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A Generalization of M-Small Modules

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Abstract

In this paper we introduce a generalization of M-small modules and discuss about the torsion theory cogenerated by this kind of modules in category $\sigma[M]$. We will use the structure of the radical of a module in $\sigma[M]$ and get some suitable results about this class of modules. Also the relation between injective hull in $\sigma[M]$ and this kind of modules will be investigated in this article. For a module $N \in \sigma[M]$ we show that N is *M-Rad* if and only if $N \subseteq Rad(\hat{N})$; where $\hat{N}$ is the M-injective hull of N. We will show that for a $\sigma-$cohereditary module M, *R[M]* is closed under extension. Let M be a module and $N \in \sigma[M]$, the torsion theory cogenerated by $R[M]$ is the reject of $R[M]$ in N, defined as $Re_{R[M]}(N) = \bigcap\{X \leq N \mid \frac{N}{X} \text{ is } M-Rad\}$. In this paper we study about the property of this torsion theory. We show that $N = Re_{R[M]}(N)$ if and only if for every nonzero homomorphism $f: N \to K$ in $\sigma[M]$, $Im(f) \not\subset Rad(K)$. Another attractive result is $N = Re_{R[M]}(N)$ if and only if $\Delta(N, A) = 0$, for all $A \in \sigma[M]$. For a module $N \in \sigma[M]$ we show that if $\frac{L}{K} \subseteq Rad(\frac{N}{K})$ for some $K \leq L \leq N$, then the inclusion $K \subseteq L$ is M-coRad and also if $N \in R[M]^{\circ}$, then for every submodule L of N and M-coRad inclusion $K \subset L$, we have $\frac{L}{K} \subseteq Rad(\frac{N}{K})$. Finally for a pseudo projective module M we show that every $N \in \sigma[M]$ with $Hom(M, N) = 0$ is *M-Rad* and if moreover $M \in R[M]^{\circ}$, then $R[M] = \{N \in \sigma[M] \mid Hom(M, N) = 0\}$.

Keywords: *M*-small module, *M-Rad* modules, Torsion theory cogenerated by *M-Rad* modules.

Introduction

Throughout this article all rings are associative with identity and all modules are unitary right R-modules except unless otherwise specified herein. We refer for basic notations to [2], [5], [7] and [13].

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The class of *M*-small modules and some generalizations of such modules are studied by some authors. Y. Talebi and N. Vanaja in [9], proceeded to investigate the *M*-small modules and torsion theory cogenerated by this kind of modules. As a generalization of *M*-small modules, Ozcan in [8] defined δ-*M*-small modules.

Let R be a ring and assume M,N are R-modules. N is called *M*-small in category $\sigma[M]$ if, $N << L$ for some $L\in\sigma[M]$ or equivalently $N << \hat{N}$, where $\hat{N}$ denotes the *M*-injective hull of N in category $\sigma[M]$. Reader can visit [9] for more information about these modules.

In this paper we proceed a generalization of *M*-small modules namely *M-Rad* modules. We characterize *M-Rad* modules and then torsion theory cogenerated by these modules is investigated. Also some suitable results about these theories are obtained.

The radical of a module has very important role in modules theory. Many researchers work in this branch and study some classes of modules which are related to radical. For more information we refer to [3], [4], [6], [11].

Y. Talebi and author in [10] studied a class of modules which is related to $\delta(M)$. There we defined the functor $\delta(M)$ by the sum of all δ-small submodules and studied a class of modules related to this functor. Recall that if M is a finitely generated modules, then $\delta(M)=Rad(M)$ and so all results about $Rad(M)$ and $\delta(M)$ are coincided in this case.

Results

In this section the class *M-Rad* modules is defined and investigated. First we characterize the $M-Rad$ modules and then we obtain some properties of this kind of modules and also the relations between some other classes of modules and *M-Rad* modules will be studied.

Definition 2.1 Let R be a ring and assume M,N are $R-$modules. Then N is called *M-Rad* if, $N\subseteq Rad(L)$ for some $L\in\sigma[M]$.

We denotes by $R[M]$ for the class of all *M-Rad* modules in $\sigma[M]$. If $N\notin R[M]$, then we say N is *non-M-Rad*.

It is clear that any M-small module in $\sigma[M]$ is *M-Rad* and if N is a finitely generated module, then N is M-small if and only if N is *M-Rad*.

Proposition 2.2 *Let* M *be a module and* $N\in\sigma[M]$. *The following statements are equivalent*

1. N is *M-Rad*;
2. $N\subseteq Rad(\hat{N})$; where $\hat{N}$ is the M-injective hull of N;
3. For any M-injective module E and any homomorphism $f:N\to E$ in $\sigma[M]$, we have $Im(f)\subseteq Rad(E)$.

• Proof. $1\Rightarrow 3$: Suppose that $N\subseteq Rad(L)$ for some $L\in\sigma[M]$. Let $f:N\to E$ be a homomorphism in $\sigma[M]$ where E is M-injective. Therefore f can be extended to a homomorphism $g:L\to E$. Now we have $f(N)=g(N)\subseteq g(Rad(L))\subseteq Rad(E)$;
as required.
$3\Rightarrow 2\Rightarrow 1$ is clear.

Recall that a module M is called *cosemisimple* if all simple modules in $\sigma[M]$ are M-injective. L. Zhongkui and J. Ahsan in [14] investigated some properties of cosemisimple modules related to injectivity. It is not difficult to see that semisimple modules are cosemisimple.

A module $P\in\sigma[M]$ is called *cohereditary* if every factor module of P is M-injective. A module M is said to be σ*-cohereditary* if every injective module in $\sigma[M]$ is cohereditary.

Note that for $N\in\sigma[M]$, the *M*-injective hull $\hat{N}$ of N is embedable in the $R-$injective hull $E(N)$, so if $N\subseteq Rad(\hat{N})$ then $N\subseteq Rad(E(N))$; i.e. every *M-Rad* module is $R-Rad$.

It is clear that every M-small module is *M-Rad* and so $S[M]\subseteq R[M]$, where $S[M]$ denote the class of all M-small modules.

Let S be a simple module in $\sigma[M]$. If S is not *M-Rad*, then S is not M-small and so there exists a module $K\subset\hat{S}$ such that $S+K=\hat{S}$. If there exists $x\in K-S$, then $S\cap xR=S$ and hence

$S \subseteq xR \subseteq K$. This implies $K = \hat{S}$, a contradiction. Thus there is no $x \in K - S$; that is $K \subseteq S$ and so $S = \hat{S}$ is an M-injective module. So we can say any simple module in category $\sigma[M]$ is either *M-Rad* or M-injective.

It is not difficult to see that the class $R[M]$ is closed under submodules, homomorphic images and infinite (direct) sum. Note that unlike of $R[M]$, the class $S[M]$ is not closed under infinite direct sum.

It is clear to see that any simple module in $\sigma[M]$ that is M-small and M-injective, is zero. Because any *M*-injective module is equal to its M-injective hull. Since zero is the only small submodule of a simple module, so an M-injective simple module is *M-Rad* iff it is M-small. Thus we can say any simple module in $\sigma[M]$ that is M-injective and *M-Rad* must be zero.

Next proposition in particular shows that if M is a cosemisimple module, then there is no non-zero M-Rad module.

Proposition 2.3 *The class* $R[M] = 0$ *if and only if* M *is cosemisimple.*

• Proof. Suppose that $R[M] = 0$; this means in paticular that simple modules are not M-*Rad* and so are M-injective. Thus M is cosemisimple.

Conversely assume M to be cosemisimple. Let $N \in R[M]$ and $x \in N$. Suppose that K is a maximal submodule of xR. Hence the simple module $\frac{xR}{K}$ is *M-Rad* and M-injective and so must be zero. Since K is a maximal submodule of xR, we must have $x = 0$, implying $N = 0$; that is $R[M] = 0$.

Example 2.4

1. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_2 \oplus \mathbb{Z}_3$. Since M is cosemisimple so we have $R[M] = 0$.

2. Let R be a ring and M an arbitrary R-module. Although there is no no-zero simple module in $\sigma[M]$ which is both *M-Rad* and M-injective, but any *M-Rad* and *M*-injective module need not be zero. Especially it is clear that the divisible Z-module $\mathbb{Q}$ has no maximal submodule and so $Rad(\mathbb{Q}) = \mathbb{Q}$; i.e. $\mathbb{Q}$ is $\mathbb{Q}-$Rad. Also it is well known that $\mathbb{Q}$ is an injective Z-module.

3. It is well known that any injective module is not M-small in $\sigma[M]$ but it may be *M-Rad*. Especially the Z-module $\mathbb{Q}$ is *Z-Rad* but it is not Z-small.

The class $R[M]$ need not be closed under extensions. But for a $\sigma-$coherediary module *M*, $R[M]$ is closed under extensions (next proposition).

Proposition 2.5 *Suppose that* M *is a* $\sigma-$ *cohereditary module. Then* $R[M]$ *is closed under extensions in* $\sigma[M]$.

• Proof. Let $0 \to K \to \frac{L}{K} \to 0$ be an exact sequence such that $L \in \sigma[M]$ and $K, \frac{L}{K} \in R[M]$. By Proposition 2.2, we conclude $K \subseteq Rad(\hat{L})$. Since M is $\sigma-$coherediary, $\frac{\hat{L}}{K}$ must be injective and *M*-injective hence $\frac{L}{K} \subseteq Rad(\frac{\hat{L}}{K}) = \frac{\hat{L}}{K}$. Again by Proposition 2.2 we have $L \subseteq Rad(\hat{L})$.

Definition 2.6 Let $K, N \in \sigma[M]$ and $f : K \to N$ be an epimorphism. Then f is called a *radical cover* of N in $\sigma[M]$ if, $Ker(f) \subseteq Rad(K)$.

Proposition 2.7 *Let* M *be a module and* $L, N \in \sigma[M]$ *be such that* N *is M-injective. Moreover let* $f : L \to N$ *be a radical cover of* N *and* K *a submodule of* L *such that* $K \not\subset Rad(L)$. *Then* $K \notin R[M]$.

• Proof. Suppose that K is *M*-Rad. Then $f(K)$ is M-*Rad* by preliminary properties of $R[M]$. Since N is injective in $\sigma[M]$, so by Proposition 2.2, $f(K) \subseteq Rad(N) = f(Rad(L))$. Hence $K \subseteq Rad(L) + Ker(f) = Rad(L)$ that is a contradiction. So K is a non-M-*Rad* module.

Let $\mathbb{A}$ be a nonempty class of modules in $\sigma[M]$.

Recall the following classes

$\mathcal{A}^{\circ} = \{B \in \sigma[M] \mid Hom(B, A) = 0; \forall A \in \mathcal{A}\} = \{B \in \sigma[M] \mid Re(B, \mathcal{A}) = B\}$

$\mathcal{A}^{\bullet} = \{B \in \sigma[M] \mid Hom(A, B) = 0; \forall A \in \mathcal{A}\} = \{B \in \sigma[M] \mid Tr(\mathcal{A}, B) = 0\}$

$\mathcal{A}^{\triangleright} = \{X \in \sigma[M] \mid Hom(U, A) = 0; \forall U \leq X, A \in \mathcal{A}\} \subseteq \mathcal{A}^{\circ}$

$\mathcal{A}^{\%} = \{X \in \sigma[M] \mid Hom(A, \frac{X}{Y}) = 0; \forall Y \leq X, A \in \mathcal{A}\} \subseteq \mathcal{A}^{\bullet}$

The class $\mathcal{A}^{\triangleright}$ defines a hereditary pretorsion class of modules and also $\mathcal{A}^{\triangleright} = \{E\}^{\circ}$ for some injective module $E \in \sigma[M]$ (for more details see [9, Proposition 9.5]). The class $\mathcal{A}^{\%}$ defines a cohereditary class of modules. It is clear that $\mathcal{A}^{\%}$ is closed under extensions and submodules but is not closed under products.

An ordered pair $(\mathcal{A}, \mathcal{B})$ of classes of modules from $\sigma[M]$ is called a *torsion theory* if $\mathcal{A} = \mathcal{B}^{\circ}$ and $\mathcal{B} = \mathcal{A}^{\bullet}$. In this case $\mathcal{A}$ is called the *torsion class* and it's elements are the torsion modules, while $\mathcal{B}$ is the *torsion free class* and it's elements are the torsion free modules. So we have the following

1. The pair $(\mathcal{A}^{\bullet\circ}, \mathcal{A}^{\bullet})$ is torsion theory, called *torsion theory generated by* $\mathcal{A}$, and the torsion class is

$$\mathcal{A}^{\bullet\circ} = \{Y \in \sigma[M] \mid Tr(\mathcal{A}, \frac{Y}{U}) \neq 0; \forall U < Y\}$$

2. The pair $(\mathcal{A}^{\circ}, \mathcal{A}^{\bullet\circ})$ is also a torsion theory, called *the torsion theory cogenerated by* $\mathcal{A}$, and the torsion free class is

$$\mathcal{A}^{\circ\bullet} = \{Y \in \sigma[M] \mid Re(U, \mathcal{A}) \neq U; \forall 0 \neq U \leq Y\}.$$

Note that $\mathcal{A} \subseteq Gen(\mathcal{A}) \subseteq \mathcal{A}^{\bullet\circ} \subseteq Cog(\mathcal{A}) \subseteq \mathcal{A}^{\circ\bullet}$.

Recall that a subfunctor τ of the identity functor for $\sigma[M]$ is a *preradical* if for each pair $N, N' \in \sigma[M]$ and each morphism $f : N \to N'$, we have $\tau(n) \leq N'$ and $\tau(f) = f|_{\tau(N)} : \tau(N) \to \tau(N')$; (i.e. $f(\tau(N)) \subseteq \tau(N')$.

Here we define the preradical generated by $M-$ modules as the trace of $N \in \sigma[M]$ by following

$Tr_{R[M]}(N) = \Sigma\{X \leq N \mid X \text{ is an } M-Rad\ module\} = \Sigma\{X \leq N \mid X \subseteq Rad(\hat{N})\}$

$= N \cap Rad(\hat{N})$, this follows from the fact that any finitely generated module K is small in $\hat{N}$ iff it is contained in $Rad(\hat{N})$.

Similarly the preradical generated by $M-$small modules is

$Tr_{\mathcal{S}}(N) = \Sigma\{X \leq N \mid X \text{ is an } M-small\ module\} = \{X \leq N \mid X = \hat{N}\}$

$= N \cap Rad(\hat{N})$.

By above statement we have the following remark.

Remark 2.8 For two modules M and $N \in \sigma[M]$ we have $Tr_{R[M]}(N) = Tr_{\mathcal{S}}(N)$ and also $Tr_{R[M]}(N) \in R[M]$.

Example 2.9 Consider the $\mathbb{Z}$-module $\mathbb{Q}$. Since $Rad(\mathbb{Q}) = \mathbb{Q}$, we have $Tr_{R[\mathbb{Z}]}(\mathbb{Q}) = \mathbb{Q}$, while $Tr_{R(\mathbb{Z})}(\mathbb{Z}^{\mathbb{N}}) = 0$. Note that $\mathbb{Q}$ is a factor module of $\mathbb{Z}^{\mathbb{N}}$. This shows that the class of modules N with $Tr_{R[M]}(N) = 0$ need not be closed under factor modules.

Proposition 2.10 *Let R be a ring, M an R-module and $N \in \sigma[M]$. The following statements are equivalent*

1. $N = Tr_{R[M]}(N)$;
2. $N = Tr_{\mathcal{S}}(N)$;
3. $N \subseteq Rad(\hat{N})$;
4. $xR = \hat{N}$ for every $x \in N$;
5. $xR \subseteq Rad(\hat{N})$ for every $x \in N$;
6. $N \in Gen(\mathcal{S})$;
7. $N \in Gen(R[M])$.

• Proof. The proof follows from the fact that $Tr_{\mathcal{S}}(N) = Tr_{R[M}(N)$ and other preliminary properties of M-*Rad* and M-small modules.

By [5, 8.5] we have the following proposition;

Proposition 2.11

1.
$\mathcal{S}^{\bullet} = \{N \in \sigma[M] \mid Tr_{\mathcal{N}}(N) = 0\} = \{N \in \sigma[M] \mid Tr_{R[M]}(N) = 0\} = R[M]^{\bullet}$

hence the class $R[M]^{\bullet}$ is cogenerated by simple M-injective modules in $\sigma[M]$.

2.

$$\mathsf{S}^{\bullet\circ} = \{N \in \sigma[M] \mid Tr_{\mathsf{S}}(\frac{N}{K}) \neq 0; \forall K \text{Ü} N\} = \{N \in \sigma[M] \mid Tr_{R[M]}(\frac{N}{K}) \neq 0; \forall K \text{Ü} N\} = R[M]^{\bullet\circ}$$

hence

$$R[M]^{\bullet\circ} = \{N \in \sigma[M] \mid N \textit{ has no simple } M-\textit{injective factor module}\}$$

3. Let $N \in \sigma[M]$, then $N \in Gen(\mathsf{S})$ iff $N = Tr_{\mathsf{S}}(N) = Tr_{R[M]}(N)$. Thus $N \in Gen(\mathsf{S})$ iff $N \in Gen(R[M])$. Now if M is σ-cohereditary, then $Gen(R[M]) = R[M]^{\bullet\circ}$.

Note that if N is an *M-Rad* module, then $Re_{R[M]}(N) = 0$ and all modules those are in $Cog(R[M])$ belong to the torsion free class $R[M]^{\circ\bullet}$. Let R be a ring. If M is a module and $N \in \sigma[M]$. Then since $Rad(N) \subseteq Rad(\hat{N})$, we have $Rad(N) \subseteq Tr_{R[M]}(N)$ and by the same token $Tr_{R[M]}(N) \subseteq Tr_{R[R]}(N)$.

Example 2.12 As Z-modules we have $\mathsf{Z} \subseteq Rad(\mathsf{Q})$ and so $Tr_{R[\mathsf{Z}]}(\mathsf{Z}) = \mathsf{Z}$. Since Z generates all Z-modules, so we have $Tr_{R[\mathsf{Z}]}(N) = N$ for every Z-module N.

Discussion

Let M be a module and $N \in \sigma[M]$, the torsion theory cogenerated by $R[M]$ is the reject of $R[M]$ in N, defined as follows

$$Re_{R[M]}(N) = \bigcap\{X \leq N \mid \frac{N}{X} \textit{ is } M-Rad\}.$$

It is clear that $Re_{R[M]}(N)$ is the smallest submodule K of N for which $\frac{N}{K}$ is cogenerated by M-*Rad* modules. Reader can see [1] to get some information about torsion theory.

By the definition of reject we conclude $Re_{R[M]}(N) = 0$ iff N is cogenerated by M-Rad modules; in this case N is called M-*Rad cogenerated*.

Also we have $\frac{Re_{R[M]}(N)+K}{K} \subseteq Re_{R[M]}(\frac{N}{K})$; for every submodule K of N in $\sigma[M]$, and $\frac{Re_{R[M]}(N)}{K} = Re_{R[M]}(\frac{N}{K})$ if $K \subseteq Re_{R[M]}(N)$. It is trivial that $N \in R[M]^{\circ}$ iff $N = Re_{R[M]}(N)$.

Assume N, K are modules in $\sigma[M]$. Define $\Delta(N,K) = \{f : N \to K \mid Im(f) \subseteq Rad(K)\}$.

Proposition 3.1 *Let M be a module and $N \in \sigma[M]$. The following conditions are equivalent*

1. $N = Re_{R[M]}(N)$;
2. If $f : N \to K$ is a nonzero homomorphism in $\sigma[M]$ and L is a submodule of $Im(f)$, then $\frac{Im(f)}{L} \subseteq Rad(\frac{K}{L})$ implies $Im(f) = L$;
3. For every nonzero homomorphism $f : N \to K$ in $\sigma[M]$, $Im(f) \not\subset Rad(K)$.

• Proof. $1 \Rightarrow 2$: Suppose that $\frac{Im(f)}{L} \subseteq Rad(\frac{K}{L})$.

Consider the map $\pi o f : N \to \frac{K}{L}$; where $\pi : K \to \frac{K}{L}$ is the natural epimorphism. Then $Im(\pi o f) = \frac{Im(f)}{L}$, and so $\pi o f$ has to be zero. Hence $Im(f) = L$.

$2 \Rightarrow 3$ is obvious.

$3 \Rightarrow 1$: Assume $f : N \to K$ to be nonzero, where $K \in R[M]$. Then the composition map $\iota o f$ is a nonzero homomorphism from N to $\hat{K}$, where $\iota : K \to \hat{K}$ is the inclusion map. Now we have $Im(\iota o f) = Im(f) \subseteq K \subseteq Rad(\hat{K})$ a contradiction. Therefore there is no nonzero homomorphism from N to M-*Rad* modules; that is $N = Re_{R[M]}(N)$.

In above proposition when condition 2 holds, we say $Im(f)$ is *Rad-coclosed* in M.

Now we have the next proposition that follows immediately from Proposition 3.1.

Proposition 3.2 *Let M be any module and $N \in \sigma[M]$. The following are equivalent*

1. $N = Re_{R[M]}(N)$;

2. If K is a nonzero homomorphic image of N, then there exists an extension module $L \in \sigma[M]$ of K such that for any $X \leq K$, $\frac{K}{X} \subseteq Rad(\frac{L}{K})$ implies $K = X$ (i.e. K is Rad-coclosed in L);

3. $\Delta(N, A) = 0$, for all $A \in \sigma[M]$.

Proposition 3.3 *Let M be a module and $N \in R[M]^{\circ}$. The following hold*

1. Every *M-Rad* proper submodule $K \subset N$ is contained in $Rad(N)$ and so $Tr_{R[M]}(N) = Rad(N)$.
2. If L is a proper extension module of N in $\sigma[M]$, then N is *Rad*-coclosed in L.
3. For any proper submodule K of N, K is *Rad*-coclosed in N iff $K \in R[M]^{\circ}$.

• Proof. 1. Suppose that K is a proper *M*-Rad submodule of N. Assume $K \not\subseteq Rad(N)$. Thus there exists an element $x \in K$ such that $x \notin Rad(N)$. Therefore $xR \not\subseteq Rad(N)$ and hence xR is not small in N. So there exists a proper submodule L of N such that $xR + L = N$. Now $\frac{xR}{L \cap xR} \cong \frac{L + xR}{L} = \frac{N}{L}$ is an $M-$Rad module. Since $N \in R[M]^{\circ}$, $\frac{N}{L}$ must be zero and so $N = L$ that is a contradiction. Hence $K \subseteq Rad(N)$.

2. Let $\frac{N}{U} \subseteq Rad(\frac{L}{U})$ where $U \subseteq N \subset L$. Hence $\frac{N}{U}$ is an *M-Rad* module. Now since $N \in R[M]^{\circ}$, there is no nonzero homomorphism from N to $\frac{N}{U}$ and so $N = U$; that is N is Rad-coclosed in L.

3. Assume $K \subset N$.

If $K \in R[M]^{\circ}$, then by (2), K is Rad-coclosed in N.

For converse suppose that $f : K \to L$ is a homomorphism for some $L \in R[M]$. So $\frac{K}{Ker(f)} \cong Im(f)$ is an *M-Rad* module and hence by (1), $\frac{K}{Ker(f)} \subseteq Rad(\frac{N}{Ker(f)})$. Now since K is Rad-coclosed in N, we must have $K = Ker(f)$ and consequently $f = 0$ as desired.

Let M be a module. Then it is clear that $R[M]^{\circ} \subseteq S[M]^{\circ}$ and $R[M]^{\bullet} \subseteq S[M]^{\bullet}$. Also if $N \in \sigma[M]$, then $N \in R[M]^{\circ}$ iff N has no nonzero $M-$Rad factor module.

Proposition 3.4 *The class $R[M]^{\circ}$ is closed under factor modules, direct sums, extensions and Rad-coclosed submodules.*

• Proof. The first three properties follow from definition and the last property follows from Proposition 3.3 (3).

Example 3.5

1. Let $M = \frac{\mathbb{Z}}{12\mathbb{Z}}$. Then $Rad(M) = \frac{6\mathbb{Z}}{12\mathbb{Z}}$ and so $\mathbb{Z} \notin R[M]^{\circ}$.
2. Suppose that M is a divisible $\mathbb{Z}$-module with no nontrivial small submodule. Then every factor module of M is contained in $R[M]^{\circ}$.

Definition 3.6 Let M be a module, $N \in \sigma[M]$ and L a submodule of N. Then the inclusion $L \subseteq N$ is called $M-coRad$ if $\frac{N}{L}$ is *M-Rad*.

Proposition 3.7 *Let $N \in \sigma[M]$. Then the following hold*

1. If $\frac{L}{K} \subseteq Rad(\frac{N}{K})$ for some $K \leq L \leq N$, then the inclusion $K \subseteq L$ is M-coRad.
2. If $N \in R[M]^{\circ}$, then for every submodule L of N and *M*-coRad inclusion $K \subset L$, we have $\frac{L}{K} \subseteq Rad(\frac{N}{K})$.

• Proof. 1. We have $\frac{N}{K} \in R[M]$ and so

$\frac{L}{K} \in R[M]$. This completes the proof.

2. By Proposition 3.4, $\frac{N}{K} \in R[M]^{\circ}$. Now applying Proposition 3.3 to get $\frac{L}{K} \subseteq Rad(\frac{N}{K})$.

Recall that a module P is called *pseudo-projective* in category $\sigma[M]$ if for any epimorphism $\alpha : N \to L$ and any homomorphism $f : P \to L$ in $\sigma[M]$, there exist an endomorphism $\beta : P \to P$ and a homomorphism $g : P \to N$ such that $\alpha o g = f o \beta$.

Proposition 3.8 *Let* M *be a pseudo-projective module in* $\sigma[M]$. *Then the following hold*

1. Every $N \in \sigma[M]$ with $Hom(M, N) = 0$ is $M-$ small and especially *M-Rad*.
2. If $M \in R[M]^{\circ}$, then
$$R[M] = \{N \in \sigma[M] \mid Hom(M, N) = 0\}.$$

- Proof. 1. See [5, 8.14].

2. By (1),
$$\{N \in \sigma[M] \mid Hom(M, N) = 0\} \subseteq R[M].$$
Since $M \in R[M]^{\circ}$, we conclude $R[M] \subseteq \{N \in \sigma[M] \mid Hom(M, N) = 0\}$; as required.

Example 3.9 Consider the $\mathbb{Z}$-module $M = \frac{\mathbb{Z}}{4\mathbb{Z}}$. Then $Rad(M) = \frac{2\mathbb{Z}}{4\mathbb{Z}}$ and so $\frac{\mathbb{Z}}{2\mathbb{Z}}$ is $M-$Rad. Hence the torsion theory cogenerated by $R[M]$ of $\frac{\mathbb{Z}}{2\mathbb{Z}}$ is zero (i.e. $Re_{R[M]}(\frac{\mathbb{Z}}{2\mathbb{Z}}) = 0$). but $Re_{R[M]}(M) \neq 0$. This means that the class of modules with zero torsion theory cogenerated by *M-Rad* modules, need not be closed under extensions.

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