Journal of Sciences, Islamic Republic of Iran 25(4): 369 - 377 (2014) http://jsciences.ut.ac.ir
University of Tehran, ISSN 1016-1104

Subpullbacks and Po-flatness Properties of S-posets

A. Golchin* and L. Nouri

Department of Mathematics, Faculty of Sciences, University of Sistan and Baluchestan, Zahedan, Islamic Republic of Iran

Received: 16 April 2014 / Revised: 1 November 2014 / Accepted: 19 November 2014

Abstract

In (Golchin A. and Rezaei P., Subpullbacks and flatness properties of S-posets. *Comm. Algebra*. **37**: 1995-2007 **(**2009)) study was initiated of flatness properties of right S-posets A_S over a pomonoid S that can be described by surjectivity of φ corresponding to certain (sub)pullback diagrams and new properties such as $(PWP)_w$ and $(WP)_w$ were discovered. In this article first of all we describe po-flatness properties of S-posets over pomonoids by po-surjectivity of φ corresponding to certain subpullback diagrams. Then we introduce three new Conditions (P_{sw}), $(WP)_{sw}$ and $(PWP)_{sw}$ and investigate the relation between them and Conditions (P), (P_w), (WP), $(WP)_w$, (PWP) and $(PWP)_w$. Also we describe these properties by po-surjectivity of φ corresponding to certain subpullback diagrams and describe Conditions (P_w), $(WP)_w$ and $(PWP)_w$ by weak po-surjectivity of φ corresponding to certain subpullback diagrams.

Keywords: Ordered monoid; S-poset; Subpullback diagram.

Introduction

In [5] the notation $P(M, N, f, g, Q)$ was introduced to denote the pullback diagram of homomorphisms $f\colon {}_SM \to {}_SQ$ and $g\colon {}_SN \to {}_SQ$ in the category of left S-acts, where S is a monoid. Tensoring such a diagram by A_S produces a diagram (in the category of sets) that may or may not be a pullback diagram, depending on whether or not the mapping φ, obtained via the universal property of pullbacks in the category of sets, is bijective. It was shown that, if we require either bijectivity or surjectivity of φ for pullback diagrams of certain types, we not only recover most of the well-known forms of flatness, but also obtain Conditions (WP) and (PWP) as well. In [4] Golchin and Rezaei extended the results from [5] to S-posets and two new Conditions $(WP)_w$ and $(PWP)_w$ were introduced.

Let S be a pomonoid. A poset A is called a *right S-poset* if S acts on A in such a way that (i) the action is monotonic in each of the variables, (ii) for all $s, t \in S$ and $a \in A$, $a(st) = (as)t$ and $a1 = a$. Left S-posets are defined similarly. The notations A_S and ${}_SB$ will often be used to denote a right or left S-poset and $\Theta_S = \{\theta\}$ is the one-element right S-poset. The S-poset morphisms are the order-preserving maps that also preserve the S-action. We denote the category of all right S-posets, with S-poset maps between them, by Pos-S. We recall from [3] that an S-poset map $f\colon A_S \to B_S$ is called an *embedding* if $f(a) \le f(a')$ implies $a \le a'$, for $a, a' \in A_S$.

We recall from [2] that in the categories of S-posets and posets the order relation on morphism sets is defined pointwise (i.e. $f \le g$ for $f, g\colon A \longrightarrow B$ if and only if $f(a) \le g(a)$ for every $\boldsymbol{a} \in A$). In such

* Corresponding author: Tel: +989153498782; Fax: +985412447166, Email: agdm@math.usb.ac.ir

categories, a diagram

$$\begin{tikzcd} P \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & M \arrow[d, "f"] \\ N \arrow[r, "g"'] & Q \end{tikzcd}$$

(P_1)

is subcommutative if $f\pi_1 \leq g\pi_2$. If $f\pi_1 \leq g\pi_2$ and for every left S-poset P' and all homomorphisms $\pi_1'\colon P' \longrightarrow M$ and $\pi_2'\colon P' \longrightarrow N$ such that $f\,\pi_1' \leq g\,\pi_2'$, there exists a unique homomorphism $h\colon P' \longrightarrow P$ such that $\pi_1 h = \pi_1'$ and $\pi_2 h = \pi_2'$, then diagram (P_1) will be called *subpullback diagram* for f and g. In the category of S-posets or the category of posets, P may in fact be realized as

$$P = \{(m,n) \in M \times N \mid f(m) \leq g(n)\}$$

where π_1 and π_2 are the first and second coordinate projections. (Notice that P is possibly empty.) The subpullback diagram (P_1) will be denoted by $P(M,N,f,g,Q)$. Tensoring the subpullback diagram $P(M,N,f,g,Q)$ by any right S-poset A one gets the subcommutative diagram

$$\begin{tikzcd} A \otimes P \arrow[r, "id_A \otimes \pi_1"] \arrow[d, "id_A \otimes \pi_2"'] & A \otimes M \arrow[d, "id_A \otimes f"] \\ A \otimes N \arrow[r, "id_A \otimes g"'] & A \otimes Q \end{tikzcd}$$

in the category of posets. For the subpullback of mappings $id_A\otimes f$ and $id_A\otimes g$ in the category of posets we may take

$$P' = \{(a \otimes m, a'\otimes n) \in (A\otimes M) \times (A\otimes N) \mid a \otimes f(m) \leq a'\otimes g(n)\}$$

with π_1' and π_2' the restrictions of the projections. It follows from the definition of subpullbacks that there exists a unique monotone mapping $\varphi\colon A \otimes_S P \longrightarrow P'$ such that, in the diagram

$$\begin{tikzcd} A \otimes P \arrow[dr, "\varphi"] \arrow[drr, "id_A \otimes \pi_1"] \arrow[ddr, "id_A \otimes \pi_2"'] & & \\ & P' \arrow[r, "\pi_1'"'] \arrow[d, "\pi_2'"] & A \otimes M \arrow[d, "id_A \otimes f"] \\ & A \otimes N \arrow[r, "id_A \otimes g"'] & A \otimes Q \end{tikzcd}$$

(P_2)

we have $\pi_1'\varphi = id_A\otimes\pi_i$ for $i = 1,2$. We shall call this mapping *the φ corresponding to the subpullback diagram* $P(M,N,f,g,Q)$. It can be seen that the mapping φ in diagram (P_2) is given by

$$\varphi(a\otimes (m,n)) = (a \otimes m, a \otimes n)$$

for all $a \in A_S$ and $(m,n) \in {}_SP$.

we define po-surjectivity and weak po-surjectivity of φ corresponding to the subpullback diagram $P(M,N,f,g,Q)$ as

$(\forall a,a' \in A_S)(\forall m \in_S M)(\forall n \in_S N)[a\otimes f(m) \leq a'\otimes g(n) \Rightarrow$
$(\exists a'' \in A_S)(\exists m' \in_S M)(\exists n' \in_S N)(f(m') \leq g(n'),$
$a\otimes m = a''\otimes m',\ a''\otimes n' \leq a'\otimes n)]$
and
$(\forall a,a' \in A_S)(\forall m \in_S M)(\forall n \in_S N)[a\otimes f(m) \leq a'\otimes g(n) \Rightarrow$
$(\exists a'' \in A_S)(\exists m' \in_S M)(\exists n' \in_S N)\ (f(m') \leq g(n'),$
$a\otimes m \leq a''\otimes m',\ a''\otimes n' \leq a'\otimes n)]$,
respectively.

Obviously, surjectivity of φ $\Rightarrow$ po-surjectivity of φ $\Rightarrow$ weak po-surjectivity of φ.

We say that an S-poset A_S satisfies *Condition* (P_{sw}) if for all $a,a' \in A_S$ and $s,s' \in S$, $as \leq a's'$ implies $a = a''u$, $a''v \leq a'$, for some $a'' \in A_S$ and $u,v \in S$, such that $us \leq vs'$. An S-poset A_S satisfies *Condition* $(WP)_{sw}$ if for all elements $s,t \in S$, all homomorphisms $f\colon {}_S(Ss \cup St) \longrightarrow {}_SS$ and all $a,a' \in A_S$, if $af(s) \leq a'f(t)$, then there exist $a'' \in A_S$, $u,v \in S$, $s',t' \in \{s,t\}$, such that $f(us') \leq f(vt')$ and $a\otimes s = a''\otimes us'$, $a''\otimes vt' \leq a'\otimes t$ in $A_S\otimes {}_S(Ss \cup St)$. An S-poset A_S satisfies *Condition* $(PWP)_{sw}$ if for all $a,a' \in A_S$ and $s \in S$, $as \leq a's$ implies $a = a''u$, $a''v \leq a'$, for some $a'' \in A_S$ and $u,v \in S$, such that $us \leq vs$.

For a pomonoid S the following relations exist among flatness properties of an S-poset.

free ⇒projective

$\Rightarrow (P) \Rightarrow (WP) \Rightarrow (PWP)$

$\Downarrow \quad \Downarrow \quad \Downarrow$

$(P_{sw}) \Rightarrow (WP)_{sw} \Rightarrow (PWP)_{sw}$

$\Downarrow \quad \Downarrow \quad \Downarrow$

$(P_{w}) \Rightarrow (WP)_{w} \Rightarrow (PWP)_{w}$

$\Downarrow \quad \Downarrow \quad \Downarrow$

$po-flat \Rightarrow$ w. po − f. $\Rightarrow p.w.po-f. \Rightarrow po-t.f.$

$\Downarrow \quad \Downarrow \quad \Downarrow \quad (incomparable)$

$flat \Rightarrow$ w.f. $\Rightarrow p.w.f. \Rightarrow t.f.$

(P_1)

In this article for a pomonoid S, using the information from [5, 4] we describe po-flatness and Conditions (P_{sw}), $(WP)_{sw}$ and $(PWP)_{sw}$ of S-posets according to po-surjectivity of φ corresponding to certain subpullback diagrams. We also describe Conditions (P_w), $(WP)_w$ and $(PWP)_w$ introduced in [7, 4] by weak po-surjectivity of φ corresponding to certain subpullback diagrams.

Results

1. Po-flatness of S-posets

In this section for a pomonoid S we give equivalences of po-flatness of $S-$posets by po-surjectivity of the mapping φ corresponding to certain subpullback diagrams.

Recall from [7] that an S-poset A_S is called *po-flat* (in Pos- S) if and only if, for all embeddings ${}_SB \longrightarrow {}_SC$ in the category of left S-posets, the induced order-preserving map $A_S\otimes {}_SB \longrightarrow A_S\otimes {}_SC$ is embedding, that is, $a\otimes b \leq a'\otimes b'$ in $A_S\otimes {}_SC$ implies $a\otimes b \leq a'\otimes b'$ in $A_S\otimes {}_SB$. The S-poset A_S is called *(principally) weakly po-flat* if the functor $A_S\otimes -$ preserves embeddings of (principal) left ideals of monoid S into S, that is, for all (principal) left ideals I of monoid S, $a, a' \in A_S$, $s, s' \in I$, $a\otimes s \leq a'\otimes s'$ in $A_S\otimes {}_SS$ implies $a \otimes s \leq a'\otimes s'$ in $A_S\otimes {}_SI$. For more po-flatness properties we refer the reader to [8].

Lemma 1.1. [7, Lemma 3.5] An S-poset A_S is po-flat if and only if for every left S-poset ${}_SB$, any $a, a' \in A_S$ and any $b, b' \in {}_SB$, $a\otimes b \leq a'\otimes b'$ in $A_S\otimes {}_SB$ implies $a\otimes b \leq a'\otimes b'$ in $A_S\otimes {}_S(Sb \cup Sb')$.

Theorem 1.2. An S-poset A_S is po-flat if and only if the corresponding φ is po-surjective for every subpullback diagram $P(M, M, \iota, \iota, Q)$, where $\iota\colon {}_SM \longrightarrow {}_SQ$ is an embedding of left S-posets.

Proof. Suppose A_S is po-flat, $\iota\colon {}_SM \longrightarrow {}_SQ$ is an embedding of left S-posets and let $a \otimes \iota(m) \leq a'\otimes \iota(m')$ in $A_S \otimes {}_SQ$, for $a, a' \in A_S$ and $m, m' \in {}_SM$. Since A_S is po-flat, $id_A \otimes \iota$ is embedding, and so $a\otimes m \leq a'\otimes m'$ in $A_S\otimes {}_SM$. Since $\iota(m) \leq \iota(m)$, $a\otimes m = a\otimes m$ and $a\otimes m \leq a'\otimes m'$, the corresponding φ is po-surjective for the subpullback diagram $P(M, M, \iota, \iota, Q)$.

Conversely, suppose the corresponding φ is po-surjective for every subpullback diagram $P(M, M, \iota, \iota, Q)$, where $\iota\colon {}_SM \longrightarrow {}_SQ$ is an embedding of left S-posets. Let ${}_SM$ be an S-poset, $a, a' \in A_S$, $m, m' \in {}_SM$ and $a\otimes m \leq a'\otimes m'$ in $A_S\otimes {}_SM$. If $i\colon {}_S(Sm \cup Sm') \longrightarrow {}_SM$ is the inclusion map, then i is an embedding (since ${}_S(Sm \cup Sm')$ is a S-subposet of ${}_SM$). Also $a\otimes i(m) = a\otimes m \leq a'\otimes m' = a'\otimes i(m')$. By po-surjectivity of φ for the subpullback diagram $P(Sm \cup Sm', Sm \cup Sm', i, i, M)$, there exist $a'' \in A_S$ and $t, t' \in {}_S(Sm \cup Sm')$ such that $i(t) \leq i(t')$, $a\otimes m = a''\otimes t$ and $a''\otimes t' \leq a'\otimes m'$ in $A_S\otimes {}_S(Sm \cup Sm')$. Thus $t \leq t'$, and so $a \otimes m = a''\otimes t \leq a''\otimes t' \leq a'\otimes m'$ in $A_S\otimes {}_S(Sm \cup Sm')$. That is, A_S is po-flat as required.

Similarly, we can prove the following theorems.

Theorem 1.3. An S-poset A_S is weakly po-flat if and only if the corresponding φ is po-surjective for every subpullback diagram $P(I, I, \iota, \iota, S)$, where I is a left ideal of S and $\iota\colon {}_SI \longrightarrow {}_SS$ is an embedding of left S-posets.

Theorem 1.4. An S-poset A_S is principally weakly po-flat if and only if the corresponding φ is po-surjective for every subpullback diagram $(Ss, Ss, \iota, \iota, S)$, where $s \in S$ and $\iota\colon {}_S(Ss) \longrightarrow {}_SS$ is an embedding of left S-posets.

We recall from [1] that an element c of a pomonoid S is called *right po-cancellable* if $sc \leq s'c$ implies $s \leq s'$, for all $s, s' \in S$. An S-poset A_S is called *po-torsion free* if $ac \leq a'c$ implies $a \leq a'$, whenever $a, a' \in A$ and c is a right po-cancellable element of S.

Theorem 1.5. An S-poset A_S is po-torsion free if and only if the corresponding φ is po-surjective for every subpullback diagram $P(S, S, \iota, \iota, S)$, where $\iota\colon {}_SS \longrightarrow {}_SS$ is an embedding of left S-posets.

Proof. Suppose A_S is po-torsion free and let $\iota\colon {}_SS \longrightarrow {}_SS$ be an embedding of left S-posets. Then $\iota(1)$ is a right po-cancellable element of S, for if $s\iota(1) \leq s'\iota(1)$, for $s, s' \in S$, then $\iota(s) \leq \iota(s')$, and so $s \leq s'$. Suppose $a \otimes \iota(t) \leq a' \otimes \iota(t')$ in $A_S \otimes {}_SS$, for $a, a' \in A_S$ and $t, t' \in S$. Then $a\iota(t) \leq a'\iota(t')$, and so $at\iota(1) \leq a't'\iota(1)$. Since $\iota(1)$ is a right po-cancellable element of S and A_S is po-torsion free, we have $at \leq a't'$, and so $a \otimes t \leq a' \otimes t'$ in $A_S \otimes {}_SS$. Since $\iota(t) \leq \iota(t)$, $a \otimes t = a \otimes t$ and $a \otimes t \leq a' \otimes t'$, φ is po-surjective for the subpullback diagram $P(S, S, \iota, \iota, S)$.

Conversely, suppose the corresponding φ is po-surjective for every subpullback diagram $P(S, S, \iota, \iota, S)$ where $\iota\colon {}_SS \longrightarrow {}_SS$ is an embedding of left S-posets. Let c be a right po-cancellable element of S and let $ac \leq$

$a'c$, for $a, a' \in A_S$. If $i\colon {}_SS \longrightarrow {}_SS$ is defined by $i(x) = xc$, for every $x \in S$, then i is an embedding, also $a \otimes i(1) = a \otimes c \leq a' \otimes c = a' \otimes i(1)$ in $A_S \otimes {}_SS$. By po-surjectivity of φ for the subpullback diagram $P(S, S, i, i, S)$, there exist $a" \in A_S$ and $u, v \in S$, such that $i(u) \leq i(v)$, $a \otimes 1 = a" \otimes u$ and $a" \otimes v \leq a' \otimes 1$ in $A_S \otimes {}_SS$. Thus $uc \leq vc$, and so $u \leq v$. Also, $a = a"u$ and $a"v \leq a'$. Hence $a = a"u \leq a"v \leq a'$, and so A_S is po-torsion free.

2. S-Posets satisfying Condition $((P_w))$ (P_{sw})

In this section we give some equivalent conditions on an S-poset A_S satisfying Condition $((P_w))$ (P_{sw}) according to (weak) po-surjectivity of φ corresponding to certain subpullback diagrams. Note that for S-posets A_S and ${}_SB$, $a \otimes b \leq a' \otimes b'$ in $A_S \otimes {}_SB$, for $a, a' \in A_S$ and $b, b' \in {}_SB$ if and only if there exist $a_1, a_2, \dots, a_n \in A_S$, $b_2, \dots, b_n \in {}_SB$, $s_1, t_1, \dots, s_n, t_n \in S$, such that

$$\begin{array}{ll} a \leq a_1 s_1 & \\ a_1 t_1 \leq a_2 s_2 & s_1 b \leq t_1 b_2 \\ \cdots & \cdots \\ a_n t_n \leq a' & s_n b_n \leq t_n b'. \end{array}$$

Lemma 2.1. An S-poset A_S satisfies Condition (P_{sw}) if and only if for all ${}_SB$ and all $a, a' \in A_S$, $b, b' \in {}_SB$, $a \otimes b \leq a' \otimes b'$ in $A_S \otimes {}_SB$ implies the existence of $a" \in A_S$ and $u, v \in S$, such that $a = a"u$, $a"v \leq a'$ and $ub \leq vb'$.

Proof. Suppose A_S satisfies Condition (P_{sw}) and let $a \otimes b \leq a' \otimes b'$ in $A_S \otimes {}_SB$, for $a, a' \in A_S$ and $b, b' \in {}_SB$. Then there exist $a_1, a_2, \dots, a_n \in A_S$, $b_2, \dots, b_n \in {}_SB$, $s_1, t_1, \dots, s_n, t_n \in S$, such that

$$\begin{array}{ll} a \leq a_1 s_1 & \\ a_1 t_1 \leq a_2 s_2 & s_1 b \leq t_1 b_2 \\ \cdots & \cdots \\ a_n t_n \leq a' & s_n b_n \leq t_n b'. \end{array}$$

Since $a_1 t_1 \leq a_2 s_2$ and A_S satisfies Condition (P_{sw}), there exist $c \in A_S$ and $u', v' \in S$, such that $a_1 = cu'$, $cv' \leq a_2$ and $u't_1 \leq v's_2$. Then

$$(u's_1)b = u'(s_1 b) \leq u'(t_1 b_2) = (u't_1)b_2 \leq (v's_2)b_2 = v'(s_2 b_2) \leq v'(t_2 b_3),$$

and so we have the following S-tossing of length $n-1$

$$\begin{array}{ll} a \leq c(u's_1) & \\ c(v't_2) \leq a_3 s_3 & (u's_1)b \leq v'(t_2 b_3) \\ \cdots & \cdots \\ a_n t_n \leq a' & s_n b_n \leq t_n b'. \end{array}$$

Continuing this procedure we have:

$$\begin{array}{ll} a \leq a_1 s_1 & \\ a_1 t_1 \leq a' & s_1 b \leq t_1 b'. \end{array}$$

Again $a \leq a_1 s_1$ implies the existence of $\bar{a} \in A_S$ and $u_1, v_1 \in S$, such that $a = \bar{a}u_1$, $\bar{a}v_1 \leq a_1$ and $u_1 \leq v_1 s_1$. Since $a_1 t_1 \leq a'$ and $\bar{a}v_1 \leq a_1$, we have $\bar{a}(v_1 t_1) \leq a'$, and so there exist $a" \in A_S$ and $u_2, v \in S$, such that $\bar{a} = a"u_2$, $a''v \leq a'$ and $u_2(v_1 t_1) \leq v$. If $u = u_2 u_1$ then $a = \bar{a}u_1 = a"(u_2 u_1) = a"u$, $a"v \leq a'$ and

$$ub = (u_2 u_1)b \leq u_2(v_1 s_1)b = u_2 v_1(s_1 b) \leq u_2 v_1(t_1 b') \leq vb'.$$

The converse is obvious.

Theorem 2.2. For an S-poset A_S the following assertions are equivalent:

(1) the corresponding φ is po-surjective for every subpullback diagram $P(M, N, f, g, Q)$;

(2) the corresponding φ is po-surjective for every subpullback diagram $P(M, M, f, g, Q)$;

(3) the corresponding φ is po-surjective for every subpullback diagram $P(I, I, f, g, S)$, where I is a left ideal of S;

(4) the corresponding φ is po-surjective for every subpullback diagram $P(Ss, Ss, f, g, S)$, $s \in S$;

(5) the corresponding φ is po-surjective for every subpullback diagram $P(S, S, f, g, S)$;

(6) the corresponding φ is po-surjective for every subpullback diagram $P(M, M, f, f, Q)$;

(7) A_S satisfies Condition (P_{sw}).

Proof. Implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) and (2) $\Longrightarrow$ (6) are obvious.

(5) $\Longrightarrow$ (7). Suppose $as \leq a's'$, for $a, a' \in A_S$ and $s, s' \in S$. Then $a \otimes s \leq a' \otimes s'$. If $f(x) = xs$ and $g(x) = xs'$, for $x \in S$, then $a \otimes f(1) = a \otimes s \leq a' \otimes s' = a' \otimes g(1)$. By assumption there exist $a" \in A_S$ and $u, v \in S$, such that $f(u) \leq g(v)$, $a \otimes 1 = a" \otimes u$ and $a" \otimes v \leq a' \otimes 1$. Thus $us \leq vs'$, $a = a"u$ and $a"v \leq a'$, and so A_S satisfies Condition (P_{sw}) as required.

(6) $\Longrightarrow$ (7). Suppose $as \leq a's'$, for $a, a' \in A_S$, $s, s' \in S$ and let ${}_SF = S \times \{x, y\}$, where $t(s, z) = (ts, z)$ and $(s, z) \leq (t, w)$ if and only if $s \leq t$ and $z = w$, for $s, t \in S$ and $z, w \in \{x, y\}$. Define $f\colon {}_SF \longrightarrow S$, as $f(1, x) = s$ and $f(1, y) = s'$. Since $as \leq a's'$, we have $a \otimes s \leq a' \otimes s'$, and so $a \otimes f(1, x) = a \otimes s \leq a' \otimes s' = a' \otimes f(1, y)$. By assumption there exist $a" \in A_S$ and $(u, i), (v, j) \in {}_SF$, such that $f(u, i) \leq f(v, j)$, $a \otimes (1, x) = a'' \otimes (u, i)$ and $" \otimes (v, j) \leq a' \otimes (1, y)$.

Then $a \otimes (1, x) = a" \otimes (u, i)$ implies that there exist $a_1, \ldots, a_n, c_1, \ldots, c_m \in A_S$, $s_1, t_1, \ldots, s_n, t_n, u_1, v_1, \ldots, u_m, v_m \in S$ and $(b_2, i_2), \ldots, (b_n, i_n), (d_2, j_2), \ldots, (d_m, j_m) \in {}_SF$, such that

$$\begin{array}{ll} a \le a_1 s_1 & \\ a_1 t_1 \le a_2 s_2 & s_1(1,x) \le t_1(b_2, i_2) \\ a_2 t_2 \le a_3 s_3 & s_2(b_2, i_2) \le t_2(b_3, i_3) \\ \ldots & \ldots \\ a_n t_n \le a" & s_n(b_n, i_n) \le t_n(u, i), \end{array}$$

$$\begin{array}{ll} a" \le c_1 u_1 & \\ c_1 v_1 \le c_2 u_2 & u_1(u, i) \le v_1(d_2, j_2) \\ c_2 v_2 \le c_3 u_3 & u_2(d_2, j_2) \le v_2(d_3, j_3) \\ \ldots & \ldots \\ c_m v_m \le a & u_m(d_m, j_m) \le v_m(1, x). \end{array}$$

Thus $i_2 = i_3 = \ldots = i_n = x$, and so $i = x$. Also $j_2 = j_3 = \ldots = j_m = x$, and so $a \otimes 1 = a" \otimes u$, that is, $a = a"u$. Similarly, $a" \otimes (v, j) \le a' \otimes (1, y)$ implies that $j = y$ and $a"v \le a'$. Since $f(u, i) \le f(v, j)$, we have $us \le vs'$, and so A_S satisfies Condition (P_{sw}) as required.

(7) $\Longrightarrow$ (1). Suppose $a \otimes f(m) \le a' \otimes g(n)$, for $a, a' \in A_S$, $m \in {}_SM$ and $n \in {}_SN$. By Lemma 2.1 there exist $a" \in A_S$ and $u, v \in S$ such that $a = a" u$, $a" v \le a'$ and $uf(m) \le vg(n)$. Thus $f(um) \le g(vn)$. If $m' = um$ and $n' = vn$, then $f(m') \le g(n')$, $a \otimes m = a''u \otimes m = a'' \otimes um = a'' \otimes m'$ and $a'' \otimes n' = a'' \otimes vn = a'' v \otimes n \le a' \otimes n$. Thus the corresponding φ is po-surjective for every subpullback diagram $P(M, N, f, g, Q)$ as required.

Recall from [7] that an S-poset A_S satisfies Condition (P_w), if $as \le a't$, for $a, a' \in A_S$ and $s, s' \in S$ implies that there exist $a" \in A_S$ and $u, v \in S$, such that $a \le a''u$, $a''v \le a'$ and $us \le vt$. Using an argument similar to that of the proof of Lemma 2.1 we have the following lemma.

Lemma 2.3. An S-poset A_S satisfies Condition (P_w) if and only if for all ${}_SB$ and all $a, a' \in A_S$, $b, b' \in {}_SB$, $a \otimes b \le a' \otimes b'$ in $A_S \otimes {}_SB$ implies the existence of $a'' \in A_S$ and $u, v \in S$, such that $a \le a''u$, $a''v \le a'$ and $ub \le vb'$.

Using Lemma 2.3 and an argument similar to that of the proof of Theorem 2.2 we have the following theorem.

Theorem 2.4. For an S-poset A_S the following assertions are equivalent:

(1) the corresponding φ is weakly po-surjective for every subpullback diagram $P(M, N, f, g, Q)$;

(2) the corresponding φ is weakly po-surjective for every subpullback diagram $P(M, M, f, g, Q)$;

(3) the corresponding φ is weakly po-surjective for every subpullback diagram $P(I, I, f, g, S)$, where I is a left ideal of S;

(4) the corresponding φ is weakly po-surjective for every subpullback diagram $P(Ss, Ss, f, g, S)$, $s \in S$;

(5) the corresponding φ is weakly po-surjective for every subpullback diagram $P(S, S, f, g, S)$;

(6) the corresponding φ is weakly po-surjective for every subpullback diagram $P(M, M, f, f, Q)$;

(7) A_S satisfies Condition (P_w).

3. *S*-posets satisfying Condition $((WP)_w)$ $(WP)_{sw}$

In this section first of all we give some equivalent conditions on an S-poset A_S satisfying Condition $((WP)_w)$ $(WP)_{sw}$ according to (weak) po-surjectivity of φ corresponding to certain subpullback diagram. Then we give a necessary and a sufficient condition for an S-poset to satisfy Condition $(WP)_{sw}$.

Theorem 3.1. An S-poset A_S satisfies Condition $(WP)_{sw}$ if and only if the corresponding φ is po-surjective for every subpullback diagram $P(I, I, f, f, S)$, where I is a left ideal of S.

Proof. Suppose A_S satisfies Condition $(WP)_{sw}$, $f : {}_SI \longrightarrow {}_SS$ be an S-poset morphism and let $a \otimes f(s) \le a' \otimes f(t)$ in $A_S \otimes {}_SS$, for $a, a' \in A_S$ and $s, t \in I$. Then, $af(s) \le a'f(t)$. If $J = Ss \cup St \subseteq I$ and $h = f|_J$ then $ah(s) \le a'h(t)$, and so by assumption there exist $a" \in A_S$, $u, v \in S$, and $s', t' \in \{s, t\}$, such that $h(us') \le h(vt')$, $a \otimes s = a" \otimes us'$ and $a'' \otimes vt' \le a' \otimes t$ in $A_S \otimes {}_SJ$. Clearly, $us', vt' \in I$, $f(us') = h(us') \le h(vt') = f(vt')$ and $J \subseteq I$ implies that $a \otimes s = a'' \otimes us'$ and $a'' \otimes vt' \le a' \otimes t$ in $A_S \otimes {}_SI$. Thus the corresponding φ is po-surjective for every subpullback diagram $P(I, I, f, f, S)$.

Conversely, suppose the corresponding φ is po-surjective for every subpullback diagram $P(I, I, f, f, S)$, $f : {}_S(Ss \cup St) \longrightarrow {}_SS$ is an S-poset morphism, for $s, t \in S$ and let $af(s) \le a' f(t)$, for $a, a' \in A_S$. Then $a \otimes f(s) \le a' \otimes f(t)$ in $A_S \otimes {}_SS$. By po-surjectivity of φ for the subpullback diagram $P(Ss \cup St, Ss \cup St, f, f, S)$, there exist $a" \in A_S$, $u, v \in S$ and $s', t' \in \{s, t\}$, such that $f(us') \le f(vt')$, $a \otimes s = a" \otimes us'$ and $a" \otimes vt' \le a' \otimes t$ in $A_S \otimes {}_S(Ss \cup St)$. Thus A_S satisfies Condition $(WP)_{sw}$ as required.

An S-poset A_S satisfies *Condition* $(WP)_w$ if for all $s, t \in S$, all homomorphisms $f : {}_S(Ss \cup St) \longrightarrow {}_SS$ and all $a, a' \in A_S$, if $af(s) \le a' f(t)$, then there exist $a" \in A_S$, $u, v \in S$, $s', t' \in \{s, t\}$ such that $f(us') \le f(vt')$ and $a \otimes s \le a" \otimes us'$, $a" \otimes vt' \le a' \otimes t$ in $A_S \otimes {}_S(Ss \cup St)$ (see [4]). Using an argument similar to that of the proof of Theorem 3.1 we can show the following theorem.

Theorem 3.2. An S-poset A_S satisfies Condition $(WP)_w$ if and only if the corresponding φ is weakly po-surjective for every subpullback diagram $P\ (I, I, f, f, S)$, where I is a left ideal of S.

Theorem 3.3. An S-poset A_S satisfies Condition $(WP)_{sw}$ if and only if for all $s, t \in S$, all homomorphisms $f: {}_S(Ss \cup St) \longrightarrow {}_SS$ and all $a, a' \in A_S$, if $af\ (s) \leq a'f(t)$, then there exist $a'', a_1, a_2, a'_1, a'_2, a'_3 \in A_S$, $u, v, p_1, p_2, q_1, q_2, p'_1, p'_2, p'_3, q'_1, q'_2, q'_3 \in S$, such that either $f(ut) \leq f(vt)$ and

$$\begin{array}{ll} a''v \leq a', & \\ a = a_1p_1 & \\ a_1q_1 \leq a_2p_2 & p_1s \leq q_1s \\ a_2q_2 = a''u & p_2s \leq q_2t, \end{array}$$

$$\begin{array}{ll} a''u = a'_1p'_1 & \\ a'_1q'_1 \leq a'_2p'_2 & p'_1t \leq q'_1t \\ a'_2q'_2 = a'_3p'_3 & p'_2t \leq q'_2s \\ a'_3q'_3 \leq a & p'_3s \leq q'_3s, \end{array}$$

or $f(us) \leq f(vs)$ and

$$a = a''u,$$

$$\begin{array}{ll} a''v \leq a_1p_1 & \\ a_1q_1 = a_2p_2 & p_1s \leq q_1t \\ a_2q_2 \leq a' & p_2t \leq q_2t, \end{array}$$

or $f(us) \leq f(vt)$ and

$$\begin{array}{ll} a = a_1p_1 & \\ a_1q_1 \leq a''u & p_1s \leq q_1s, \end{array}$$

$$\begin{array}{ll} a''u = a'_1p'_1 & \\ a'_1q'_1 \leq a & p'_1s \leq q'_1s, \end{array}$$

$$\begin{array}{ll} a''v = a_2p_2 & \\ a_2q_2 \leq a' & p_2t \leq q_2t. \end{array}$$

Proof. Necessity. Let $f: {}_S(Ss \cup St) \longrightarrow S$ be a homomorphism, for $s, t \in S$ and $af(s) \leq a'f(t)$, for $a, a' \in A_S$. By assumption there exist $a'' \in A_S$, $u, v \in S$ and $s', t' \in \{s, t\}$, such that $f(us') \leq f(vt')$, $a \otimes s = a'' \otimes us'$ and $a'' \otimes vt' \leq a' \otimes t$. Then $a \otimes s = a''u \otimes s'$ and $a''v \otimes t' \leq a' \otimes t$, and so there exist $a_1, \ldots, a_n, c_1, \ldots, c_{n'}, c'_1, \ldots, c'_{m'} \in A_S$, $p_1, q_1, \ldots, p_n, q_n, u_1, v_1, \ldots, u_{n'}, v_{n'}, u'_1, v'_1, \ldots, u'_{m'}, v'_{m'} \in S$ and $z_2, \ldots, z_n, d_2, \ldots, d_{n'}, d'_2, \ldots, d'_{m'} \in \{s, t\}$, such that

$$\begin{array}{ll} a \leq a_1p_1 & \\ a_1q_1 \leq a_2p_2 & p_1s \leq q_1z_2 \\ \cdots & \cdots \\ a_nq_n \leq a''u & p_nz_n \leq q_ns', \end{array}$$

$$\begin{array}{ll} a''u \leq c_1u_1 & \\ c_1v_1 \leq c_2u_2 & u_1s' \leq v_1d_2 \\ \cdots & \cdots \\ c_{n'}v_{n'} \leq a & u_{n'}d_{n'} \leq v_{n'}s, \end{array}$$

$$\begin{array}{ll} a''v \leq c'_1u'_1 & \\ c'_1v'_1 \leq c'_2u'_2 & u'_1t' \leq v'_1d'_2 \\ \cdots & \cdots \\ c'_{m'}v'_{m'} \leq a' & u'_{m'}d'_{m'} \leq v'_{m'}t. \end{array}$$

If $z_1 = s$, $z_{n+1} = s'$, $d_1 = s'$, $d_{n'+1} = s$, $d'_1 = t'$ and $d'_{m'+1} = t$, then there are three cases as follows:

Case 1. $s' = t$. In this case there exist $k \in \{1,\ldots,n\}$ and $k' \in \{1, \ldots, n'\}$, such that $z_k = s$, $z_{k+1} = z_{k+2} = \ldots = z_{n+1} = s' = t$, $d_{k'+1} = s$, $d_{k'} = d_{k'-1} = \cdots = d_1 = s' = t$. Then $a_kq_kf(t) \leq a_{k+1}p_{k+1}f(t) = a_{k+1}f(p_{k+1}z_{k+1}) \leq a_{k+1}f(q_{k+1}z_{k+2}) \leq \cdots \leq a_nf(q_nz_{n+1}) = a_nq_nf(z_{n+1}) \leq a''uf(s') = a''f(us') \leq a''f(vt') = a''vf(t') \leq c'_1u'_1f(t') = c'_1f(u'_1t') \leq c'_1f(v'_1d'_2) \leq \cdots \leq c'_{m'}f(v'_{m'}d'_{m'+1}) = c'_{m'}v'_{m'}f(t) \leq a'f(t)$.

Since A_S satisfies Condition $(PWP)_{sw}$, there exist $d_1 \in A_S$ and $x_1, x_2 \in S$, such that $a_kq_k = d_1x_1$, $d_1x_2 \leq a'$ and $x_1f(t) \leq x_2f(t)$. Since also

$$as \leq a_1p_1s \leq a_1q_1z_2 \leq a_2p_2z_2 \leq \cdots \leq a_kp_kz_k = a_kp_ks$$

there exist $d_2 \in A_S$ and $y_1, y_2 \in S$, such that $a = d_2y_1$, $d_2y_2 \leq a_kp_k$ and $y_1s \leq y_2s$. Thus $f(x_1t) \leq f(x_2t)$ and

$$d_1x_2 \leq a',$$

$$\begin{array}{ll} a = d_2y_1 & \\ d_2y_2 \leq a_kp_k & y_1s \leq y_2s \\ a_kq_k = d_1x_1 & p_ks \leq q_kt. \end{array}$$

Also,

$$\begin{aligned} d_1x_1t = a_kq_kt &\leq a_{k+1}p_{k+1}z_{k+1} \leq a_{k+1}q_{k+1}z_{k+2} \leq \cdots \\ &\leq a_nq_nz_{n+1} \leq a''us' \leq c_1u_1s' \\ &\leq c_1v_1d_2 \leq c_2u_2d_2 \leq \cdots \leq c_{k'}u_{k'}d_{k'} \\ &= c_{k'}u_{k'}t \end{aligned}$$

and since A_S satisfies Condition $(PWP)_{sw}$, there exist $d'_1 \in A_S$ and $x'_1, x'_2 \in S$, such that $d_1x_1 = d'_1x'_1$, $d'_1x'_2 \leq c_{k'}u_{k'}$ and $x'_1t \leq x'_2t$. On the other hand,

$$\begin{aligned} c_{k'}v_{k'}s \leq c_{k'+1}u_{k'+1}s &= c_{k'+1}u_{k'+1}d_{k'+1} \\ &\leq c_{k'+1}v_{k'+1}d_{k'+2} \\ &\leq c_{k'+2}u_{k'+2}d_{k'+2} \leq \cdots \leq c_{n'}u_{n'}d_{n'} \\ &\leq c_{n'}v_{n'}s \leq as \end{aligned}$$

and so there exist $d'_2 \in A_S$ and $y'_1, y'_2 \in S$, such that $c_{k'}v_{k'} = d'_2y'_1$, $d'_2y'_2 \leq a$ and $y'_1s \leq y'_2s$.

Thus

$$d_1x_1 = d'_1x'_1$$
$$d'_1x'_2 \leq c_{k'}u_{k'} \qquad x'_1t \leq x'_2t$$
$$c_{k'}v_{k'} = d'_2y'_1 \qquad u_{k'}t \leq v_{k'}s$$
$$d'_2y'_2 \leq a \qquad y'_1s \leq y'_2s,$$

as required.

Case 2. $t' = s$. In this case there exists $k' \in \{1,...,m'\}$, such that $d'_{k'+1} = t$ and $d'_{k'} = d'_{k'-1} = \cdots = d'_1 = s$. Thus

$$\begin{aligned} af(s) &\leq a_1p_1f(s) = a_1f(p_1s) \leq a_1f(q_1z_2) \leq \cdots \\ &\leq a_nf(q_nz_{n+1}) = a_nq_nf(s') \\ &\leq a''uf(s') = a''f(us') \leq a''f(vt') \\ &= a''vf(t') \leq c'_1u'_1f(t') \\ &= c'_1f(u'_1t') \leq c'_1f(v'_1d'_2) \\ &= c'_1v'_1f(d'_2) \leq \cdots \\ &\leq c'_{k'-1}v'_{k'-1}f(d'_{k'}) \\ &\leq c'_{k'}u'_{k'}f(s) \end{aligned}$$

Since A_S satisfies Condition $(PWP)_{sw}$, there exist $b \in A_S$ and $w, w' \in S$, such that $a = bw$, $bw' \leq c'_{k'}u'_{k'}$ and $wf(s) \leq w'f(s)$. Since also
$c'_{k'}v'_{k'}t \leq c'_{k'+1}u'_{k'+1}t = c'_{k'+1}u'_{k'+1}d'_{k'+1} \leq c'_{k'+1}v'_{k'+1}d'_{k'+2} \leq \cdots \leq c'_{m'}v'_{m'}d'_{m'+1} \leq a't$,
there exist $b' \in A_S$ and $w_1, w'_1 \in S$, such that $c'_{k'}v'_{k'} = b'w_1$, $b'w'_1 \leq a'$ and $w_1t \leq w'_1t$. Thus $f(ws) \leq f(w's)$ and

$$a = bw,$$

$$bw' \leq c'_{k'}u'_{k'}$$
$$c'_{k'}v'_{k'} = b'w_1 \qquad u'_{k'}s \leq v'_{k'}t$$
$$b'w'_1 \leq a' \qquad w_1t \leq w'_1t,$$

as required.

Case 3. $s' = s$ and $t' = t$. Then $f(us) \leq f(vt)$, $a \otimes s = a''u \otimes s$ and $a''v\otimes t \leq a' \otimes t$. Since $as = a''us$ and A_S satisfies Condition $(PW P)_{sw}$, there exist $d_1, d_2 \in A_S$ and $x_1, x_2, y_1, y_2 \in S$, such that

$$a = d_1x_1$$
$$d_1y_1 \leq a''u \qquad x_1s \leq y_1s,$$

$$a''u = d_2x_2$$
$$d_2y_2 \leq a \qquad x_2s \leq y_2s.$$

Since $a''vt \leq a't$, there exist $d_3 \in A_S$ and $x_3, y_3 \in S$, such that

$$a''v = d_3x_3$$
$$d_3y_3 \leq a' \qquad x_3t \leq y_3t,$$

Sufficiency. Suppose $f\colon {}_S(Ss\cup St) \longrightarrow S$ is a homomorphism, for $s, t \in S$ and let $af(s) \leq a'f(t)$, for $a, a' \in A_S$. By assumption there are three cases as follows:

Case 1. $f(ut) \leq f(vt)$ and

$$a''v \leq a',$$
$$a = a_1p_1$$
$$a_1q_1 \leq a_2p_2 \qquad p_1s \leq q_1s$$
$$a_2q_2 = a''u \qquad p_2s \leq q_2t,$$

$$a''u = a'_1p'_1$$
$$a'_1q'_1 \leq a'_2p'_2 \qquad p'_1t \leq q'_1t$$
$$a'_2q'_2 = a'_3p'_3 \qquad p'_2t \leq q'_2s$$
$$a'_3q'_3 \leq a \qquad p'_3s \leq q'_3s.$$

Thus

$$\begin{aligned} a \otimes s = a_1p_1 \otimes s &= a_1 \otimes p_1s \leq a_1 \otimes q_1s \\ &= a_1q_1 \otimes s \leq a_2p_2\otimes s = a_2\otimes p_2s \\ &\leq a_2\otimes q_2t = a_2q_2\otimes t = a''u\otimes t \\ &= a''\otimes ut \end{aligned}$$

and
$a''\otimes ut = a''u\otimes t = a'_1p'_1\otimes t = a'_1\otimes p'_1t \leq a'_1\otimes q'_1t = a'_1q'_1\otimes t \leq a'_2p'_2\otimes t = a'_2\otimes p'_2t \leq a'_2 \otimes q'_2s = a'_2q'_2 \otimes s = a'_3p'_3 \otimes s = a'_3 \otimes p'_3s \leq a'_3\otimes q'_3s = a'_3q'_3\otimes s \leq a\otimes s$,

and hence $a\otimes s = a''\otimes ut$. Since $a''v \leq a'$, we have $a''v\otimes t \leq a'\otimes t$, and so $a''\otimes vt \leq a'\otimes t$. Thus A_S satisfies Condition $(WP)_{sw}$ as required.

Case 2. $f\,(us) \leq f\,(vs)$ and

$$a = a''u,$$
$$a''v \leq a_1p_1$$
$$a_1q_1 = a_2p_2 \qquad p_1s \leq q_1t$$
$$a_2q_2 \leq a' \qquad p_2t \leq q_2t.$$

Thus $a\otimes s = a''u \otimes s = a'' \otimes us$ and
$a''\otimes vs = a''v\otimes s \leq a_1p_1 \otimes s = a_1\otimes p_1s \leq a_1\otimes q_1t = a_1q_1\otimes t = a_2p_2\otimes t = a_2\otimes p_2t \leq a_2\otimes q_2t = a_2q_2\otimes t \leq a'\otimes t$,
as required.

Case 3. $f(us) \leq f(vt)$ and

$$a = a_1p_1$$
$$a_1q_1 \leq a''u \qquad p_1s \leq q_1s,$$

$$a''u = a'_1p'_1$$
$$a'_1q'_1 \leq a \qquad p'_1s \leq q'_1s,$$

$$a''v = a_2p_2$$
$$a_2q_2 \leq a' \qquad p_2t \leq q_2t.$$

Thus

$$a \otimes s = a_1 p_1 \otimes s = a_1 \otimes p_1 s \leq a_1 \otimes q_1 s$$
$$= a_1 q_1 \otimes s \leq a'' u \otimes s = a'' \otimes us$$

and

$a''\otimes us = a''u\otimes s = a'_1 p'_1 \otimes s = a'_1 \otimes p'_1 s \leq a'_1 \otimes q'_1 s = a'_1 q'_1 \otimes s \leq a \otimes s$,

and so $a\otimes s = a''\otimes us$. Also,

$a''\otimes vt = a''v\otimes t = a_2 p_2 \otimes t = a_2 \otimes p_2 t \leq a_2 \otimes q_2 t = a_2 q_2 \otimes t \leq a' \otimes t$,

as required.

4. *S*-Posets satisfying Condition $((PWP)_w)$ $(PWP)_{sw}$

In this section we give some equivalent conditions on an S-poset A_S satisfying Condition $((PWP)_w)$ $(PWP)_{sw}$ according to (weak) po-surjectivity of φ corresponding to certain subpullback diagrams.

Theorem 4.1. For an S-poset A_S the following assertions are equivalent:

(1) the corresponding φ is po-surjective for every subpullback diagram $P(Ss, Ss, f, f, S)$, where $s \in S$;

(2) the corresponding φ is po-surjective for every subpullback diagram $P(S, S, f, f, S)$;

(3) A_S satisfies Condition $(PWP)_{sw}$.

Proof. Implication (1) $\Longrightarrow$ (2) is clear.

(2) $\Longrightarrow$ (3). Suppose $as \leq a' s$, for $a, a' \in A_S$ and $s \in S$. Then, $a \otimes s \leq a' \otimes s$ in $A_S \otimes {}_S S$. If ρ_s: ${}_S S \longrightarrow {}_S S$ is defined as $\rho_s(x) = xs$, for $x \in S$, then $a \otimes \rho_s(1) = a \otimes s \leq a' \otimes s = a' \otimes \rho_s(1)$. By po-surjectivity of φ for the subpullback diagram $P(S, S, \rho_s, \rho_s, S)$, there exist $a'' \in A_S$ and $t, t' \in {}_S S$, such that $\rho_s(t) \leq \rho_s(t')$, $a \otimes 1 = a'' \otimes t$ and $a'' \otimes t' \leq a' \otimes 1$ in $A_S \otimes {}_S S$. Thus $ts \leq t' s$, $a = a''t$ and $a''t' \leq a'$, and so A_S satisfies Condition $(PWP)_{sw}$ as required.

(3) $\Longrightarrow$ (1). Suppose $f : {}_S Ss \longrightarrow {}_S S$ is an S-poset morphism and let $a \otimes f(ts) \leq a' \otimes f(t's)$ in $A_S \otimes {}_S S$, for $a, a' \in A_S$ and $s, t, t' \in S$. Then $(ts) \leq a' f(t's)$, and so $atf(s) \leq a' t' f(s)$. Since A_S satisfies Condition $(PWP)_{sw}$ there exist $a'' \in A_S$ and $u, v \in S$, such that $at = a''u$, $a''v \leq a't'$ and $uf(s) \leq vf(s)$. Thus $f(us) \leq f(vs)$. If $m = us$ and $m' = vs$, then

$$a \otimes ts = at \otimes s = a'' u \otimes s = a'' \otimes us = a'' \otimes m,$$

and

$$a'' \otimes m' = a'' \otimes vs = a'' v \otimes s \leq a' t' \otimes s = a' \otimes t's,$$

and so the corresponding φ is po-surjective for the subpullback diagram $P(Ss, Ss, f, f, S)$.

An S-poset A_S satisfies *Condition* $(PWP)_w$, if $at \leq a't$, for $a, a' \in A_S$ and $t \in S$, implies that there exist $a'' \in A_S$ and $u, v \in S$, such that $a \leq a''u$, $a''v \leq a'$ and $ut \leq vt$ (see [4]). Using an argument similar to that of the proof of Theorem 4.1 we can prove the following theorem.

Theorem 4.2. For an S-poset A_S the following assertions are equivalent:

(1) the corresponding φ is weakly po-surjective for every subpullback diagram $P(Ss, Ss, f, f, S)$, where $s \in S$;

(2) the corresponding φ is weakly po-surjective for every subpullback diagram $p(S, S, f, f, S)$;

(3) A_S satisfies Condition $(PWP)_w$.

5. Relations of the properties

In this section we demonstrate that with one possible exception ($(P_w) \Longrightarrow$ po-flat), all of the implications in Figure (P_1) are strict.

The proof of the following theorem is clear.

Theorem 5.1. Let G be an ordered group. Then all G-posets satisfy Condition (P_{sw}).

Theorem 5.2. [4, Theorem 4.6.] For any pomonoid S, there exists an S-poset that does not satisfy Condition (PWP).

Recall from [7], [8], [1], [4], that an S-poset A_S satisfies *Condition* (P), if $as \leq a't$, for $a, a' \in A_S$ and $s, t \in S$, implies that there exist $a'' \in A_S$ and $u, v \in S$, such that $a = a''u$, $a''v = a'$ and $us \leq vt$. An S-poset A_S is called *flat* (in SPOS) if and only if, for all embeddings ${}_S B \longrightarrow {}_S C$ in the category of left S-posets, the induced order-preserving map $A_S \otimes {}_S B \longrightarrow A_S \otimes {}_S C$ is injective. An S-poset A_S is called *weakly flat* if the induced morphism $A_S \otimes I \longrightarrow A_S \otimes S$ is injective for all embeddings of left ideals into ${}_S S$. A pomonoid S is called *weakly right* reversible in case $\cap$ $(St] \neq \emptyset$, for all $s, t \in S$ (if X is a subset of a poset P, $(X] := \{p \in P : p \leq x$ for some $x \in X\}$ is the down-set of X, or the order ideal generated by X). An S-poset A_S satisfies *Condition* (WP) if the corresponding φ is surjective for every subpullback diagram $P(I, I, f, f, S)$, where I is a left ideal of S.

Using [4, Theorem 6.2.] the following theorem is clear.

Theorem 5.3. For a pomonoid S the following assertions are equivalent:

(1) Θ_S satisfies Condition (P);
(2) Θ_S satisfies Condition (P_{sw});
(3) Θ_S satisfies Condition (P_w);
(4) Θ_S is po-flat;
(5) Θ_S is flat;
(6) Θ_S satisfies Condition (WP);
(7) Θ_S satisfies Condition $(WP)_{sw}$;
(8) Θ_S satisfies Condition $(WP)_w$;

(9) Θ_S is weakly po-flat;
(10) Θ_S is weakly flat;
(11) S is weakly right reversible.

The crucial things we need for distinctness of the properties are $(WP) \nRightarrow$flat (from Example 3 of [5]), $(PWP) \nRightarrow w.f$ (from Theorem 5.3), $(P_{sw}) \nRightarrow (PW\ P)$ (from Theorems 5.1 and 5.2) and $(P_w) \nRightarrow (PWP)_{sw}$ (from the following example).

Example 5.4. ($(P_w) \nRightarrow (PWP)_{sw}$) Let $S = \{1, x, e\}$ be the monoid with the following table

	1	x	e
1	1	x	e
x	x	x	x
e	e	e	e

and trivial order, and let $A = \{a, b, c\}$ be the set with the following order:

$$\leq_A := \{(a,a),(b,b),(c,c),(c,a),(a,b),(c,b)\}.$$

We give the covering relation " $\prec$ " and the figure of A as follows:

$$\prec_A = \{(c,a),(a,b)\}.$$

Define $x{*}s = \begin{cases} x & s = 1 \\ b & s \neq 1 \end{cases}$, for all $x \in A$ and $s \in S$. Indeed A_S is an S-poset. We claim that A_S does not satisfy Condition $(PWP)_{sw}$. Otherwise, $a * x \leq c * x$ implies that there exist $a'' \in A_S$ and $u, v \in S$, such that $a = a'' * u$, $a'' * v \leq c$ and $u \cdot x \leq v \cdot x$. Then $a = a'' * u$ implies that $a'' = a$ and $u = 1$, and so for every $v \in S$, $a'' * v \nleq c$, which is a contradiction. It can be shown that A_S satisfies Condition (P_w) (see [6] pages 121 and 122).

References

1. Bulman-Fleming S., Gilmour A., Gutermuth D. and Kilp M., Flatness properties of S-posets. *Comm. Algebra*. **34**: 1291-1317 (2006).
2. Bulman-Fleming S. and Laan V., Lazard's theorem for S-posets. *Math. Nachr*. **278**: 1743-1755 (2005).
3. Bulman-Fleming S. and Mahmoudi M., The category of S-posets. *Semigroup Forum*. **71**: 443-461 (2005).
4. Golchin A. and Rezaei P., Subpullbacks and flatness properties of S-posets. *Comm. Algebra*. **37**: 1995-2007 (2009).
5. Laan V., Pullbacks and flatness properties of acts I. *Comm. Algebra*. **29**: 829-850 (2001).
6. Nouri L., On flatness properties of product acts and posets. Ph.D Thesis, Zahedan (2014).
7. Shi X., Strongly flat and po-flat S-posets. *Comm. Algebra*. **33**: 4515-4531 (2005).
8. Shi X., Liu Z., Wang F. and Bulman-Fleming S., Indecomposable, projective, and flat S-posets. *Comm. Algebra*. **33**: 235-251 (2005).