JCAMECH
Vol. 46, No. 2, July 2015, pp 117- 131

Numerical free vibration analysis of higher-order shear deformable beams resting on two-parameter elastic foundation

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Received: 26 November 2014; Accepted: 2 May 2015

Abstract
Free vibration analysis of higher-order shear deformation beam resting on one- and two-parameter elastic foundation is studied using differential transform method (DTM) as a part of a calculation procedure. First, the governing differential equations of beam are derived in a general form considering the shear-free boundary conditions (zero shear stress conditions at the top and bottom of a beam). Using DTM the derived equations governing beams, followed by higher-order shear deformation model, Timoshenko model and Bernoulli-Euler model are transformed to algebraic forms and a set of recurrence formulae is then derived. Upon imposing the boundary conditions of the beam at hand, a set of algebraic equations are obtained and expressed in matrix form. Finally, the transverse natural frequencies of the beam are calculated through an iterative procedure. Several numerical examples have been carried out to demonstrate the competency of the present method and the results obtained by DTM are in good agreement with those in the literature. Afterward, the free vibration of beams followed up by different models (i.e. Bernoulli-Euler, Timoshenko and different higher-order models) are taken into consideration.

Keywords: *differential transform method (DTM), elastic foundation, free vibration, higher-order beam theory (HOBT).*

1. Introduction

Many researchers have used classical beam theory (CBT) or Bernoulli-Euler beam model for beam analysis over the years in both static and dynamic analyses [1-5]. In this model, it is assumed that the plain sections of the cross section remain plain and perpendicular to the beam's neutral axis as shown in Figure 1a.

To overcome restrictions of the CBT (neglecting the shear deformation), another model widely used by researchers is the Timoshenko beam theory (TBT) in which the effects of shear and rotational inertia are taken into account [6]. In this theory, the plain sections of the cross section remain plain but not necessarily perpendicular to the beam's neutral axis as shown in Figure 1b.

However, the problem is the presence of the shear correction coefficient, κ, which was introduced to correct the contradictory shear stress distribution over the cross section of the beam [7] that does not satisfy shear-free boundary conditions [8]. This is because an accurate estimation of the shear correction

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coefficient is quite complex for different cross sections. Timoshenko beam model has been widely employed to study the behavior of beams in both static and dynamic analyses [9-17]. However, Levinson [18] and Bickford [19] presented a shear deformation theory for rectangular beams called higher-order beam theory (HOBT). Figure 1c schematically shows deformation of a Higher-Order beam. Levinson's third order shear beam theory meets the requirement of the zero shear stress conditions at the top and bottom of a beam. As reported in [20], by using Levinson’s kinematics, Bickford [19] presented a variational consistent sixth-order beam theory and Reddy [21] developed a variational consistent third-order shear plate theory in which the plate kinematics are identical to those of Levinson [18] and then Wang and Wang [22] and Gao and Wang [23] proposed a complicated beam theory that was not able to properly account for the constraints restriction at a clamped end of shear deformable beams. Different Higher-Order theories emerging gradually over the years have been well presented by Bhimaraddi and Chandrashekhara [24]. In addition, different Higher-Order models have been applied to analyze various applications of beam and plate [25- 29].

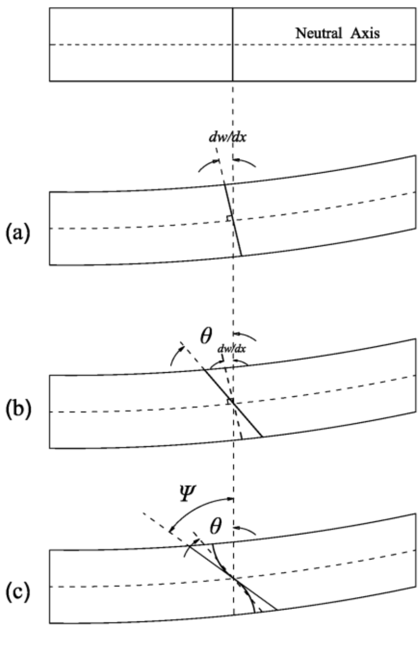

Fig. 1.Cross section displacement in different beam theories: (a) Beroulli-Euler model; (b) Timoshenko model; (c) Higher-Order model

The problem of a beam on elastic foundation is important in civil and mechanical engineering fields as it constitutes a practical idealization of many problems. Various types of foundation models, such as those of Winkler, Pasternak, Flinenko-Borodich, Hensity, and Vlasov and Leontev have been presented for the analysis of structures on elastic foundation. First, Winkler [30] proposed a simple model with only one parameter but the vertical deformation characteristics of the foundation were defined by means of continuous, independent, discrete, linearly and closely spaced springs which is its most important deficiency. To overcome this problem, Pasternak [31], Filonenko-Borodich [32], Hetenyi [33], Hetenyi [34] and Vlasov and Leontev [35] proposed a two parameter foundation model in which the effects of the interaction between springs and continuity and cohesion of the soil are taken into account. In the two parameter model, the first parameter is the same as in the Winkler model and the second one is the stiffness of the shearing layer. The importance of the problem is indicated by the large number of papers which have appeared in the literature about beams on one- and two-parameter elastic foundations using different beam models i.e. Bernoulli-Euler [1,3,4,36, 37], Timoshenko [9, 10, 11, 12, 14, 17] and different Higher-Order Models [29, 38]. Among them, Sayyad [38] used unified shear deformation theory to analyze simply supported thick isotropic beams and the results of displacement, stresses, natural bending and thickness shear mode frequencies for the beam were presented and discussed critically with those of exact solution and other higher order theories. Not all problems can be solved by theoretical approach because of its complexity. Therefore many researchers have used different efficient numerical methods to study various structural elements resting on elastic foundations [10, 11, 14, 38]. Differential Transform method is one of these numerical methods.

Differential Transform Method (DTM) is an efficient numerical method for solving ordinary and partial differential equations, based on Taylor series expansion of the main variables and coefficients to derive solutions in polynomial form. The concept of one-dimensional DTM was first proposed and applied to solve linear and nonlinear initial value problems in electric circuit analysis by

Zhao [39]. Using one-dimensional differential transform technique, Chen and Ho [40] proposed a method to solve eigenvalue problems. Other authors applied this method to different beam and plate problems [41-47] among which were Yalcin et al. [43] who derived the governing differential equation of circular plate in terms of radial coordinate and studied free vibration of circular plates via one-dimensional DTM. Semnani et al. [47] extended DTM to two-dimensional DTM (2D-DTM) to investigate free vibration analysis of Kirochhof plate.

In this paper, considering a higher-order shear deformation beam, conventional Bernoulli-Euler and Timoshenko beam resting on one- and two-parameter elastic foundations, one-dimensional differential transform method (DTM) is employed to derive transverse natural frequencies of the beam. First, the governing equations of a beam are derived in a general form after which by using DTM, the derived equations and boundary conditions of the beam are transformed to a set of algebraic equations and expressed in matrix form. Finally, the unknown transverse natural frequencies of the beam are calculated through an iterative procedure. Several numerical examples have been carried out to prove the competency of the present method and results are discussed for different models.

2. Structural Model

Consider a general beam resting on one- and two-parameter elastic foundation. The geometrical and material properties of the considered beam are: the rectangular cross-sectional area A, second moment of inertia I, modulus of elasticity E and mass density ρ. The considered beam is oriented in Cartesian co-ordinate (xyz) as shown in Figure 2. Also, the elastic foundation parameters are Winkler elastic foundation parameter k_w, and second foundation parameter k_φ. In this section, considering a general beam, the constitutive equations are presented first and then the governing differential equations are derived. Finally, the derived governing equations are reduced for different beam models i.e. Bernoulli-Euler, Timoshenko and four Higher-order models.

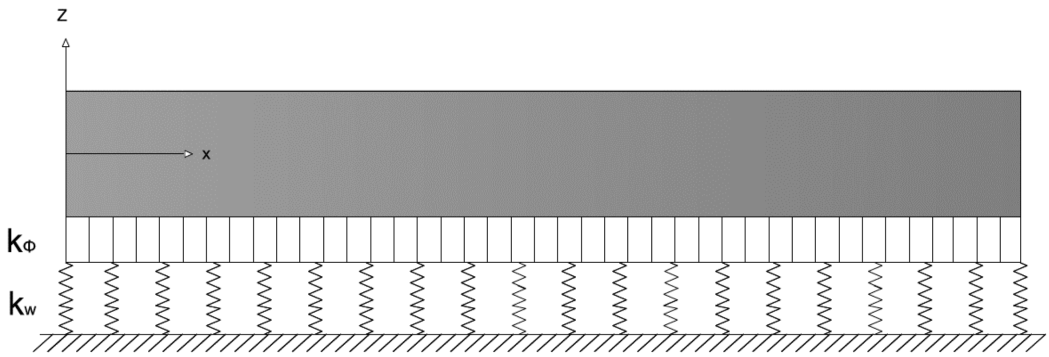

Fig. 2. A general beam resting on two-parameter elastic foundation (Winkler foundation, k_w and Second foundation parameter, k_φ)

Assuming shear strain γ_{xz} has the general form

$$\gamma_{xz} = \varphi\gamma \tag{1}$$

where γ is the shear strain at the mid-plane of the beam. Taking into account the shear-traction free boundary condition at the top and bottom surfaces of the beam, the trial distribution of the transverse shear strain φ should satisfy the required boundary conditions i.e.

$$\varphi(-h/2) = \varphi(h/2) = 0 \tag{2}$$

Substituting Equation (1) into the following strain compatibility equation for infinitesimal strains, we obtain:

$$\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x} = \gamma_{xz} \tag{3}$$

Afterwards, integrating both sides with respect to z (the coordinate along the thickness of the beam), the longitudinal displacement function, U could be derived as:

$$U = \psi\,\gamma - z\frac{\partial W}{\partial x} \tag{4}$$

where

$$\psi = \int \varphi dz \tag{5}$$

Introducing the rotation angle of cross-section perpendicular to the mid-plane as $\theta = \partial U / \partial z$ at $z = 0$, then substituting into Equation (3), we obtain:

$$\gamma = \theta + \frac{\partial W}{\partial x} \tag{6}$$

Thus the normal strain is derived as:

$$\varepsilon_{xx} = \frac{\partial U}{\partial x} = \psi\frac{\partial \theta}{\partial x} + (\psi - z)\frac{\partial^2 W}{\partial x^2} \tag{7}$$

Ignoring body forces, the equations of motion are [48]:

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = \rho\frac{\partial^2 W}{\partial t^2} \tag{8}$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = \rho\frac{\partial^2 U}{\partial t^2} \tag{9}$$

where ρ is mass density of the beam and $\sigma_{zz} = 0$.

Integrating Equation (8) with respect to z over the beam cross-section, we obtain:

$$\frac{\partial V}{\partial x} + q - k_w W + k_\varphi\frac{\partial^2 W}{\partial x^2} = \rho A\frac{\partial^2 W}{\partial t^2} \tag{10}$$

where q is applied distributed load on the beam.

Afterward, multiplying both sides of Equation (9) by z and substituting U from Equation (4), then integrating with respect to z over the beam cross-section, it can be derived:

$$\frac{\partial M}{\partial x} - V = \rho\psi_1\frac{\partial^2 \theta}{\partial t^2} + \rho(\psi_1 - 1)\frac{\partial^3 W}{\partial x \partial t^2} \tag{11}$$

In Equations (10) and (11), V, the shear force resultant, M, the bending moment, ψ_1 and I_2 are defined as:

$$V = \int_{-h/2}^{h/2} \tau_{xz} b dz \;;\; M = \int_{-h/2}^{h/2} \sigma_{xx} z b dz \tag{12}$$

$$\psi_1 = \int_{-h/2}^{h/2} \psi z b dz \;;\; I_2 = \int_{-h/2}^{h/2} z^2 b dz \tag{13}$$

Also, strain-stress relations can be expressed as (Hook's law):

$$\sigma_{xx} = E\,\varepsilon_{xx} \;;\; \tau_{xz} = G\gamma_{XZ} \tag{14}$$

Substituting Equations (1), (6), (7) and (14) into (12), we obtain:

$$V = Gb\left(\theta + \frac{\partial W}{\partial x}\right)\left[\psi\left(\frac{h}{2}\right) - \psi\left(-\frac{h}{2}\right)\right] \tag{15}$$

$$M = E\left[\psi_1\frac{\partial \theta}{\partial x} + (\psi_1 - I_2)\frac{\partial^2 W}{\partial x^2}\right] \tag{16}$$

where b is width of the beam.

Finally, substituting V and M from Equations (15) and (16) into Equations (10) and (11), the governing differential equations can be derived in terms of W and θ as [49, 50].

$$\begin{aligned}&\left\{Gb\left[\psi\left(\frac{h}{2}\right) - \psi\left(-\frac{h}{2}\right)\right] + k_\phi\right\}\frac{\partial^2 W}{\partial x^2} + \\ &Gb\left[\psi\left(\frac{h}{2}\right) - \psi\left(-\frac{h}{2}\right)\right]\frac{\partial \theta}{\partial x} - \\ &k_w W + q = \rho A\frac{\partial^2 W}{\partial t^2}\end{aligned} \tag{17}$$

$$\begin{aligned}&E(\psi_1 - I_2)\frac{\partial^3 W}{\partial x^3} + E\psi_1\frac{\partial^2 \theta}{\partial x^2} - \\ &Gb\left[\psi\left(\frac{h}{2}\right) - \psi\left(-\frac{h}{2}\right)\right]\frac{\partial W}{\partial x} - \\ &Gb\left[\psi\left(\frac{h}{2}\right) - \psi\left(-\frac{h}{2}\right)\right]\theta = \\ &\rho\psi_1\frac{\partial^2 \theta}{\partial x^2} + \rho(\psi_1 - 1)\frac{\partial^3 W}{\partial x\,\partial t^2}\end{aligned} \tag{18}$$

The function of ψ for Bernoulli-Euler, Timoshenko and different higher-order beam models is shown in Table 1 in which Model 1 is a parabolic shear deformation beam model of Reddy [21], Model 2 is a sinusoidal shear deformation beam model of Touratier [51], Model 3 is an exponential shear deformation beam model of Soldatos [52] and Model 4 is a new higher-order shear deformation model of Aydogdu [53]. Substituting corresponding ψ from Table 1 into Equations (8) and (9), the governing differential equation of any theory can be obtained.

Table 1. Function of ψ for different models

Model	Function
CBT	$\psi = 0$
TBT	$\psi = z$
Model 1	$\psi = z\left(1-4z^2/3h^2\right)$
Model 2	$\psi = \dfrac{h}{\pi}\sin\left(z\pi/h\right)$
Model 3	$\psi = ze^{-2\left(z/h\right)^2}$
Model 4	$\psi = z\alpha^{\frac{-2\left(z/h\right)^2}{\ln\alpha}}$ with $\alpha = 3$

3. Differential Transform Method

At this stage, the basic definitions and theorems of One-Dimensional DTM are given according to Arikoglu et al. [54].

$$W(k) = \frac{1}{k!}\frac{\partial^k W(x)}{\partial x^k}\bigg|_{x=x_0} \tag{19}$$

where $W(x)$ is the original function and $W(k)$ is the transformed function that is called T-function. Also, the inverse differential transform of $W(k)$ is defined as:

$$W(x) = \sum_{k=0}^{\infty}\left(x-x_0\right)^k W(k) \tag{20}$$

It is concluded from Equations (19) and (20) that

$$W(x) = \sum_{k=0}^{\infty}\frac{\left(x-x_0\right)^k}{k!}\frac{\partial^k W(x)}{\partial x^k}\bigg|_{x=x_0} \tag{21}$$

Since infinite terms are not attainable in practice, in real applications the function $W(x)$ is expressed by a finite series. Thus, Equation (21) can be rewritten as:

$$W(x) = \sum_{k=0}^{\alpha}\frac{\left(x-x_0\right)^k}{k!}\frac{\partial^k W(x)}{\partial x^k}\bigg|_{x=x_0} \tag{22}$$

where α determines the exactness of the results. Appropriate value of α for each problem could be determined by trial and error.

Some fundamental operations of the one-dimensional differential transform to be applied later in this text are shown below through Theorems [1-4]:

Theorem [1] If $w(x) = \lambda u(x)$, then $W(k) = \lambda U(k)$. λ is a constant.

Theorem [2] If $w(x) = u(x) \pm v(x)$, then $W(k) = U(k) \pm V(k)$.

Theorem [3] If $w(x) = \partial^r u(x)/\partial x^r$, then $W(k) = (k+r)!/k!\,U(k+r)$.

Theorem [4] If $w(x) = u(x)v(x)$, then $W(k) = \sum_{r=0}^{k} U(r)V(k-r)$.

4. Application of Differential Transform Method

In general, the following steps are carried out to solve one or a set of PDEs with boundary conditions via DTM.

Step 1. Using the concept of DTM, the governing differential equation(s) of the problem is transformed to algebraic equation(s) and a set of recurrence formula is obtained which presents all Differential Transform terms (DT) in terms of a minimum of DT terms namely, Fundamental DT terms (FDT).

Step 2. The boundary conditions of the problem are transformed to a set of algebraic equations by using DTM.

Step 3. Using recurrence formula(s) obtained in Step 1, this set of algebraic equations is presented in terms of Fundamental DT terms. Afterward, these equations can be represented in matrix form.

Step 4.Through an iterative procedure, the natural frequencies of the problem in hand are derived.

In what follows, DTM is employed for a beam resting on one- and two-parameter elastic foundations considering different beam models, i.e. Bernoulli-Euler, Timoshenko and different higher-order models, respectively, and how to derive the recurrence formula for any model is shown.

4.1. Recurrence Formula for Bernoulli-Euler beam model

Due to neglecting shear strains in this theory; shear force resultant cannot be defined. Consequently, V is eliminated from equilibrium equations. In order to achieve this purpose, one can differentiate Equation (11) and then adding it to Equation (10) results in the following:

$$\frac{\partial^2 M}{\partial x^2} - k_w W + k_\varphi \frac{\partial^2 W}{\partial x^2} + q = \rho A \frac{\partial^2 W}{\partial t^2} + \rho\left(\psi_1 - 1\right)\frac{\partial^4 W}{\partial x^2 \partial t^2} + \rho\psi_1 \frac{\partial^3 \theta}{\partial x\, \partial t^2} \tag{23}$$

Substituting M from Equation (8) into Equation (15), the governing differential equation is derived in terms of W and θ as follows:

$$E\left(\psi_1 - 1\right)\frac{\partial^4 W}{\partial x^4} + E\psi_1 \frac{\partial^3 \theta}{\partial x^3} - k_w W + k_\varphi \frac{\partial^2 W}{\partial x^2} + q = \rho A \frac{\partial^2 W}{\partial t^2} + \rho\left(\psi_1 - 1\right)\frac{\partial^4 W}{\partial x^2 \partial t^2} + \rho\psi_1 \frac{\partial^3 \theta}{\partial x\, \partial t^2} \tag{24}$$

Substituting $\psi = 0$ into Equation (24), the governing differential equation is reduced to

$$-EI\frac{\partial^4 W}{\partial x^4} - k_w W + k_\varphi \frac{\partial^2 W}{\partial x^2} = \rho A \frac{\partial^2 W}{\partial t^2} + \rho\frac{\partial^4 W}{\partial x^2 \partial t^2} \tag{25}$$

In order to carry out a free vibration analysis, q is set to zero and by assuming a harmonic variation of W with circular frequency ω,

$$W(x,t) = W(x)e^{i\omega t} \tag{26}$$

and substituting Equation (26) into Equation (25), the governing equation becomes

$$-EI\frac{\partial^4 W}{\partial x^4} + \left(\rho A\omega^2 - k_w\right)W + \left(k_\varphi - \rho\omega^2\right)\frac{\partial^2 W}{\partial x^2} = 0 \tag{27}$$

Using DTM, the obtained governing differential equation can be transformed into the following algebraic equation

$$-EI\left(i+4\right)\left(i+3\right)\left(i+2\right)\left(i+1\right)W_{i+4} + \left(\rho A\omega^2 - k_w\right)W_i + \left(k_\varphi - \rho\omega^2\right)\left(i+2\right)\left(i+1\right)W_{i+2} = 0 \ , \quad i = 0,1,2,\ldots \tag{28}$$

The recursive formula readily can be derived as follows

$$W_{i+4} = \frac{\left(\rho A\omega^2 - k_w\right)W_i + \left(k_\varphi - \rho\omega^2\right)\left(i+2\right)\left(i+1\right)W_{i+2}}{EI\left(i+4\right)\left(i+3\right)\left(i+2\right)\left(i+1\right)} \tag{29}$$

4.2. Recurrence Formula for Timoshenko beam model

Substituting $\psi = z$ into Equations (17) and (18), the governing differential equations are reduced to:

$$\left(GA + k_\varphi\right)\frac{\partial^2 W}{\partial x^2} + GA\frac{\partial \theta}{\partial x} - k_w W + q = \rho A\frac{\partial^2 W}{\partial t^2} \tag{30}$$

$$EI\frac{\partial^2 \theta}{\partial x^2} - GA\frac{\partial W}{\partial x} - GA\theta = \rho I\frac{\partial^2 \theta}{\partial t^2} \tag{31}$$

The Timoshenko beam theory assumes constant shear strain and stress over the cross-section, thus the shear-traction free boundary condition at the top and bottom surfaces of the beam is violated. Although it does not correct this fault, the use of a shear correction factor can adjust the solutions to match the static or dynamic behavior of the beam [9]. Including the shear correction factor, κ , the governing Equations (30) and (31) can be rewritten in the form:

$$\left(\kappa GA + k_\varphi\right)\frac{\partial^2 W}{\partial x^2} + \kappa GA\frac{\partial \theta}{\partial x} - k_w W + q = \rho A\frac{\partial^2 W}{\partial t^2} \tag{32}$$

$$EI\frac{\partial^2 \theta}{\partial x^2} - \kappa GA\frac{\partial W}{\partial x} - \kappa GA\theta = \rho I\frac{\partial^2 \theta}{\partial t^2} \tag{33}$$

In order to carry out a free vibration analysis, q is set to zero and a sinusoidal variation of W and θ is assumed with circular frequency ω:

$$W(x,t) = W(x)e^{i\omega t} \tag{34}$$

$$\theta(x,t) = \theta(x)e^{i\omega t} \tag{35}$$

Substituting Equations (34) and (35) into Equations (32) and (33), the governing equations become

$$\left(\kappa GA + k_\varphi\right)\frac{\partial^2 W}{\partial x^2} + \kappa GA\frac{\partial \theta}{\partial x} + \left(\rho A\omega^2 - k_w\right)W = 0 \tag{36}$$

$$EI\frac{\partial^2 \theta}{\partial x^2} - \kappa GA\frac{\partial W}{\partial x} + \left(\rho I\omega^2 - \kappa GA\right)\theta = 0 \tag{37}$$

Consequently, the DT form of derived governing differential equations can be expressed as

$$\left(\kappa GA + k_\varphi\right)\left(i+1\right)\left(i+2\right)W_{i+2} + \kappa GA\left(i+1\right)\theta_{i+1} + \left(\rho A\omega^2 - k_w\right)W_i = 0 \tag{38}$$

$$EI\left(i+1\right)\left(i+2\right)\theta_{i+2} + \left(\rho I\omega^2 - \kappa GA\right)\theta_{i+1} - \kappa GA\left(i+1\right)W_{i+1} = 0 \quad ; \quad i = 0,1,2,... \tag{39}$$

Afterwards, the following recurrence formulas can be obtained

$$W_{i+2} = \frac{\left(k_w - \rho A\omega^2\right)W_i - \kappa GA\left(i+1\right)\theta_{i+1}}{\left(\kappa GA + k_\varphi\right)\left(i+1\right)\left(i+2\right)} \tag{40}$$

$$\theta_{i+2} = \frac{\left(\kappa GA - \rho I\omega^2\right)\theta_{i+1} + \kappa GA\left(i+1\right)W_{i+1}}{EI\left(i+1\right)\left(i+2\right)} \tag{41}$$

4.3. Recurrence Formula for Higher-Order beam model

At this stage, the recursive formula is obtained for different Higher-Order beam theories named Model 1, 2, 3 and 4.

- **Model 1**

Substituting $\psi = z\left(1 - 4z^2/3h^2\right)$ into Eqs. (17) and (18), the governing differential equations are reduced to:

$$\frac{2}{3}GA\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial\theta}{\partial x}\right) + k_\varphi\frac{\partial^2 W}{\partial x^2} - k_w W + q = \rho A\frac{\partial^2 W}{\partial x^2} \tag{42}$$

$$\frac{1}{5}EI\left(4\frac{\partial^2\theta}{\partial x^2} - \frac{\partial^3 W}{\partial x^3}\right) - \frac{2}{3}GA\left(\frac{\partial W}{\partial x} + \theta\right) = \frac{1}{5}\rho I\left(4\frac{\partial^2\theta}{\partial t^2} - \frac{\partial^3 W}{\partial x\,\partial t^2}\right) \tag{43}$$

Imposing the free vibration conditions i.e. $q = 0$ and by assuming a sinusoidal variation of W and θ with circular frequency ω

$$W(x,t) = W(x)e^{i\omega t} \tag{44}$$

$$\theta(x,t) = \theta(x)e^{i\omega t} \tag{45}$$

and substituting Equations (44) and (45) into Equations (42) and (43), the governing differential equations become

$$\frac{2}{3}GA\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial\theta}{\partial x}\right) + k_\varphi\frac{\partial^2 W}{\partial x^2} - k_w W + \rho A\omega^2 W = 0 \tag{46}$$

$$\frac{1}{5}EI\left(4\frac{\partial^2\theta}{\partial x^2} - \frac{\partial^3 W}{\partial x^3}\right) - \frac{2}{3}GA\left(\frac{\partial W}{\partial x} + \theta\right) + \frac{1}{5}\rho I\omega^2\left(4\theta - \frac{\partial W}{\partial x}\right) = 0 \tag{47}$$

If one differentiates Equation (46) with respect to x, multiply it by $F = \left(1/5EI\right)/\left(2/3GA + k_\varphi\right)$ and then add it to Equation (47), after rearranging the governing equations take the form

$$\left(\frac{2}{3}GA + k_\varphi\right)\frac{\partial^2 W}{\partial x^2} + \frac{2}{3}GA\frac{\partial\theta}{\partial x} + (\rho A\omega^2 - k_w)W = 0 \tag{48}$$

$$\left(\frac{2}{3}GAF + \frac{4}{5}EI\right)\frac{\partial^2\theta}{\partial x^2} + \left(\frac{4}{5}\rho I\omega^2 - \frac{2}{3}GA\right)\theta + \left[\frac{1}{5}\rho I\omega^2 - \frac{2}{3}GA + F(\rho A\omega^2 - k_w)\right]\frac{\partial W}{\partial x} = 0 \tag{49}$$

Using DTM, the two governing differential equations can be transformed to a set of algebraic equations as follows

$$\left(\frac{2}{3}GA + k_\varphi\right)(i+1)(i+2)W_{i+2} + \frac{2}{3}GA(i+1)\theta_{i+1} + (\rho A\omega^2 - k_w)W_i = 0 \tag{50}$$

$$\left(\frac{2}{3}GAF + \frac{4}{5}EI\right)(i+1)(i+2)\theta_{i+2} + \left(\frac{4}{5}\rho I\omega^2 - \frac{2}{3}GA\right)\theta_i + \left[\frac{1}{5}\rho I\omega^2 - \frac{2}{3}GA + F(\rho A\omega^2 - k_w)\right](i+1)W_{i+1} = 0 \quad ; \quad i = 0,1,2,... \tag{51}$$

in which

$$F = \frac{\left(\frac{1}{5}E1\right)}{\left(\frac{2}{3}GA + k_\varphi\right)} \tag{51-1}$$

Therefore, the following recurrence formulae can be readily derived

$$W_{i+2} = \frac{(k_w - \rho A\omega^2)W_i - \frac{2}{3}GA(i+1)\theta_{i+1}}{\left(\frac{2}{3}GA + k_\varphi\right)(i+1)(i+2)} \tag{52}$$

$$\theta_{i+2} = \frac{(\frac{2}{3}GA - \frac{4}{5}\rho I\omega^2)\theta_i + \left[\frac{2}{3}GA - \frac{1}{5}\rho I\omega^2 + F(k_w - \rho A\omega^2)\right](i+1)W_{i+1}}{\left(\frac{2}{3}GAF + \frac{4}{5}EI\right)(i+1)(i+2)} \tag{53}$$

- **Model 2**

Following the same procedures as in the case of Model 1, the two governing differential equations of Model 2 by assuming $\psi = {h}/{\pi}\sin(z\pi/h)$ are obtained:

$$(\frac{2}{\pi}GA + k_\varphi)\frac{\partial^2 W}{\partial x^2} + \frac{2}{\pi}GA\frac{\partial\theta}{\partial x} - k_w W + q = \rho A\frac{\partial^2 W}{\partial t^2} \tag{54}$$

$$E(\frac{24}{\pi^3}-1)\frac{\partial^3 W}{\partial x^3} + \frac{24}{\pi^3}EI\frac{\partial^2\theta}{\partial x^2} - \frac{2}{\pi}GA\frac{\partial W}{\partial x} - \frac{2}{\pi}GA\theta = \frac{24}{\pi^3}\rho I\frac{\partial^2\theta}{\partial t^2} + EI(\frac{24}{\pi^3}-1)\frac{\partial^3 W}{\partial x\,\partial t^2} \tag{55}$$

and the two recursive formulae of the free vibration analysis can be obtained

$$W_{i+2} = \frac{(k_w - \rho A\omega^2)W_i - \frac{2}{\pi}GA(i+1)\theta_{i+1}}{\left(\frac{2}{\pi}GA + k_\varphi\right)(i+1)(i+2)} \tag{56}$$

$$\theta_{i+2} = \frac{(\frac{2}{\pi}GA - \frac{24}{\pi^3}\rho I\omega^2)\theta_i + \left[\rho I\omega^2(1-\frac{24}{\pi^3}) + \frac{2}{\pi}GA + F(k_w - \rho A\omega^2)\right](i+1)W_{i+1}}{\left(\frac{2}{\pi}GAF + \frac{24}{\pi^3}EI\right)(i+1)(i+2)} \tag{57}$$

in which

$$F = \frac{E(1-\frac{24}{\pi^3})}{(\frac{2}{\pi}GA + k_\varphi)} \quad ; \quad i = 0,1,2,... \tag{58}$$

- **Model 3**

Similarly, the two governing differential equations can be expressed as follows by considering $\psi = ze^{-2(z/h)^2}$ for this model

$$(e^{\frac{-1}{2}}GA + k_\varphi)\frac{\partial^2 W}{\partial x^2} + e^{\frac{-1}{2}}GA\frac{\partial\theta}{\partial x} - k_w W + q = \rho A\frac{\partial^2 W}{\partial t^2} \tag{59}$$

$$E(\psi_1 - 1)\frac{\partial^3 W}{\partial x^3} + E\psi_1\frac{\partial^2\theta}{\partial x^2} - e^{\frac{-1}{2}}GA\frac{\partial W}{\partial x} - e^{\frac{-1}{2}}GA\theta = \rho\psi_1\frac{\partial^2\theta}{\partial t^2} + \rho(\psi_1 - 1)\frac{\partial^3 W}{\partial x\partial t^2} \tag{60}$$

Also, the recursive formulae can be derived as

$$W_{i+2} = \frac{(k_w - \rho A\omega^2)W_i - e^{\frac{-1}{2}}GA(i+1)\theta_{i+1}}{\left(e^{\frac{-1}{2}}GA + k_\varphi\right)(i+1)(i+2)} \tag{61}$$

$$\theta_{i+2} = \frac{(e^{\frac{-1}{2}}GA - \rho\psi_1\omega^2)\theta_i + \left[\rho\omega^2(1-\psi_1) + e^{\frac{-1}{2}}GA + F(k_w - \rho A\omega^2)\right](i+1)W_{i+1}}{\left(e^{\frac{-1}{2}}GAF + E\psi_1\right)(i+1)(i+2)} \tag{62}$$

in which

$$F=\frac{E\,(1-\psi_1)}{(e^{\frac{-1}{2}}GA+k_\varphi)}\qquad;\qquad i=0,1,2,... \tag{63}$$

and $\psi_1=\int_{-h/2}^{h/2} z^2 e^{-2(z/h)^2} bdz$ can be evaluated numerically.

- **Model 4**

Similarly, the two differential equations governing this model are obtained by imposing $\psi = z\alpha^{-2(z/h)^2/\ln\alpha}$ with $\alpha=3$ as:

$$(\alpha^{\frac{-1}{2L\ln\alpha}}GA+k_\varphi)\frac{\partial^2 W}{\partial x^2}+e^{\frac{-1}{2L\ln\alpha}}GA\frac{\partial\theta}{\partial x}-k_w W+q=\rho A\frac{\partial^2 W}{\partial t^2} \tag{64}$$

$$E\,(\psi_1-1)\frac{\partial^3 W}{\partial x^3}+E\,\psi_1\frac{\partial^2\theta}{\partial^2 x}-\alpha^{\frac{-1}{2L\ln\alpha}}GA\frac{\partial W}{\partial x}-\alpha^{\frac{-1}{2L\ln\alpha}}GA\,\theta=\rho\psi_1\frac{\partial^2\theta}{\partial^2 t}+\rho(\psi_1-1)\frac{\partial^3 W}{\partial x\,\partial t^2} \tag{65}$$

Afterwards, the following recursive formulas are obtained

$$W_{i+2}=\frac{(k_w-\rho A\omega^2)W_i-\alpha^{\frac{-1}{2L\ln\alpha}}GA(i+1)\theta_{i+1}}{\left(\alpha^{\frac{-1}{2L\ln\alpha}}GA+k_\varphi\right)(i+1)(i+2)} \tag{66}$$

$$\theta_{i+2}=\frac{\left(\alpha^{\frac{-1}{2L\ln\alpha}}GA-\rho\psi_1\omega^2\right)\theta_i+\left[\rho(1-\psi_1)\omega^2+\alpha^{\frac{-1}{2L\ln\alpha}}GA+F(k_w-\rho A\omega^2)\right](i+1)W_{i+1}}{\left(\alpha^{\frac{-1}{2L\ln\alpha}}GAF+E\psi_1\right)(i+1)(i+2)} \tag{67}$$

in which

$$F=\frac{E\,(1-\psi_1)}{\alpha^{\frac{-1}{2L\ln\alpha}}GA+k_\varphi}\qquad;\qquad i=0,1,2,... \tag{68}$$

and $\psi_1=\int_{-h/2}^{h/2} z\alpha^{-2(z/h)^2/\ln\alpha} bdz$ can be evaluated numerically.

4.4. Derivation of the transverse natural frequencies

At this stage, the boundary conditions of the beam are transformed to a set of algebraic equations and then, using the obtained recurrence formula, these algebraic equations can be presented in terms of Fundamental DT terms. The transformed boundary conditions for different cases of boundary conditions of the left and right side of the beam are shown in Table 2. There exist four transformed boundary conditions for the beam which can be presented in matrix form by regarding Fundamental DT terms as follows

$$\left[\bar{C}_\omega^m\right]\left[\overline{FDT}\right]=0 \tag{69}$$

where $\bar{C}_\omega^m$ and $\overline{FDT}$ are called coefficient matrix and Fundamental matrix, respectively, which include unknown natural frequency (ω) and Fundamental DT terms $(\overline{FDT}_i, i=1,2,3,4)$. For non-trivial solution of Equation (69), determinant of matrix $\bar{C}_\omega^m$, i.e. $\left|\bar{C}_\omega^m\right|$, must be zero. Accordingly, applying an iterative procedure, natural frequencies can be derived to the desired precision. The convergence criterion is

$$\left|\omega^m-\omega^{m-1}\right|\le\varepsilon \tag{70}$$

where the superscript m, implies the number of DT terms taken into account and ε is a small positive value, such as $\varepsilon=0.001$.

5. Numerical Examples

In this section, several numerical examples are carried out to verify the results obtained with one-dimensional DTM and the competency of the method is demonstrated. First, using recurrence formula(s) obtained in section 4 and imposing corresponding boundary conditions of the beam from Table 2, the coefficient matrix is constructed. Then, setting the determinant of the mentioned matrix equal to zero, the natural frequencies of the beam at hand are calculated for different models mentioned in section 4. Any case of boundary conditions could be considered for the beam such as: Simply supported beam (SS), Clamped-Free beam (CF) and Clamped-Simply supported beam (CS). In Example 1, a simply supported (SS) beam is considered and the first five natural frequencies are shown for Timoshenko beam model and different Higher-Order models. The obtained results are shown in Table 3 and compared with those reported in Ruta [13] and Attarnejad et al. [14]. For different values of L/h (L and h denote the whole length and height of the beam, respectively) and different elastic foundation parameter i.e. k_φ and k_w, the first transverse natural frequencies are calculated for a simply supported beam (SS) through Example 2 and the results are shown in Table 4. Different theories are considered in the Example 2 and it is shown that the results obtained by DTM are in good agreement with those reported in Matsunaga [29].

Table 2. Properties of differential transform for boundary conditions ($\xi = x/l$)

At $\xi=0$		At $\xi=1$	
Original	**Transformed**	**Original**	**Transformed**
$f(0)=c$	$\bar{F}(0)=c$	$f(1)=c$	$\sum_{k=0}^{\infty}\bar{F}(k)=c$
$\frac{df}{d\xi}(0)=c$	$\bar{F}(1)=c$	$\frac{df}{d\xi}(1)=c$	$\sum_{k=0}^{\infty}k\bar{F}(k)=c$
$\frac{d^2f}{d\xi^2}(0)=c$	$2\bar{F}(2)=c$	$\frac{d^2f}{d\xi^2}(1)=c$	$\sum_{k=0}^{\infty}k.(k-1)\bar{F}(k)=c$
$\frac{d^3f}{d\xi^3}(0)=c$	$6\bar{F}(3)=c$	$\frac{d^3f}{d\xi^3}(1)=c$	$\sum_{k=0}^{\infty}k.(k-1).(k-2).\bar{F}(k)=c$

Table 3. The first five natural frequencies of Simply Supported (SS) Timoshenko and different Higher-Order beams ($E = 2.1\times10^{11}$ Pa, $G = \frac{3}{8}E$, $\upsilon = 0.3$, $\rho = 7850\frac{Kg}{m^3}$, $L = 0.4m$, $b = 0.02m$, $h = 0.08m$, $k_w = k_\varphi = 0$, $\kappa = 2/3$)

Simple Supported (SS)		Frequency number				
		1	2	3	4	5
[13]	TBT	6838.8336	23190.827	43443.493	64939.185	86710.889
[14]	TBT	6838.8333	23190.8264	43443.4922	64939.1839	86710.8977
DTM	CBT	7368.0740	29472.3330	66312.7189	117889.5010	184203.0784
	TBT	6838.8333	23190.8265	43443.4923	64939.1842	86710.8238
	Model 1	7481.0913	24373.4153	47528.8753	72175.3324	98725.7733
	Model 2	7545.0993	24388.5839	47647.3762	72322.4613	99054.4827
	Model 3	6907.3135	23852.8268	45395.1112	68674.063	92517.9678
	Model 4	6907.3135	23852.8268	45395.1112	68674.063	92517.9678

Table 4. The first dimensionless natural frequency of the Simply Supported (SS) beam on elastic foundation for different values of L/h ($\upsilon = 0.3$, $E = 2.1\times10^{11}$, $\Omega = \omega L^2\sqrt{\rho A/EI}$, $\kappa = 5/6$)

$L/h = 2$	k_φ	0				1			
	k_w	0	10	100	1000	0	10	100	1000
[29]	CBT	9.8696	10.3638	14.0502	33.1272	13.9577	14.3115	17.1703	34.5661
	TBT	7.4127	8.0106	12.1084	29.0828	12.0106	12.3836	15.3153	29.9225
	HOBT	7.4664	8.0102	11.282	17.5208	11.2136	11.4721	13.2672	17.663
Present	TBT	7.4127	8.0106	12.1084	29.0828	12.0106	12.3836	15.3153	29.9225
	Model 1	8.202	8.7788	12.847	27.1746	12.7546	13.1316	16.1257	27.1746
	Model 2	8.2835	8.8599	12.9354	26.9969	12.8444	13.2227	16.2311	26.9969
	Model 3	7.3799	7.9812	12.0941	26.8188	11.9961	12.3701	15.3083	26.8188
	Model 4	7.3799	7.9812	12.0941	26.8188	11.9961	12.3701	15.3083	26.8188
$L/h = 10$	k_φ	0				1			
	k_w	0	10	100	1000	0	10	100	1000
[29]	CBT	9.8696	10.3638	14.0502	33.1271	13.9577	14.3115	17.1703	34.5661
	TBT	9.7071	10.2057	13.9086	32.9615	13.8162	14.1709	17.0326	34.3963
	HOBT	9.7121	10.2078	13.8941	32.8494	13.8018	14.1548	17.0046	34.2741
Present	TBT	9.7074	10.2058	13.909	32.9615	13.8162	14.1708	17.0326	34.3963
	Model 1	10.4547	10.9204	14.4514	33.2375	14.3634	14.7058	17.4886	34.666
	Model 2	10.5427	11.0048	14.5167	33.2716	14.4293	14.7704	17.544	34.6995
	Model 3	9.7043	10.2028	13.9068	32.9607	13.814	14.1686	17.0309	34.3956
	Model 4	9.7043	10.2028	13.9068	32.9607	13.814	14.1686	17.0309	34.3956
$L/h = 5$	k_φ	0				1			
	k_w	0	10	100	1000	0	10	100	1000
[29]	CBT	9.8696	10.3638	14.0502	33.1272	13.9577	14.3115	17.1703	34.5661
	TBT	9.274	9.7848	13.5408	32.5378	13.4473	13.8045	16.6781	33.9613
	HOBT	9.2903	9.7912	13.4726	31.6171	13.3812	13.7307	16.5354	32.9239
Present	TBT	9.274	9.7848	13.5408	32.5378	13.4473	13.8045	16.6781	33.9613
	Model 1	10.0325	10.5121	14.1134	32.9487	14.0251	14.3721	17.1824	34.3741
	Model 2	10.1185	10.595	14.1804	33.0002	14.093	14.4389	17.2426	34.4262
	Model 3	9.2631	9.7746	13.5337	32.5363	13.4402	13.7976	16.6727	33.96
	Model 4	9.2631	9.7746	13.5337	32.5363	13.4402	13.7976	16.6727	33.96

6. Conclusion

In DTM, the number of DT terms chosen to expand the unknown functions directly affects the accuracy of the results. Figure 3 shows the convergence of the first four transverse natural frequencies with the number of DT terms for a simply supported beam resting on an elastic foundation (Model 1). Figure 4 depicts the same for Timoshenko beam. As shown in Figures 3 and 4, DTM exhibits fast convergence in first mode but by increasing the mode number, the required DT terms increase (10, 20, 30 and 40 required DT terms for first, second, third and fourth mode, respectively). As a result, it is concluded that the present method has the advantage of yielding accurate results through a simple iterative procedure with fast convergence and little computational effort.

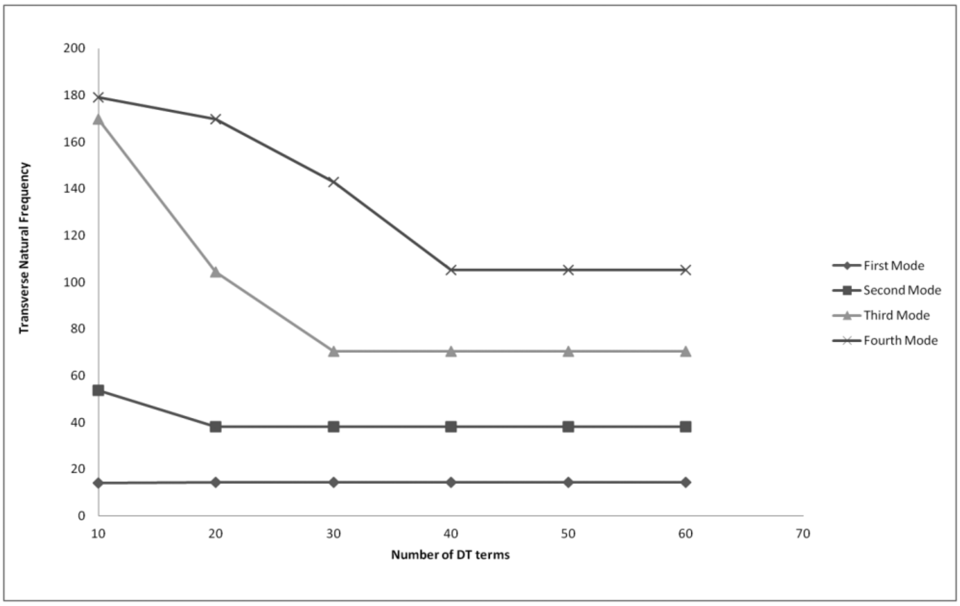

Fig. 3. Convergence of of the first four dimensionless natural frequencies(Ω) with the number of DT terms for a simply supported beam followed up Timoshenko model (Dimensionless parameter: $k_w = k_w L^4/EI$, $k_\varphi = k_\varphi L^2/\pi^2 EI$, $\Omega = \omega L^2\sqrt{\rho A/EI}$)

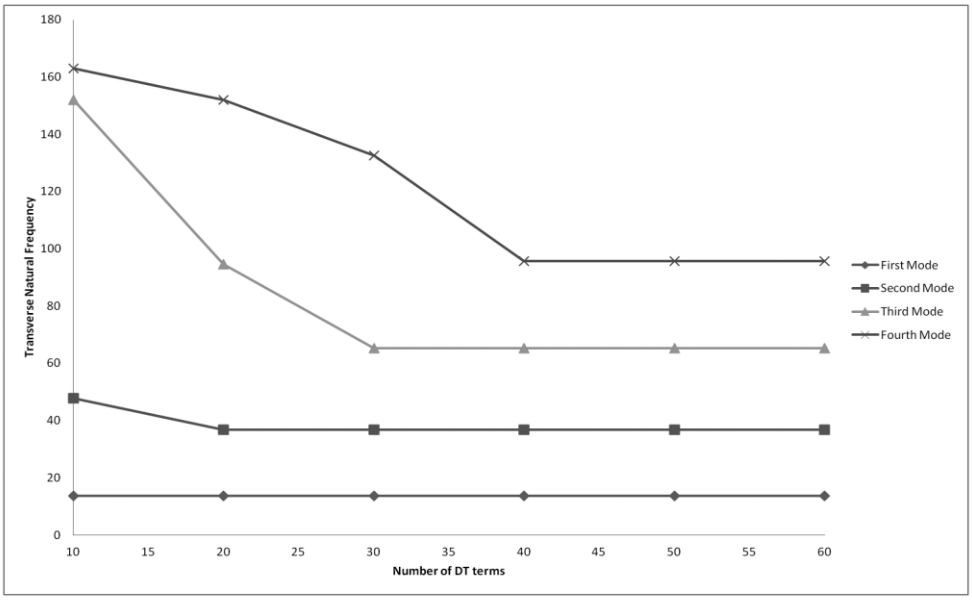

Fig. 4. Convergence of of the first four dimensionless natural frequencies(Ω) with the number of DT terms for a simply supported beam followed up Model 1. (Dimensionless parameter: $k_w = k_w L^4/EI$, $k_\varphi = k_\varphi L^2/\pi^2 EI$, $\Omega = \omega L^2\sqrt{\rho A/EI}$)

To depict the difference between the first natural frequency of different higher-order beam and Timoshenko beam with the Winkler elastic foundation stiffness model, Figure 5 is presented. It can thus be concluded that the natural frequencies calculated by employing Models 3 and 4 have approximately the same values as Timoshenko beam. It seemed Models 1 and 2 show about 3.29% of error while Models 3 and 4, 0.03% error compared with the Timoshenko beam results. Consequently, using Timoshenko beam theory is recommended both economically and accuracy in comparison with Models 3 and 4 for free vibration analysis of a beam resting on one- and two-parameter elastic foundation.

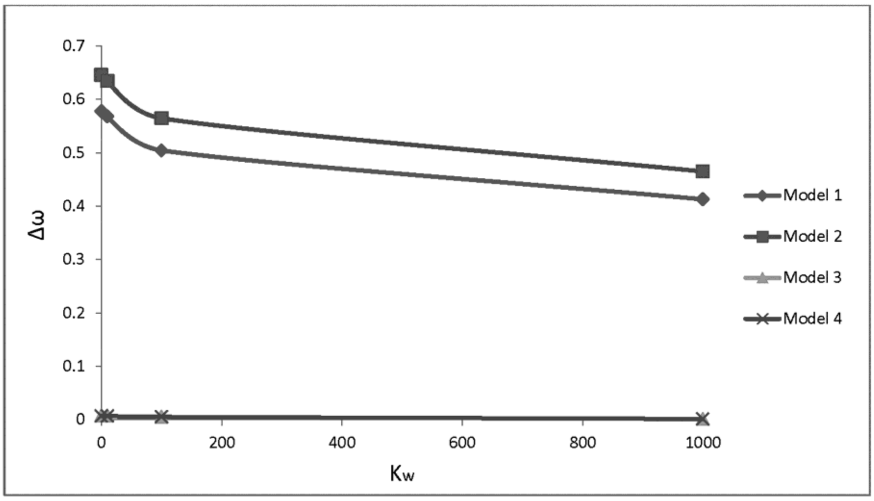

Fig. 5. Difference between the first natural frequency of different Higher-Order model and Timoshenko model with the Winkler elastic foundation stiffness ($k_\varphi = 1$ and $L/h = 5$, simply supported beam)

Differential transform method (DTM) was employed to solve the free vibration problem of a beam resting on one- and two-parameter elastic foundations followed up by different beam models i.e. Bernoulli-Euler model, Timoshenko model and different higher-order models (model 1, 2, 3 and 4). First, the governing equations of a beam were derived in a general form by considering the shear-free boundary conditions. Then using DTM, the recurrence formulas were obtained from the derived governing equations. Finally, imposing the boundary conditions of the beam, the unknown transverse natural frequencies were calculated through an iterative procedure for all mentioned models. To verify the results and show the competency of the method, several numerical examples were carried out and the results presented. Based on the results, it was concluded that the DTM shows fast convergence in the free vibration analysis of the beam with higher-order and other models mentioned in this study. Also, the results showed that Models 3 and 4 produced approximately the same results as Timoshenko beam model in the free vibration analysis of a beam resting on an elastic foundation. Finally, the effects of the different higher-order beam on mode shapes are shown in Figure 6.

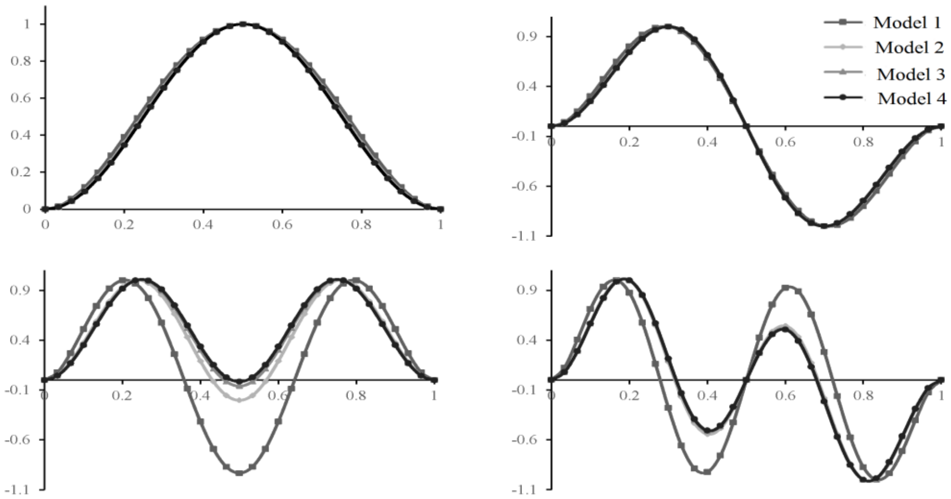

Fig. 6. First normalized mode shapes of different higher order beams as described in Example 1($k_w = 1$, $k_\varphi = 1$ and $L/h = 5$, simply supported beam)

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