to the modified Zubov's partial differential equation,  $N_{i(x)}$  should be continuously differentiable all over the integration domain,  $\Gamma$ . An example of basis functions applied to  $n$ -dimensional space  $x$  is proposed in the previous section.

### Step 3

After constructing the approximate Lyapunov function, the residual error can be minimized using the method of weighted residual. To do this, the weighting functions are chosen. Consequently, the coefficients of the linear Eq. (36) are obtained using Eqs. (37) and (38). As previously mentioned, the Eq. (36) is not linearly independent. In order to achieve the value of the Lyapunov function at particular point  $x_i$ , the boundary condition (39) is added to (36). As a result, the approximation of  $V_{(x)}$  will be derived using Eq. (31).

### Step 4

At the final step, domain  $\Omega$  is obtained. In this domain, the time derivative of the Lyapunov function,  $\dot{V}_{(x)}$ , is negative semi-definite. Afterward, the region of attraction will be in hand by finding the maximum value of  $c^*$  which keeps  $S$  in  $\Omega$  (Eq. (10)). In this way, the theory of moments transforms the global optimization problem, Eq. (11), into a sequence of the convex LMI problem, Eq. (20).

## NUMERICAL EXAMPLES

In this section, some problems are numerically solved to show the qualifications of the suggested method. Hence, the stability of five multiple scalar differential equations including an asymptotically stable equilibrium point at the origin are investigated. Most of the examples are large deformation problems in mechanical and structural engineering. Since Zubov's construction procedure is only applicable to polynomial nonlinear systems, non-polynomial differential equations are not concerned in this paper. Furthermore, in order to have a better comparison between various methods, the same format is

assumed for function  $\varphi_{(x)}$  in the modified Zubov's PDE:

$$\varphi_{(x)} = \sum_{q=1}^n x_q^2 \quad (60)$$

It is needless to say that all approaches mentioned in this survey obtain the region of attraction conservatively. Therefore, the largest estimated stability domain is closer to the exact solution compared to the other estimations. In diagrams presented here, solid and dashed lines indicate the boundary of stability domains given by the suggested method and Zubov's construction procedure, respectively. In all examples, the number of terms used in Zubov's method is greater than the number of terms constituting the Lyapunov functions.

### Example 1

A simple example of the nonlinear dynamical system with one degree of freedom is shown in Figure 5 (Thompson and Hunt, 1984). In this system, a rigid body under vertical loading is supported by a truss bar. Large deformation in this member makes the equation of motion nonlinear.

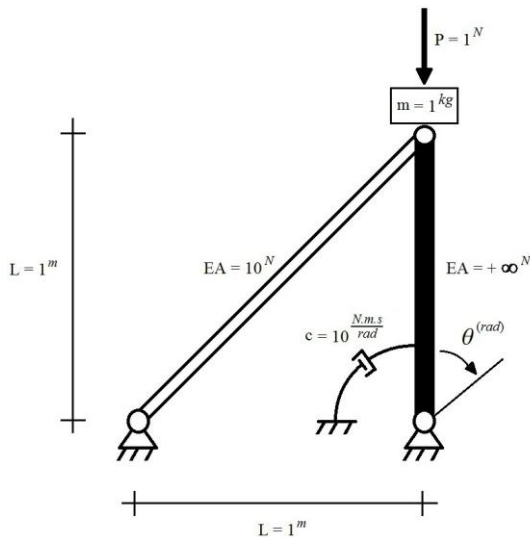


Fig. 5. A rigid body supported by a truss bar.

All mechanical properties of the dynamical system are specified in Figure 5. If  $\theta$  is assumed to be the only degree of freedom, the energy terms are as follows:

$$T = \frac{1}{2} m (L\dot{\theta})^2 \quad (61)$$

$$U = \frac{\sqrt{2}}{8} EA L \sin^2 \theta \quad (62)$$

$$W_e = PL(1 - \cos \theta) \quad (63)$$

$$W_{nc} = -\int_0^\theta (c\dot{\theta}) d\theta \quad (64)$$

where  $T$  and  $U$  are the kinetic and strain energies, respectively. In Eq. (62), it is presumed that the displacement along the truss bar varies linearly. Hence,  $U$  is a function of the Green strain  $\varepsilon$  (Crisfield, 1991):

$$U = \frac{1}{2} EA L_{e0} \varepsilon^2 \quad (65)$$

Where

$$\varepsilon = \frac{L_e^2 - L_{e0}^2}{2L_{e0}^2} = \frac{\sin \theta}{2} \quad (66)$$

In addition, the existence of the external work,  $W_e$ , and the non-conservative work,  $W_{nc}$ , are due to the external load,  $P$ , and the rotational damper,  $c$ , respectively. The Lagrange equation of motion is as follows (Wiggins, 2003):

$$mL^2 \ddot{\theta} + c\dot{\theta} + \frac{\sqrt{2}}{8} EA L \sin 2\theta - PL \sin \theta = 0 \quad (67)$$

This equation shows that  $\theta = 0^{rad}$  is an asymptotically stable equilibrium point for  $P < P_{cr}$  ( $P_{cr} = +0.35355 EA$ ). In order to

convert differential Eq. (67) to an autonomous system, the following change of variables is required:

$$\begin{cases} x_1 = \theta \\ x_2 = \dot{\theta} \end{cases} \quad (68)$$

By substituting the values of  $m, c, EA, L$  and  $P$  into Eq. (67) and considering Eq. (68), an autonomous system including an asymptotically stable equilibrium point at the origin will be at hand:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -10x_2 + \sin x_1 - 1.7678 \sin 2x_1 \end{cases} \quad (69)$$

Eq. (69) is not polynomial nonlinear. Therefore, the expansion of Taylor series around the origin up to the 5<sup>th</sup> power could be helpful:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -2.5355 x_1 - 10 x_2 + 2.1904 x_1^3 - 0.46307 x_1^5 \end{cases} \quad (70)$$

Figure 6 displays stability domains estimated using the proposed methods and Zubov's construction procedure. As seen, the Collocation technique provides the

largest region of attraction. On the other hand, there is not a solution for Eq. (35) and boundary condition, Eq. (39), when the method of Moments is applied. The estimation of Zubov's construction procedure is contained in all other computed stability domains. The application of Galerkin and Sub-domain methods provide similar results. Finally, the region of attraction obtained using Least Squares has an area between estimations resulted from Galerkin technique and Zubov's method.

Here, the integration domain  $\Gamma$  is assumed to be a square which sides have length 2 with the center at the origin. In this example,  $3 \times 3$  nodes are considered over the integration domain for the method of weighted residuals. Consequently, the Lyapunov functions computed using the suggested techniques include 9 terms. Hence, the maximum power of  $x_q$  is equal to 2. Furthermore, the modified Zubov's PDE in Zubov's construction procedure is satisfied up to the 3<sup>rd</sup> power. This means that the Lyapunov function calculated using Zubov's method includes 10 terms. Figure 7 shows the computed terms in Lyapunov functions and the largest possible value of  $c^*$ .

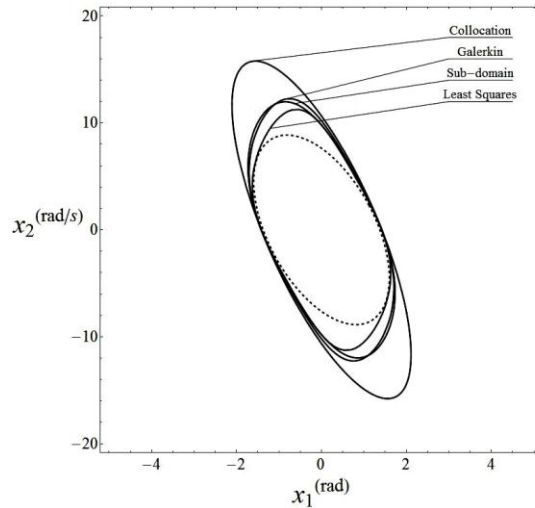
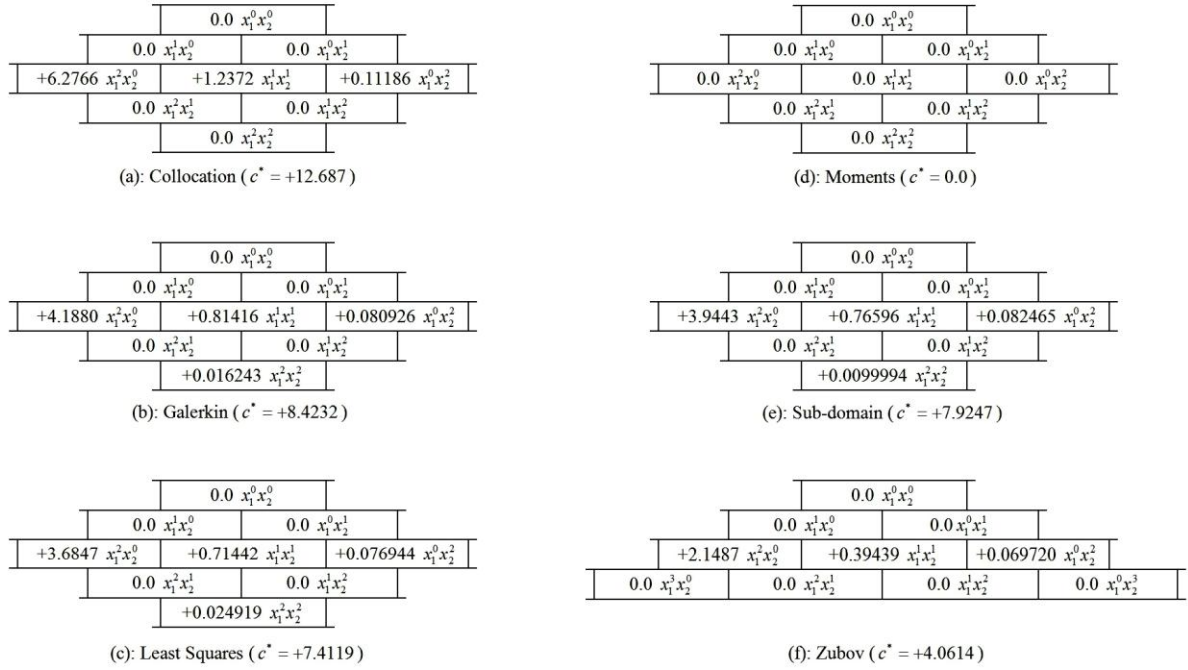


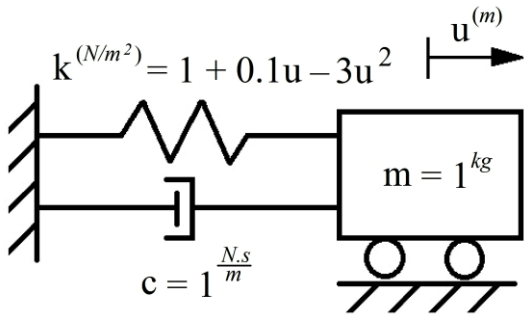
Fig. 6. Stability domain computed by the suggested techniques (solid) and Zubov's method (dashed).



**Fig. 7.** Lyapunov function terms and  $c^*$  for (a) Collocation (b) Galerkin (c) Least Squares (d) Moments (e) Sub-domain (f) Zubov.

**Example 2**

The nonlinear spring-mass system in Figure 8 is an example of autonomous systems (Khalil, 2002). This system is composed of a mass  $m$  connected to the support using a nonlinear spring,  $k$ , and a linear damper,  $c$ .



**Fig. 8.** A spring-mass system.

This dynamical system has one degree of freedom,  $u$ . Therefore, the equation of motion is as follows:

$$m\ddot{u} + c\dot{u} + ku = 0 \tag{71}$$

The equation of motion becomes simpler by substituting the values of  $m$ ,  $c$  and  $k$  in (Eq. (71)):

$$\ddot{u} + \dot{u} + u + 0.1u^2 - 3u^3 = 0 \tag{72}$$

Similar to Eq. (68), the change of variables obtains two scalar differential equations:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_2 - 0.1x_1^2 + 3x_1^3 \end{cases} \tag{73}$$

where

$$\begin{cases} x_1 = u \\ x_2 = \dot{u} \end{cases} \tag{74}$$

The stability domains estimated using the suggested techniques and Zubov's construction procedure are displayed in Figure 9. Although the method of Moments gives a Lyapunov function, it concludes the null area (similar to the previous example). The result of Zubov's method is contained in the estimated domain using the Least Squares technique. In addition, Galerkin and sub-domain yield a similar region of attraction, which is located in Collocation stability domain.

For this system, the Lyapunov function in Zubov's construction procedure is calculated up to the 3<sup>rd</sup> power. In the method of weighted residuals, the integration domain is a square which sides have length 0.5 and includes  $3 \times 3$  nodes. Consequently, the number of terms used in the suggested techniques and Zubov's method are 9 and 10, respectively. As mentioned in Section 6, a linear change of variables could make a suitable impression on the weighted

residuals techniques. Here,  $\Gamma$  is assumed to be rotated by an angle  $\theta = 45^\circ$  around the origin:

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \cdot \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} \quad (75)$$

Therefore, Eq. (73) is transformed into the following form:

$$\begin{cases} \dot{y}_1 = -0.5y_1 + 1.5y_2 + 0.035y_1^2 + 0.071y_1y_2 + 0.035y_2^2 - \\ \quad 0.75y_1^3 - 2.25y_1^2y_2 - 2.25y_1y_2^2 - 0.75y_2^3 \\ \dot{y}_2 = -0.5y_1 - 0.5y_2 - 0.035y_1^2 - 0.071y_1y_2 - 0.035y_2^2 + \\ \quad 0.75y_1^3 + 2.25y_1^2y_2 + 2.25y_1y_2^2 + 0.75y_2^3 \end{cases} \quad (76)$$

Figure 10 shows the applied terms in Lyapunov functions obtained and the maximum value of  $c^*$ .

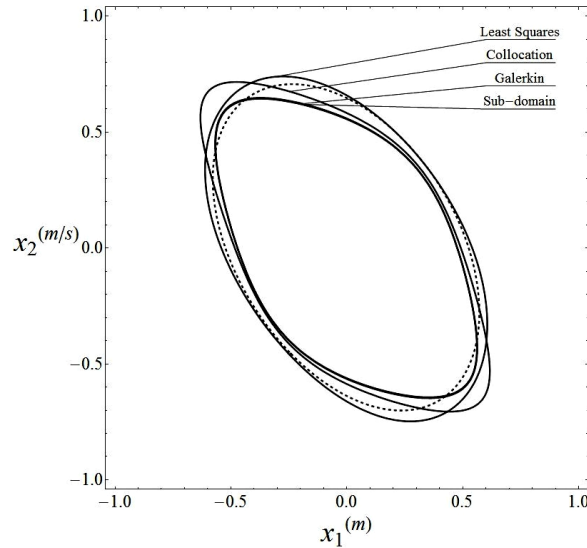
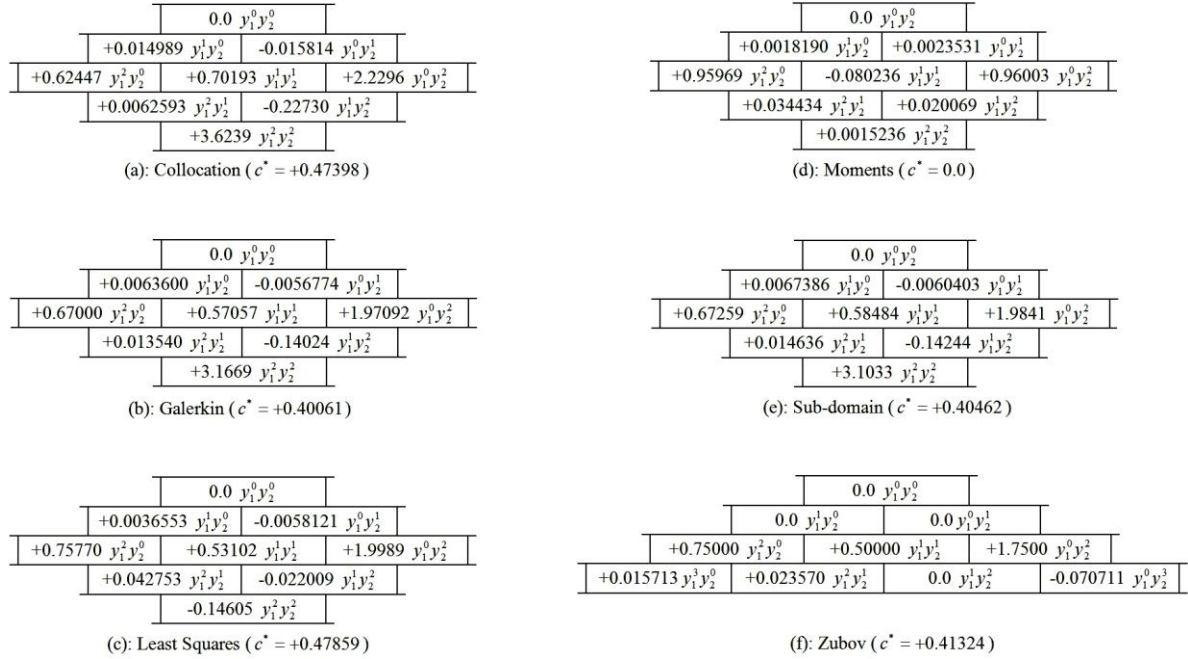


Fig. 9. Stability domain computed by the suggested techniques (solid) and Zubov's method (dashed).





**Fig. 10.** Lyapunov function terms and  $c^*$  for (a) Collocation (b) Galerkin (c) Least Squares (d) Moments (e) Sub-domain (f) Zubov.

After calculating the Lyapunov functions in space  $y$ , one can achieve the stability domains in space  $x$  by using Eq. (75) (see Figure 9).

**Example 3**

The differential equation of Vander Pol oscillator includes an asymptotically stable equilibrium point at the origin confined by an unstable limit circle as below (Grosman and Lewin, 2009):

$$\ddot{u} - \varepsilon(u^2 - 1)\dot{u} + u = 0 \quad (77)$$

where,  $\varepsilon$  is equal to 1. This equation can change into a set of two scalar differential equations as follows:

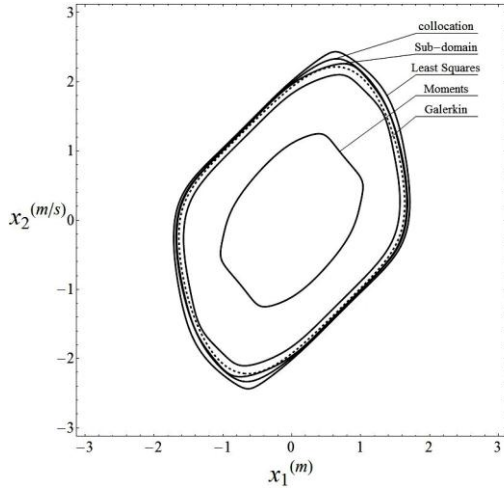
$$\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = x_1 + (x_1^2 - 1)x_2 \end{cases} \quad (78)$$

Similar to the previous example, the change of variables, Eq. (75), can be

performed to have a rotation in the integration domain,  $\Gamma$ , for weighted residuals techniques:

$$\begin{cases} \dot{y}_1 = -0.5 y_1 - 0.5 y_2 + 0.25 y_1^3 + 0.25 y_1^2 y_2 - \\ \quad 0.25 y_1 y_2^2 - 0.25 y_2^3 \\ \dot{y}_2 = +1.5 y_1 - 0.5 y_2 - 0.25 y_1^3 - 0.25 y_1^2 y_2 + \\ \quad 0.25 y_1 y_2^2 + 0.25 y_2^3 \end{cases} \quad (79)$$

The angle of rotation,  $\theta$ , is presumed to be 45 degrees. In this example,  $\Gamma$  is a square which sides have length 2 and contains  $7 \times 7$  nodes. Furthermore, Zubov's construction procedure is calculated up to the 9<sup>th</sup> power implying the related Lyapunov function includes 55 terms (6 terms greater than the Lyapunov functions in suggested methods). Figure 11 displays the estimated regions of attraction.

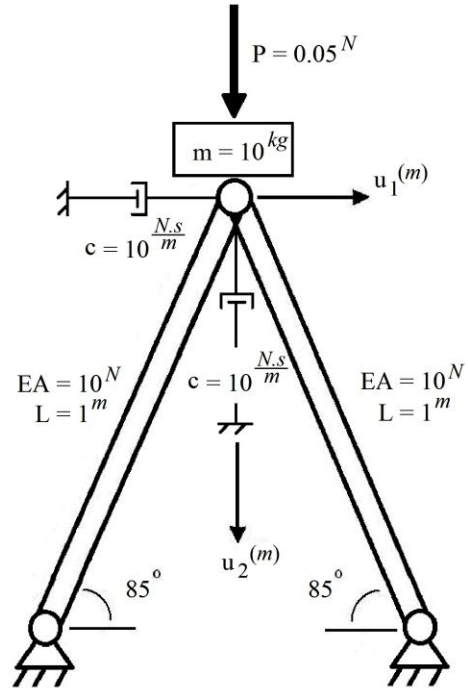


**Fig. 11.** Stability domain computed by the suggested techniques (solid) and Zubov's method (dashed).

As Figure 11 shows, Least Squares and Moments give the largest and the smallest stability domain, respectively. Here, the result of Collocation and Sub-domain are close to each other and contain the region of attraction achieved using Galerkin and Zubov's method.

#### Example 4

A two-bar non-shallow arch in Figure 12 includes a symmetric bifurcation point in its equilibrium path (Wu, 2000). This structure is subjected to a vertical load,  $P$ , at the top node. While the value of  $P$  is less than the critical load,  $P_{cr}$ , the equilibrium path is composed of asymptotically stable equilibrium points. All mechanical characteristics are specified in this figure. In the state of static loading, while the vertical load is not reaching the critical load,  $P_{cr} = 0.15018 N$ , the value of  $u_1$  is equal to zero. For  $P = 0.05 N$ , the vertical displacement of the top node,  $u_2$ , is  $0.002529 m$ . In this example, it is assumed that the Green strain is constant along the bar axis.



**Fig. 12.** A two-bar non-shallow arch.

Similar to example 1, the same procedure can be applied to obtain the equations of motion:

$$\begin{cases} m\ddot{u}_1 + c\dot{u}_1 + \frac{EA}{L^3}(3q_1^2 - L^2 + u_1^2 + \\ (u_2 - q_2)^2)u_1 = 0 \\ m\ddot{u}_2 + c\dot{u}_2 + \frac{EA}{L^3}(q_1^2 - L^2 + u_1^2 + \\ (u_2 - q_2)^2)(u_2 - q_2) = P \end{cases} \quad (80)$$

where

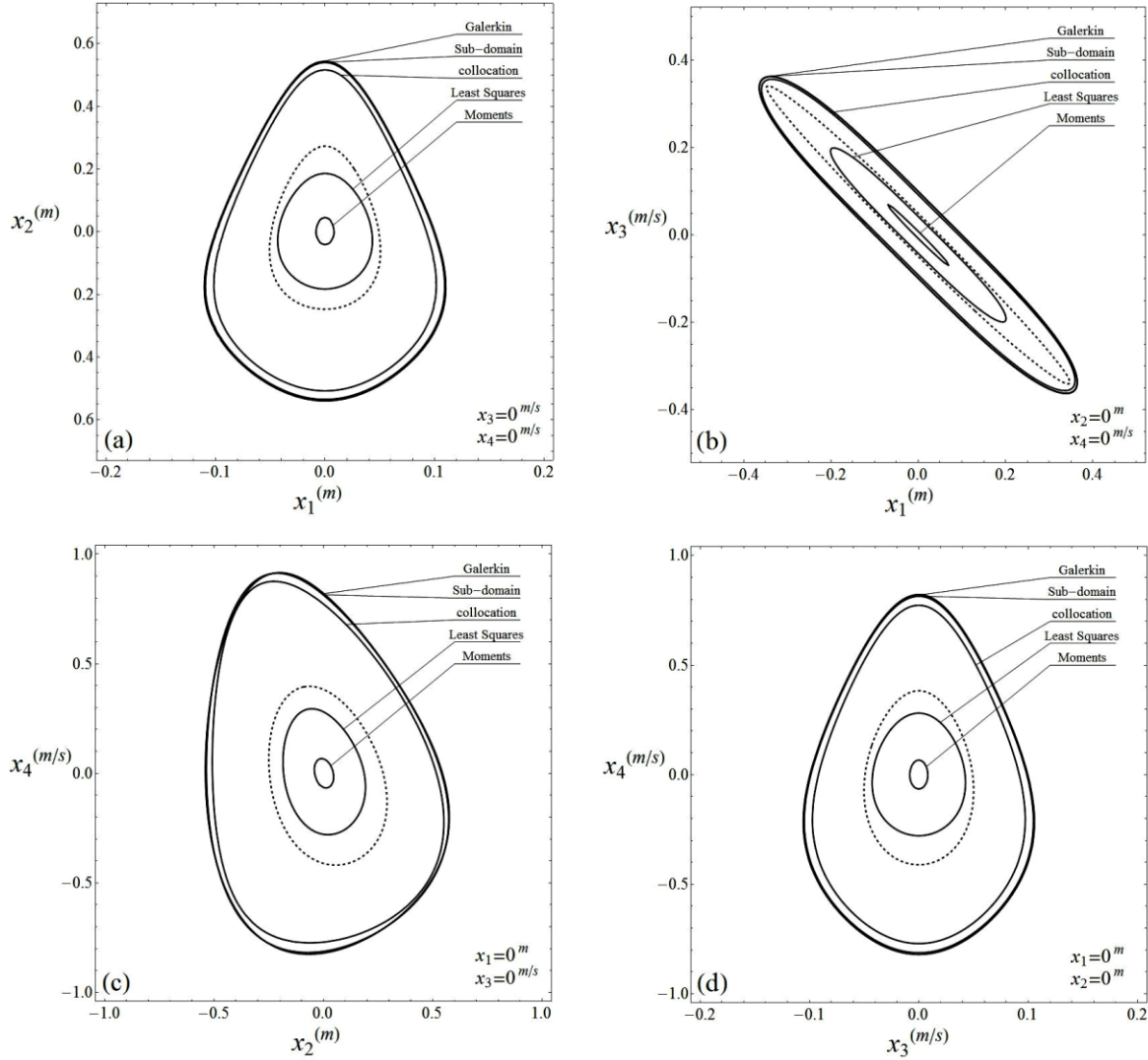
$$\begin{cases} q_1 = L \cos 85^\circ \\ q_2 = L \sin 85^\circ \end{cases} \quad (81)$$

In order to investigate the stability of the truss under the particular loading,  $P = 0.05 N$ , Eq. (80) should be changed into first-order ordinary differential equations:

$$\begin{cases} \dot{x}_1 = x_3 \\ \dot{x}_2 = x_4 \\ \dot{x}_3 = -0.01016 x_1 - x_3 + 1.9873 x_1 x_2 - x_1^3 - x_1 x_2^2 \\ \dot{x}_4 = -1.9697 x_2 - x_4 + 0.9937 x_1^2 + 2.981 x_2^2 - x_1^2 x_2 - x_2^3 \end{cases} \quad (82)$$

$$\begin{cases} x_1 = u_1 \\ x_2 = u_2 - 0.002529 \\ x_3 = \dot{u}_1 \\ x_4 = \dot{u}_2 \end{cases} \quad (83)$$

where the variables  $x_q$  are defined as follows:



**Fig. 13.** Four cross sections of 4D stability domain computed by the suggested techniques (solid) and Zubov’s method (dashed) (a)  $x_3 = x_4 = 0$  (b)  $x_2 = x_4 = 0$  (c)  $x_1 = x_3 = 0$  (d)  $x_1 = x_2 = 0$ .

Here, the integration domain,  $\Gamma$ , for the method of weighted residuals is presumed to be a four-dimensional cube which sides have length 0.25 and includes  $3 \times 3 \times 3 \times 3$  nodes. This means that there are 81 terms in the Lyapunov functions which are computed using weighted residuals techniques. In this state, the maximum power of each variable,  $x_q$ , equals 2. On the other hand, the Lyapunov function calculated using Zubov's construction procedure satisfies the modified Zubov's PDE up to the 5<sup>th</sup> power. Consequently, this function is composed of 126 terms (45 terms greater than the Lyapunov functions in the suggested methods). Figure 13 shows some cross sections of four-dimensional stability domains provided by the proposed techniques and Zubov's construction procedure.

Figures 13(a) and 13(d) display the regions of attraction without initial velocity  $x_3 = x_4 = 0 \text{ m/s}$  and initial displacement  $x_1 = x_2 = 0 \text{ m}$ , respectively. If there are not initial velocity and displacement in one degree of freedom, the stability domains in the other direction are drawn in Figures 13(b) and 13(c). As seen, Galerkin and Subdomain techniques take the largest area compared with the other techniques. The Collocation technique approximates the region of attraction close to these methods. Unlike the previous examples, the estimation of Least Squares is contained in the stability domain provided by Zubov's construction procedure. Finally, the Moments technique results in an inappropriate area.

### Example 5

Figure 14 shows a two-bar truss with two degrees of freedom. All mechanical properties of the nonlinear dynamical system are displayed in this figure. Large deformation in bars makes the equations of motion nonlinear.

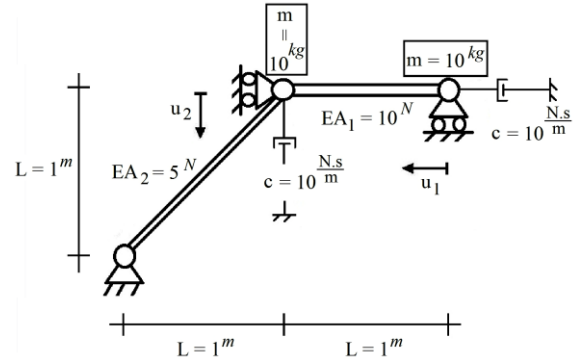


Fig. 14. A two-bar truss.

Similar to example 1, the same procedure can obtain the equations of motion:

$$\begin{cases} m\ddot{u}_1 + c\dot{u}_1 + \frac{EA_1}{2L^3}(u_1^2 + u_2^2 - 2Lu_1)(u_1 - L) = 0 \\ m\ddot{u}_2 + c\dot{u}_2 + \frac{EA_1}{2L^3}(u_1^2 + u_2^2 - 2Lu_1)u_2 + \\ \frac{\sqrt{2}EA_2}{8L^3}(2L^2 - 3Lu_2 + u_2^2)u_2 = 0 \end{cases} \quad (84)$$

By substituting the values of  $m$ ,  $c$ ,  $L$ ,  $EA_1$  and  $EA_2$  into Eq. (84), the equations of motion can be rewritten in the form of first-order ordinary differential equations:

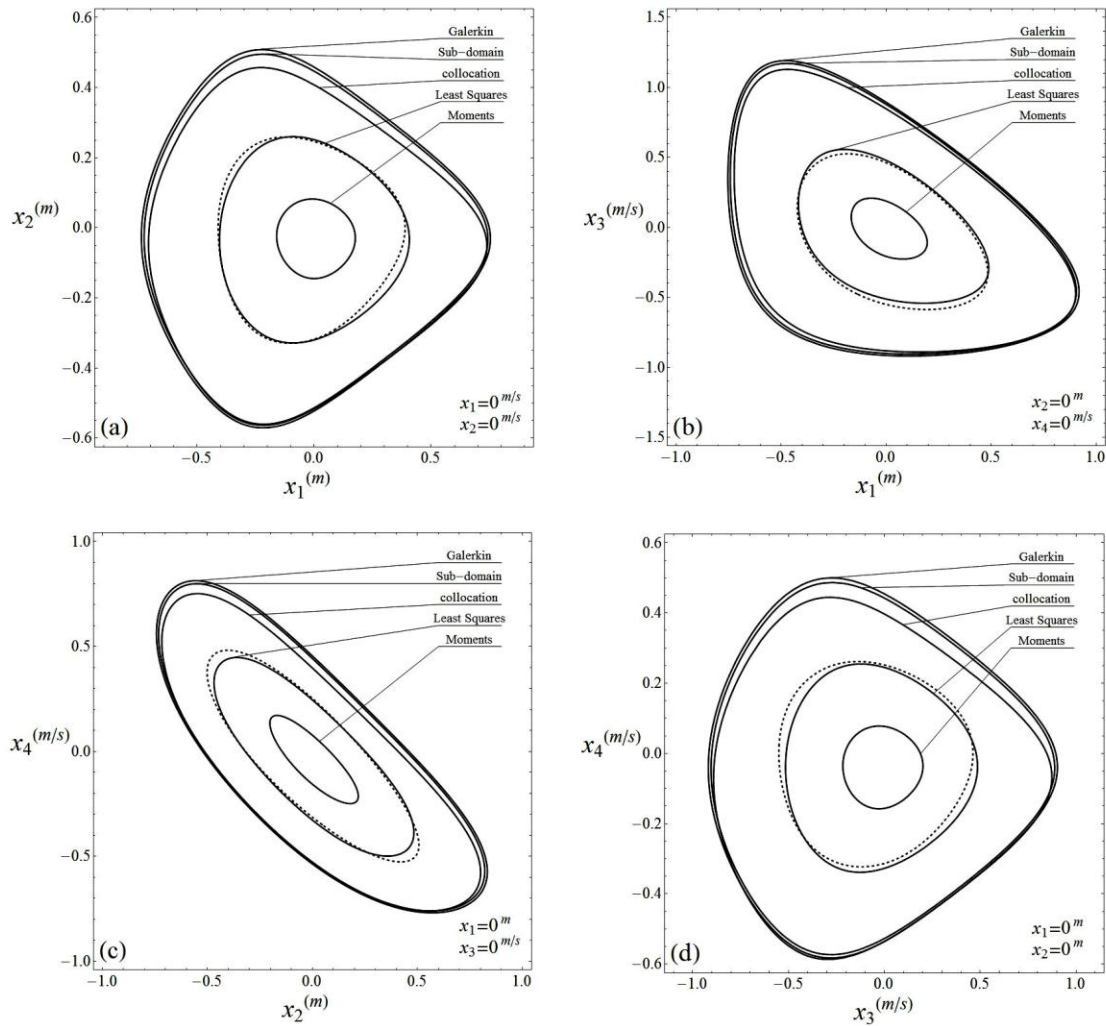
$$\begin{cases} \dot{x}_1 = x_3 \\ \dot{x}_2 = x_4 \\ \dot{x}_3 = -x_1 - x_3 + 1.5x_1^2 + 0.5x_2^2 - \\ \quad 0.5x_1^3 - 0.5x_1x_2^2 \\ \dot{x}_4 = -0.1768x_2 - x_4 + x_1x_2 + \\ \quad 0.2652x_2^2 - 0.5x_1^2x_2 - 0.5884x_2^3 \end{cases} \quad (85)$$

where

$$\begin{cases} x_1 = u_1 \\ x_2 = u_2 \\ x_3 = \dot{u}_1 \\ x_4 = \dot{u}_2 \end{cases} \quad (86)$$

In this example, for the suggested method, the integration domain,  $\Gamma$ , is a four-dimensional cube which sides have length 0.5 and includes  $3 \times 3 \times 3 \times 3$  nodes. In addition, Zubov's construction procedure is calculated up to the 5<sup>th</sup> power. Therefore, the number of applied terms in both techniques is similar to the previous example. Figure 15 illustrates four cross sections of four-dimensional stability domains given by the proposed and Zubov's method. In this

figure, the estimated stability domains for each degree of freedom, initial velocity and initial displacement are drawn. According to Figure 15, Galerkin and Sub-domain techniques take the largest region of attraction. Furthermore, Collocation gives a suitable area compared to Galerkin. Least Squares and Zubov's construction procedure obtain a similar result and are contained in Collocation. The smallest estimation belongs to the method of Moments.



**Fig. 15.** Four cross sections of 4D stability domain computed by the suggested techniques (solid) and Zubov's method (dashed) (a)  $x_3 = x_4 = 0$  (b)  $x_2 = x_4 = 0$  (c)  $x_1 = x_3 = 0$  (d)  $x_1 = x_2 = 0$ .

## CONCLUSIONS

In this paper, an analytical procedure is proposed to estimate a conservative stability domain around the asymptotically stable equilibrium point in autonomous nonlinear systems. To do this, the method of weighted residuals is suggested to solve the modified Zubov's partial differential equation in some neighborhood of the origin. For this purpose, the Lyapunov function is approximated using a linear combination of basis functions. For nonlinear systems including  $n$  independent variables, a set of suitable basis functions is defined, which can be used for all subclasses of the weighted residuals technique. To extend the investigation, strategies such as Collocation, Galerkin, Least squares, Moments and Sub-domain are utilized in the proposed algorithm. Finally, the residual function is minimized by a number of weighted integrals over the integration domain. Unlike Zubov's construction procedure, the suggested method is also applicable to non-polynomial dynamical systems.

Numerical examples show that Collocation has an appropriate and stable procedure for the estimation of stability domains compared with the other techniques, while Moments is *not* a reliable method for the approximation of the Lyapunov function. Considering this, it can be concluded that Lyapunov functions (especially in structural and mechanical engineering problems) are smooth functions and do not need weighting functions such that extremely vary all over the integration domain. Least Squares and Zubov's construction procedure obtain a similar region of attraction. In most cases, the estimation of Zubov's method is contained in the stability domain given by Least Squares. The final results of Galerkin and Sub-domain are strongly close to each other.

In some cases, they take a larger region of attraction compared to Collocation.

Finally, the number of terms used in Zubov's construction procedure is *greater* than the number of terms constituting the Lyapunov functions in the proposed techniques in all examples. This result convinces the analyst for the robustness and ability of the suggested algorithm. It is worth adding that depending on the nature of the dynamical system, other types of basis functions could be potentially applied to achieve a better estimation of stability domain.

## REFERENCES

- Akin, J.E. (2005). *Finite element analysis with error estimators: An introduction to the FEM and adaptive error analysis for engineering students*. Butterworth-Heinemann, Oxford.
- Aulbach, B. (1983). "Asymptotic stability regions via extensions of Zubov's method-I", *Nonlinear Analysis*, 7, 1431-1440.
- Aulbach, B. (1983). "Asymptotic stability regions via extensions of Zubov's method-II", *Nonlinear Analysis*, 7, 1441-1454.
- Camilli, F., Grüne, L. and Wirth, F. (2008). "Control Lyapunov functions and Zubov's method", *SIAM Journal on Control and Optimization*, 47, 301-326.
- Chesi, G. (2007). "Estimating the domain of attraction via union of continuous families of Lyapunov estimates", *Systems & Control Letters*, 56, 326-333.
- Chesi, G., Garulli, A., Tesi, A. and Vicino, A. (2005). "LMI-based computation of optimal quadratic Lyapunov functions for odd polynomial systems", *International Journal of Robust Nonlinear Control*, 15, 35-49.
- Crisfield, M.A. (1991). *Non-Linear finite element analysis of solids and structures, Vol. 1: Essentials*. John Wiley & Sons, Chichester.
- Dubljević, S. and Kazantzis, N. (2002). "A new Lyapunov design approach for nonlinear systems based on Zubov's method (brief paper)", *Automatica*, 38, 1999-2007.
- Fermín Guerrero-Sánchez, W., Guerrero-Castellanos, J.F. and Alexandrov, V.V. (2009). "A computational method for the determination of attraction regions", 6<sup>th</sup> *International Conference*



- on *Electrical Engineering Computing Science and Automatic Control (CCE 2009)*, 1-7.
- Genesio, R., Tartaglia, M. and Vicino, A. (1985). "On the estimation of asymptotic stability regions: State of the art and new proposals", *IEEE Transactions on Automatic Control*, 30, 747-755.
- Giesl, P. (2007). *Construction of global Lyapunov functions using radial basis functions*. Lecture Notes in Mathematics, Vol. 1904, Springer, Berlin.
- Giesl, P. (2008). "Construction of a local and global Lyapunov function using radial basis functions", *IMA Journal of Applied Mathematics*, 73, 782-802.
- Giesl, P. and Wendland, H. (2011). "Numerical determination of the basin of attraction for exponentially asymptotically autonomous dynamical systems", *Nonlinear Analysis*, 74, 3191-3203.
- Gilsinn, D.E. (1975). "The method of averaging and domains of stability for integral manifolds", *SIAM Journal on Applied Mathematics*, 29, 628-660.
- Grosman, B. and Lewin, D.R. (2009). "Lyapunov-based stability analysis automated by genetic programming (brief paper)", *Automatica*, 45, 252-256.
- Hachicho, O. (2007). "A novel LMI-based optimization algorithm for the guaranteed estimation of the domain of attraction using rational Lyapunov functions", *Journal of the Franklin Institute*, 344, 535-552.
- Hetzler, H., Schwarzer, D. and Seemann, W. (2007). "Analytical investigation of steady-state stability and Hopf-bifurcations occurring in sliding friction oscillators with application to low-frequency disc brake noise", *Communications in Nonlinear Science and Numerical Simulation*, 12, 83-99.
- Johansen, T.A. (2000). "Computation of Lyapunov functions for smooth nonlinear systems using convex optimization", *Automatica*, 36, 1617-1626.
- Kaslik, E., Balint, A.M. and Balint, St. (2005). "Methods for determination and approximation of the domain of attraction", *Nonlinear Analysis*, 60, 703-717.
- Kaslik, E., Balint, A.M., Grigis, A. and Balint, St. (2005). "Control procedures using domains of attraction", *Nonlinear Analysis*, 63, 2397-2407.
- Khalil, H.K. (2002). *Nonlinear systems*. 3<sup>rd</sup> Ed. Prentice-Hall.
- Kormanik, J. and Li, C.C. (1972). "Decision surface estimate of nonlinear system stability domain by Lie series method", *IEEE Transactions on Automatic Control*, 17, 666-669.
- Lasserre, J.B. (2001). "Global optimization with polynomials and the problem of moments", *SIAM Journal on Optimization*, 11, 796-817.
- Lewis, A.P. (2002). "An investigation of stability in the large behaviour of a control surface with structural non-linearities in supersonic flow", *Journal of Sound and Vibration*, 256, 725-754.
- Lewis, A.P. (2009). "An investigation of stability of a control surface with structural nonlinearities in supersonic flow using Zubov's methods", *Journal of Sound and Vibration*, 325, 338-361.
- Margolis, S.G. and Vogt, W.G. (1963). "Control engineering applications of V.I. Zubov's construction procedure for Lyapunov functions", *IEEE Transactions on Automatic Control*, 8, 104-113.
- O'Shea, R.P. (1964). "The extension of Zubov's method to sampled data control systems described by nonlinear autonomous difference equations", *IEEE Transactions on Automatic Control*, 9, 62-70.
- Pavlović, R., Kozić, P., Rajković, P. and Pavlović, I. (2007). "Dynamic stability of a thin-walled beam subjected to axial loads and end moments", *Journal of Sound and Vibration*, 301, 690-700.
- Peet, M.M. (2009). "Exponentially stable nonlinear systems have polynomial Lyapunov functions on bounded regions", *IEEE Transactions on Automatic Control*, 54, 979-987.
- Ralston, A., and Rabinowitz, P. (1978). *A first course in numerical analysis*, 2<sup>nd</sup> Ed., McGraw-Hill, New York.
- Rezaiee-Pajand, M. and Moghaddasie, B. (2012). "Estimating the region of attraction via collocation for autonomous nonlinear systems", *Structural Engineering and Mechanics*, 41, 263-284.
- Tan, W. (2006). *Nonlinear Control Analysis and Synthesis using Sum-of-Squares Programming*. PhD dissertation, University of California, Berkeley, California.
- Tan, W. and Packard, A. (2008). "Stability Region Analysis Using Polynomial and Composite Polynomial Lyapunov Functions and Sum-of-Squares Programming", *IEEE Transactions on Automatic Control*, 53, 565-571.
- Thompson, J.M.T. and Hunt, G.W. (1984). *Elastic instability phenomena*, John Wiley & Sons, New York.
- Tylikowski, A. (2005). "Lyapunov functionals application to dynamic stability analysis of continuous systems", *Nonlinear Analysis*, 63, e169-e183.
- Vannelli, A. and Vidyasagar, M. (1985). "Maximal Lyapunov functions and domains of attraction for

autonomous nonlinear systems”, *Automatica*, 21, 69-80.

Wiggins, S. (2003). *Introduction to applied nonlinear dynamical systems and chaos*. 2<sup>nd</sup> Ed., Springer, New York.

Wu, B. (2000). “Direct calculation of buckling strength of imperfect structures”, *International Journal of Solids and Structures*, 37, 1561-1576.

Yang, X.D., Tang, Y.Q., Chen, L.Q. and Lim, C.W. (2010). “Dynamic stability of axially accelerating Timoshenko beam: Averaging method”, *European Journal of Mechanics - A/Solids*, 29, 81-90.

Zienkiewicz, O.C. and Taylor, R.L. (2000). *The finite element method, Vol. 1: The basis*. 5<sup>th</sup> Ed. Butterworth-Heinemann, Oxford.