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Three-Dimensional Interfacial Green's Function for Exponentially Graded Transversely Isotropic Bi-Materials

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ABSTRACT: By virtue of a complete set of two displacement potentials, an analytical derivation of the elastostatic Green's functions of an exponentially graded transversely isotropic bi-material full-space was presented. Three-dimensional point-load Green's functions for stresses and displacements were given in line-integral representations. The formulation included a complete set of transformed stress-potential and displacement-potential relations, with the utilization of Fourier series and Hankel transform. As illustrations, the present Green's functions were analytically degenerated into special cases, such as exponentially graded half-space and homogeneous full-space bi-material Green's functions. Owing to the complicated integrand functions, the integrals were evaluated numerically, and in computing the integrals numerically, a robust and effective methodology was laid out which provided the necessary account of the presence of singularities of integration. Some typical numerical examples were also illustrated to demonstrate the general features of the exponentially graded bi-material Green's functions which will be recognized by the effect of degree of variation of material properties.

Keywords: Bi-Material, Displacement Potential, Exponentially Graded, Functionally Graded Material, Green's Function, Transversely Isotropic.

INTRODUCTION

The continuous variation of the mechanical characteristics of functionally graded composite materials have broad application in industrial engineering, including thermal barriers, abradable seals, wear-resistant and oxidation resistant coatings, due to better residual stress distribution, thermal properties and greater fracture toughness. Also, functionally graded materials (FGMs) which possess the desired variation of material properties in spatial directions are widely used in different applications, such as aerospace and automobile industries (Eskandari and Shodja, 2010). For an inclusive review of the recent developments in the theory and application of FGMs, and

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the subjects of further research, one might refer to Birman and Byrd (2007). In addition to the manufactured functionally graded composite materials, the more deep understanding of the deposited soil and rock formations have attracted researchers to the area of analysis of stress transfer in this type of materials, and the effective utilization of their advantages. As many natural soils deposited through a period of time exhibit the anisotropic and inhomogeneous characteristics, a more precise modeling of soil medium with such behavior would be of considerable interest in foundation engineering and geomechanics, etc. However, among the different kinds of inhomogeneity, the exponential variation of the elasticity tensor is widely used for FGMs in the engineering literature and is evident in the list of references provided by Martin et al. (2002).

It was recognized that the performance of these materials was closely related to the effects occurring at the interface between their different components. Issues, such as interfacial fracture and crack problems, in functionally graded bi-material systems, are at the forefront of many investigations (Lambros and Rosakis, 1995). A powerful approach for the analysis of the foregoing FGMs and mechanics problems is the integral equations or boundary element methods. Central to their success is the availability of suitable Green's functions.

For a detailed review of studies in the field of inhomogeneous isotropic and anisotropic solutions, one might refer to Wang et al. (2003) who also presented the Green's functions for the point load acting in an exponentially graded transversely isotropic half-space. Later, the fundamental solutions of an exponentially graded transversely half-space subjected to uniform and non-uniform vertical patch loads were derived, respectively (Wang et al., 2006, 2009). Martin et al. (2002) presented a Green's function of three-dimensional exponentially graded elastic isotropic solids. Pan and Yang (2003) derived three-dimensional static interfacial Green's functions in anisotropic bi-materials. Chan et al. (2004) presented a general analytical technique for obtaining the Green's function for two-dimensional exponentially graded elastic isotropic solids. Kashatalyan and Rushchitsky (2009) presented two displacement potential functions in three-dimensional elasticity of a transversely isotropic inhomogeneous media with the assumption of a constant Poisson ratio and functionally graded Young and shear moduli. Eskandari-Ghadi et al. (2009a), by virtue of Hu–Nowacki–Lekhnitskii potentials, presented the elastostatic Green's functions for an arbitrary internal load in a transversely isotropic bi-material full-space. Eskandari-Ghadi et al. (2009b) and Eskandari-Ghadi et al. (2008) presented elastodynamic solution for a tri-material transversely isotropic full-space and a coating-subgrade under surface loads, respectively. Sallah et al. (2010) obtained the Green's function for the three-dimensional exponentially graded elasticity. Eskandari and Shodja (2010) derived an exponentially graded transversely isotropic half-space Green's functions under static different loads acting in an arbitrary depth of the medium. Eskandari-Ghadi and Amiri-Hezaveh (2014) presented the wave propagations in an exponentially graded transversely isotropic half-space with the aid of Fourier series and Hankel transform. Khojasteh et al. (2008a, 2013), with the aid of displacement potential functions and Hankel transform, derived three-dimensional Green's functions in transversely isotropic bi-material and tri-material full-space, respectively. Selvadurai and Katebi (2013) studied the axisymmetric response of an incompressible elastic half-space with the exponential variation of the linear elastic

shear modulus along with the depth. Noijen et al. (2012) presented a semi-analytic method for crack kinking analysis at isotropic bi-material interfaces. Kalantari et al. (2015) presented a theoretical formulation for the determination of the dynamic interaction of a horizontally loaded inextensible circular membrane embedded at the interface of a transversely isotropic bi-material full-space. Zhao et al. (2015) studied the Green's functions and extended displacement discontinuity method for interfacial cracks in three-dimensional transversely isotropic magneto-electro-elastic bi-materials. Li et al. (2015) presented axisymmetric problems of a penny-shaped crack at the interface of a bi-material under shear and compression.

This paper presents the Green's function of an exponentially graded transversely isotropic bi-material by utilizing the method presented by Khojasteh et al. (2013) together with the new displacement potential functions presented by Eskandari-Ghadi and Amiri-Hezaveh (2014). An arbitrary point load is assumed to be applied at the interface between two half-spaces. The formulation includes a complete set of transformed stress-potential and displacement-potential relations, together with the application of Fourier series and Hankel transform. The potential methods applied in this paper are the same with the pervious works (Eskandari-Ghadi, 2005, 2007; Khojasteh et al., 2008a,b, 2006; Ardeshir-Behrestaghi and Eskandari-Ghadi, 2009). The complete set of point-load Green's functions of displacements and stresses are given in terms of real-plane line-integral representations. The elastic constants of materials are assumed to vary exponentially along the axis of symmetry of the solid. The Green's functions are confirmed to be in exact agreement with the previous degenerate homogeneous transversely isotopic solutions by Khojasteh et al. (2008a) and the result by Eskandari and Shodja (2010) for a heterogeneous half-space. Also, the accuracy of the numerical result is confirmed by the comparison with the solution by Selvadurai and Katebi (2013) for the case of an incompressible heterogeneous isotropic solid. The effect of the material inhomogeneity is elucidated by several numerical displays. With the aid of the Green's functions presented herein, treatments by boundary-integral-equation formulations for the analysis of interfacial inclusions and cracks in bi-material FGMs can be facilitated, which can also be useful in a number of foundation–soil interaction and earthquake engineering problems.

GOVERNING EQUATIONS IN DISPLACEMENT POTENTIALS

The governing equilibrium equations for a vertically heterogeneous transversely isotropic elastic solid which its material properties vary exponentially along the axis of symmetry of the solid, in terms of displacements and in the absence of the body forces can be expressed as (Wang et al., 2003).

$$C_{11}\left(\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}\right)+C_{66}\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}+C_{44}\frac{\partial^2 u_r}{\partial z^2}-\left(C_{11}+C_{66}\right)\frac{1}{r^2}\frac{\partial u_\theta}{\partial \theta}$$
$$+\left(C_{12}+C_{66}\right)\frac{1}{r}\frac{\partial^2 u_\theta}{\partial r\partial \theta}+\left(C_{13}+C_{44}\right)\frac{\partial^2 u_z}{\partial r\partial z}+2\beta C_{44}\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right)=0,$$
$$C_{66}\left(\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r^2}\right)+C_{11}\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+C_{44}\frac{\partial^2 u_\theta}{\partial z^2}+\left(C_{11}+C_{66}\right)\frac{1}{r^2}\frac{\partial u_r}{\partial \theta}$$

$$+\left(C_{12}+C_{66}\right)\frac{1}{r}\frac{\partial^2 u_r}{\partial r\partial\theta}+\left(C_{13}+C_{44}\right)\frac{1}{r}\frac{\partial^2 u_z}{\partial r\partial z}+2\beta C_{44}\left(\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial\theta}\right)=0,$$
$$C_{44}\left(\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial\theta^2}\right)+C_{33}\frac{\partial^2 u_z}{\partial z^2}+\left(C_{13}+C_{44}\right)\left(\frac{\partial^2 u_r}{\partial r\partial z}+\frac{1}{r}\frac{\partial u_r}{\partial z}+\frac{1}{r}\frac{\partial^2 u_\theta}{\partial z\partial\theta}\right)$$
$$+2\beta C_{33}\frac{\partial u_z}{\partial z}+2\beta C_{13}\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)=0. \tag{1}$$

in a cylindrical coordinate system (r,θ,z), where z-axis is the axis of symmetry of the solid; u_r, u_θ and u_z: are the displacement components in the r, θ and z directions, respectively; C_{ij}: are elasticity constants of the solid corresponding to the depth $z=0$ with the relation $C_{66}=(C_{11}-C_{12})/2$; and β: is the exponential factor characterizing the degree of the material gradient in the z-direction. Here, it is assumed that the elastic constants of the medium vary exponentially in the z-direction as:

$$C_{ij}^{\mathrm{I}}(z)=C_{ij}^{\mathrm{I}}e^{2\beta^{\mathrm{I}}z} \tag{2}$$

in medium I, and

$$C_{ij}^{\mathrm{II}}(z)=C_{ij}^{\mathrm{II}}e^{2\beta^{\mathrm{II}}z} \tag{3}$$

in medium II, where C_{ij}^{I} and C_{ij}^{II}: indicate the z-independent elastic constants corresponding to the depth, with $z=0$. It is obvious that $\beta=0$ corresponds to the homogeneous transversely isotropic bi-materials. In order to solve the coupled partial differential Eq. (1) a set of complete potential functions F and χ introduced by Eskandari-Ghadi and Amiri-Hezaveh (2014) is used. These two potential functions are related to the displacement components, u_r, u_θ, and u_z as:

$$u_r(r,\theta,z)=-\alpha_3\frac{\partial^2 F}{\partial r\partial z}-2\alpha_2\beta\frac{\partial F}{\partial r}-\frac{1}{r}\frac{\partial\chi}{\partial\theta},$$
$$u_\theta(r,\theta,z)=-\frac{\alpha_3}{r}\frac{\partial^2 F}{\partial\theta\partial z}-\frac{2\alpha_2\beta}{r}\frac{\partial F}{\partial\theta}+\frac{\partial\chi}{\partial r},$$
$$u_z(r,\theta,z)=\left[(1+\alpha_1)\nabla_{r\theta}^2+\alpha_2\frac{\partial}{\partial z}\left(2\beta+\frac{\partial}{\partial z}\right)\right]F \tag{4}$$

where

$$\nabla_{r\theta}^2=\frac{\partial}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \tag{5}$$

and

$$\alpha_1=\frac{C_{12}+C_{66}}{C_{66}},$$
$$\alpha_2=\frac{C_{44}}{C_{66}}, \tag{6}$$
$$\alpha_3=\frac{C_{13}+C_{44}}{C_{66}}$$

The substitution of Eq. (4) into Eq. (1) result in two separate partial differential equations (PDEs) which are the governing equations for the potential function F and χ:

$$\left[\nabla_1^2\nabla_2^2-4\frac{\alpha_3-\alpha_2}{1+\alpha_1}\beta^2\nabla_{r\theta}^2\right]F=0 \tag{7}$$
$$\nabla_0^2\chi=0 \tag{8}$$

where

$$\nabla_i^2 = \nabla_{r\theta}^2 + \frac{1}{s_i^2}\left(\frac{\partial^2}{\partial z^2}\right)_\beta - \frac{\beta^2}{s_i^2} \tag{9}$$

$$i = 0,1,2$$

$$\left(\frac{\partial^2}{\partial z^2}\right)_\beta Y = \frac{\partial^2}{\partial z^2}\left(e^{\beta z} Y\right) \tag{10}$$

$$\alpha_4\alpha_2 s^4 + \left(\alpha_3^2 - \alpha_2^2 - (1+\alpha_1)\alpha_4\right)s^2 + \alpha_2(1+\alpha_1) = 0 \tag{11}$$

Here $s_0 = 1/\sqrt{\alpha_2}$ and s_1 and s_2: are the roots of the following equation, which in view of the positive-definiteness of the strain energy, are not zero or pure imaginary numbers (Lekhnitskii, 1963).

By virtue of Fourier expansion, with respect to the angular coordinate θ, one may express (Sneddon, 1951) Eq. (12) with similar expressions for the displacement and stress components.

$$\left[F(r,\theta,z), \chi(r,\theta,z)\right] = \sum_{m=-\infty}^{\infty}\left[F_m(r,z), \chi_m(r,z)\right]e^{im\theta} \tag{12}$$

Moreover, utilizing the mth order Hankel transform pair sufficiently regular function $f(\mathrm{r},\mathrm{z})$ with respect to the radial coordinate as (Sneddon, 1972) Eq. (13), the following ordinary differential equations for F and χ can be obtained (Eqs. (14-16)).

$$\tilde{f}^{\mathrm{m}}(\xi,\mathrm{z}) = \int_0^\infty f(\mathrm{r},\mathrm{z}) r J_m(r\xi) dr,$$
$$f(r,\mathrm{z}) = \int_0^\infty \tilde{f}^{\mathrm{m}}(\xi,\mathrm{z})\xi J_m(r\xi) d\xi \tag{13}$$

$$\left[\nabla_{1m}^2\nabla_{2m}^2 + 4\frac{\alpha_3-\alpha_2}{1+\alpha_1}\beta^2\xi^2\right]F_m^m = 0 \tag{14}$$

$$\nabla_{0m}^2 \chi_m^m = 0 \tag{15}$$

where

$$\nabla_{im}^2 = -\xi^2 + \frac{1}{s_i^2}\left(\frac{d^2}{dz^2}\right)_\beta - \frac{\beta^2}{s_i^2} \tag{16}$$

$$i = 0,1,2$$

The general solution of Eqs. (14) and (15) can be written as:

$$\tilde{F}_m^m(\xi,z) = e^{-\beta z}\left[A_m(\xi)e^{\lambda_1 z} + D_m(\xi)e^{-\lambda_2 z}\right] + \left[B_m(\xi)e^{\lambda_2 z} + E_m(\xi)e^{-\lambda_2 z}\right] \tag{17}$$

$$\tilde{\chi}_m^m(\xi,z) = e^{-\beta z}\left[C_m(\xi)e^{\lambda_3 z} + F_m(\xi)e^{-\lambda_3 z}\right] \tag{18}$$

where

$$\lambda_1 = \sqrt{a\xi^2 + b + \frac{1}{2}\sqrt{c\xi^4 + d\xi^2}},$$
$$\lambda_2 = \sqrt{a\xi^2 + b - \frac{1}{2}\sqrt{c\xi^4 + d\xi^2}}, \tag{19}$$
$$\lambda_3 = \sqrt{\beta^2 + s_0^2\xi^2},$$

$$a = \frac{1}{2}\left(s_1^2 + s_2^2\right),$$
$$b = \beta^2,$$
$$c = \left(s_1^2 - s_2^2\right)^2, \tag{20}$$
$$d = -16\frac{\alpha_3-\alpha_2}{\alpha_4}\beta^2.$$

It is worth mentioning at this point that in view of the positive-definiteness of the strain

energy, s_1 and s_2 are neither zero nor pure imaginary numbers (Lekhnitskii, 1963). The values of λ_1, λ_2, and λ_3 are selected such that $\mathrm{Re}(\lambda_j) \geq |\beta|$. Under these choices, the $e^{\lambda_1 z}$, $e^{\lambda_2 z}$ and $e^{\lambda_3 z}$ terms become inadmissible and are thus omitted in Eqs. (17) and (18), while $A_m,...,F_m$ are constants of integration to be determined from boundary conditions.

By means of Eq. (4) and the identities involving Hankel transforms, the transformed displacement-potential relations may be compactly expressed as:

$$
\begin{aligned}
&\tilde{u}_{zm}^{m} = \\
&\left[\alpha_2 \frac{\partial}{\partial z}\left(2\beta + \frac{\partial}{\partial z}\right) - \xi^2\left(1+\alpha_1\right)\right]\tilde{F}_m^m, \\
&\tilde{u}_{rm}^{m+1} + i\tilde{u}_{\theta m}^{m+1} = \\
&\alpha_3 \xi \frac{\partial \tilde{F}_m^m}{\partial z} + 2\alpha_2 \beta \xi \tilde{F}_m^m - i\,\xi \tilde{\chi}_m^m, \\
&\tilde{u}_{rm}^{m-1} - i\tilde{u}_{\theta m}^{m-1} = \\
&-\alpha_3 \xi \frac{\partial \tilde{F}_m^m}{\partial z} - 2\alpha_2 \beta \xi \tilde{F}_m^m - i\,\xi \tilde{\chi}_m^m
\end{aligned}
\tag{21}
$$

while the transformed stress-potential relationships can be written as:

$$
\begin{aligned}
&\tilde{\tau}_{zzm}^{m} = \left(C_{33}\frac{d}{dz}\left(\alpha_2 \frac{d}{dz}(2\beta + \frac{d}{dz}) - \xi^2\left(1+\alpha_1\right) + \alpha_3 \frac{C_{13}}{C_{33}}\xi^2\right)\tilde{F}_m^m + 2\alpha_2\beta C_{13}\xi^2 \tilde{F}_m^m\right)e^{2\beta z}, \\
&\tilde{\tau}_{zrm}^{m+1} + i\tilde{\tau}_{z\theta m}^{m+1} = \left(C_{44}\xi\left(-\alpha_2 \frac{d}{dz}(2\beta + \frac{d}{dz}) + \xi^2\left(1+\alpha_1\right) + 2\alpha_2\beta\frac{d}{dz} + \alpha_3\frac{d^2}{dz^2}\right)\tilde{F}_m^m - iC_{44}\xi\frac{d\tilde{\chi}_m^m}{dz}\right)e^{2\beta z}, \\
&\tilde{\tau}_{zrm}^{m-1} - i\tilde{\tau}_{z\theta m}^{m-1} = \left(-C_{44}\xi\left(-\alpha_2 \frac{d}{dz}(2\beta + \frac{d}{dz}) + \xi^2\left(1+\alpha_1\right) + 2\alpha_2\beta\frac{d}{dz} + \alpha_3\frac{d^2}{dz^2}\right)\tilde{F}_m^m - iC_{44}\xi\frac{d\tilde{\chi}_m^m}{dz}\right)e^{2\beta z}, \\
&\tilde{\tau}_{rrm}^{m} + 2C_{66}e^{2\beta z}\left\{(\frac{\tilde{u}_{r_m}}{r})^m + im(\frac{\tilde{u}_{\theta_m}}{r})^m\right\} = \\
&e^{2\beta z}\left\{\frac{d}{dz}\left[\alpha_3 C_{11}\xi^2 - C_{13}\xi^2(1+\alpha_1) + \alpha_2 C_{13}\frac{d}{dz}(2\beta + \frac{d}{dz})\right] + 2\alpha_2\beta C_{11}\xi^2\right\}\tilde{F}_m^m, \\
&\tilde{\tau}_{\theta\theta m}^{m} - 2C_{66}e^{2\beta z}\left\{(\frac{\tilde{u}_{r_m}}{r})^m + im(\frac{\tilde{u}_{\theta_m}}{r})^m\right\} = \\
&e^{2\beta z}\left\{\frac{d}{dz}\left[\alpha_3 C_{12}\xi^2 - C_{13}\xi^2(1+\alpha_1) + \alpha_2 C_{13}\frac{d}{dz}(2\beta + \frac{d}{dz})\right] + 2\alpha_2\beta C_{12}\xi^2\right\}\tilde{F}_m^m, \\
&\tilde{\tau}_{r\theta m}^{m} + 2C_{66}e^{2\beta z}\left\{(\frac{\tilde{u}_{r_m}}{r})^m - im(\frac{\tilde{u}_{\theta_m}}{r})^m\right\} = -C_{66}e^{2\beta z}\xi^2\tilde{\chi}_m^m
\end{aligned}
\tag{22}
$$

With the aid of Eqs. (17), (18), (20) and (22), the imposition of the loading, interfacial and regularity conditions associated with a bi-material FGM is greatly facilitated, as will be illustrated in the ensuing sections.

STATEMENT OF THE PROBLEM

Consider the physical domain of interest to be composed of two dissimilar transversely isotropic half-spaces with exponential variation of elastic moduli along its axis of symmetry and fully bonded across the plane $z=0$. Figure 1 depicts a cylindrical coordinate system (r,θ,z) in such a way that z-axis is normal to the horizontal interface of the domain, therefore it serves as the common axis of symmetry of both media. Let the upper half-space ($z<0$) be occupied by medium I and the lower half-space ($z>0$) be occupied by medium II. The elasticity constants of the upper half-space (referred to as medium I) will be denoted as $C_{ij}^{\mathrm{I}}(z)$ and those of the lower half-space (referred to as medium II) as $C_{ij}^{\mathrm{II}}(z)$. Hereafter, the superscripts I and II denotes the quantities in media I and II, respectively. An arbitrary interfacial traction is assumed to be distributed on a finite region $\prod_0$ which is located at the interface of the domain. The action of this arbitrary distributed source can be represented as a set of prescribed stress discontinuities across the interface (Khojasteh et al., 2008a), i.e.

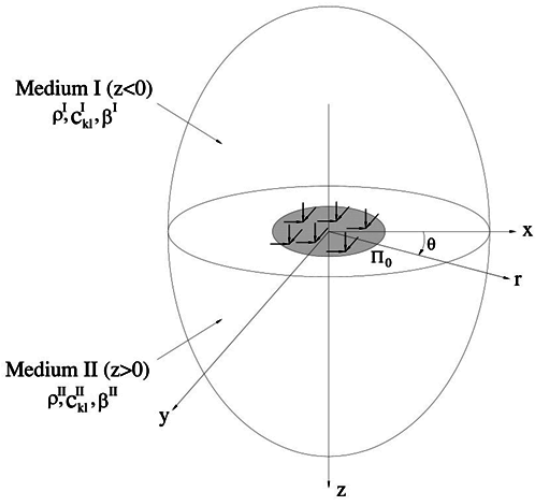

Fig. 1. Functionally graded transversely isotropic bi-material full-space under arbitrary interfacial load

$$
\begin{aligned}
&\tau_{zr}(r,\theta,0^-)-\tau_{zr}(r,\theta,0^+)=\\
&\begin{cases}P(r,\theta), & (r,\theta)\in\prod_0\\ 0, & (r,\theta)\notin\prod_0\end{cases}\\
&\tau_{z\theta}(r,\theta,0^-)-\tau_{z\theta}(r,\theta,0^+)=\\
&\begin{cases}Q(r,\theta), & (r,\theta)\in\prod_0\\ 0, & (r,\theta)\notin\prod_0\end{cases}\\
&\tau_{zz}(r,\theta,0^-)-\tau_{zz}(r,\theta,0^+)=\\
&\begin{cases}R(r,\theta), & (r,\theta)\in\prod_0\\ 0, & (r,\theta)\notin\prod_0\end{cases}
\end{aligned}
\tag{23}
$$

where $P(r,\theta)$, $Q(r,\theta)$ and $R(r,\theta)$: are the specified interfacial traction distributions in radial, angular, and axial directions, respectively. In consistency with the regularity condition at infinity, the general solutions (17) and (18) for F and χ can be rearranged as Eqs. (24) and (25), in medium I, and Eqs. (26) and (27), in medium II.

$$\widetilde{F}_m^m(\xi,z)=e^{\beta^{\mathrm{I}}z}[A_m^{\mathrm{I}}(\xi)e^{\lambda_1^{\mathrm{I}}z}+B_m^{\mathrm{I}}(\xi)e^{\lambda_2^{\mathrm{I}}z}] \tag{24}$$

$$\widetilde{\chi}_m^m(\xi,z)=e^{\beta^{\mathrm{I}}z}[C_m^{\mathrm{I}}(\xi)e^{\lambda_3^{\mathrm{I}}z}] \tag{25}$$

$$\tilde{F}_m^m(\xi,z)= e^{-\beta^{II}z}\left[A_m^{II}(\xi)e^{-\lambda_1^{II}z}+B_m^{II}(\xi)e^{-\lambda_2^{II}z}\right] \quad (26)$$

$$\tilde{\chi}_m^m(\xi,z)=e^{-\beta^{II}z}[C_m^{II}(\xi)e^{-\lambda_3^{II}z}] \quad (27)$$

where $A_m^{I},\ldots,\ C_m^{II}$: are the integration constants to be determined using the boundary conditions. For the general exponentially graded bi-material full-space problem of interest, an exact solution therefore requires the determination of six coefficients. With the aid of Eqs. (21) and (22), interfacial traction conditions (Eq. (23)), together with the continuity of displacements across the interface provide six equations required for the solution of the six unknown coefficients $A_m^{I},\ldots,\ C_m^{II}$. Substituting the result into Eq. (21) gives the transformed Fourier components of the displacement field in the form of Eq. (28).

$$\begin{aligned}
&\tilde{u}_{zm}^m = \\
&\Omega_1(z,\xi)\left(\frac{X_m-Y_m}{2C_{44}}\right)+\Omega_2(z,\xi)\left(\frac{Z_m}{C_{44}}\right),\\
&\tilde{u}_{rm}^{m+1}+i\tilde{u}_{\theta m}^{m+1} = \\
&\gamma_1(z,\xi)\left(\frac{X_m-Y_m}{2C_{44}}\right)+\\
&\gamma_2(z,\xi)\left(\frac{X_m+Y_m}{2C_{44}}\right)-\gamma_3(z,\xi)\left(\frac{Z_m}{C_{44}}\right),\\
&\tilde{u}_{rm}^{m-1}-i\tilde{u}_{\theta m}^{m-1} = \\
&\gamma_1(z,\xi)\left(\frac{X_m-Y_m}{2C_{44}}\right)+\\
&\gamma_2(z,\xi)\left(\frac{X_m+Y_m}{2C_{44}}\right)+\gamma_3(z,\xi)\left(\frac{Z_m}{C_{44}}\right)
\end{aligned} \quad (28)$$

Analogously, Eq. (22) yields the transformed Fourier components of the stress field as:

$$\begin{aligned}
&\tilde{\tau}_{zzm}^m=\left[\left\{C_{33}\frac{d\Omega_1}{dz}-C_{13}\xi\gamma_1\right\}\left(\frac{X_m-Y_m}{2C_{44}}\right)+\left\{C_{33}\frac{d\Omega_2}{dz}-C_{13}\xi\gamma_3\right\}\left(\frac{Z_m}{C_{44}}\right)\right]e^{2\beta|z|},\\
&\tilde{\tau}_{zrm}^{m+1}+i\tilde{\tau}_{z\theta m}^{m+1}=\left[-C_{44}\left\{\xi\Omega_1+\frac{d\gamma_1}{dz}\right\}\left(\frac{X_m-Y_m}{2C_{44}}\right)+C_{44}\left\{\frac{d\gamma_2}{dz}\right\}\left(\frac{X_m+Y_m}{2C_{44}}\right)-C_{44}\left\{\xi\Omega_2+\frac{d\gamma_3}{dz}\right\}\left(\frac{Z_m}{C_{44}}\right)\right]e^{2\beta|z|},\\
&\tilde{\tau}_{zrm}^{m-1}-i\tilde{\tau}_{z\theta m}^{m-1}=\left[C_{44}\left\{\xi\Omega_1+\frac{d\gamma_1}{dz}\right\}\left(\frac{X_m-Y_m}{2C_{44}}\right)+C_{44}\left\{\frac{d\gamma_2}{dz}\right\}\left(\frac{X_m+Y_m}{2C_{44}}\right)+C_{44}\left\{\xi\Omega_2+\frac{d\gamma_3}{dz}\right\}\left(\frac{Z_m}{C_{44}}\right)\right]e^{2\beta|z|},\\
&\tilde{\tau}_{rrm}^m+2C_{66}e^{2\beta z}\{(\frac{u_{r_m}}{r})^m+im(\frac{u_{\theta_m}}{r})^m\}=\left[\left\{C_{13}\frac{d\Omega_1}{dz}-C_{11}\xi\gamma_1\right\}\left(\frac{X_m-Y_m}{2C_{44}}\right)+\left\{C_{13}\frac{d\Omega_2}{dz}-C_{11}\xi\gamma_3\right\}\left(\frac{Z_m}{C_{44}}\right)\right]e^{2\beta|z|},\\
&\tilde{\tau}_{\theta\theta m}^m-2C_{66}e^{2\beta z}\{(\frac{u_{r_m}}{r})^m+im(\frac{u_{\theta_m}}{r})^m\}=\left[\left\{C_{13}\frac{d\Omega_1}{dz}-C_{12}\xi\gamma_1\right\}\left(\frac{X_m-Y_m}{2C_{44}}\right)+\left\{C_{13}\frac{d\Omega_2}{dz}-C_{12}\xi\gamma_3\right\}\left(\frac{Z_m}{C_{44}}\right)\right]e^{2\beta|z|},\\
&\tilde{\tau}_{r\theta m}^m+2C_{66}e^{2\beta z}\{(\frac{u_{r_m}}{r})^m-im(\frac{u_{\theta_m}}{r})^m\}=\left[iC_{66}\xi\gamma_2\left(\frac{X_m+Y_m}{2C_{44}}\right)\right]e^{2\beta|z|}
\end{aligned} \quad (29)$$

In the aforementioned equations,

$$\gamma_1(z,\xi)=-\frac{1}{S(\xi)}\left(\varphi_1 l_2 e^{-(\lambda_1+\beta)|z|}-\varphi_2 l_1 e^{-(\lambda_2+\beta)|z|}\right),$$
$$\gamma_2(z,\xi)=\frac{C_{44}}{C_{44}^{\mathrm{I}}(\lambda_3^{\mathrm{I}}+\beta^{\mathrm{I}})+C_{44}^{\mathrm{II}}(\lambda_3^{\mathrm{II}}+\beta^{\mathrm{II}})}\left(e^{-(\lambda_3+\beta)|z|}\right),$$
$$\gamma_3(z,\xi)=-\frac{C_{44}\xi}{S(\xi)}\frac{z}{|z|}\left(\varphi_1 k_2 e^{-(\lambda_1+\beta)|z|}-\varphi_2 k_1 e^{-(\lambda_2+\beta)|z|}\right), \tag{30}$$
$$\Omega_1(z,\xi)=-\frac{1}{\xi S(\xi)}\frac{z}{|z|}\left(\vartheta_1 l_2 e^{-(\lambda_1+\beta)|Z|}-\vartheta_2 l_1 e^{-(\lambda_2+\beta)|z|}\right),$$
$$\Omega_2(z,\xi)=\frac{C_{44}}{S(\xi)}\left(\vartheta_1 k_2 e^{-(\lambda_1+\beta)|z|}-\vartheta_2 k_1 e^{-(\lambda_2+\beta)|z|}\right)$$

here,

$$X_m=\widetilde{P}_m^{m-1}(\xi)-i\widetilde{Q}_m^{m-1}(\xi);\quad Y_m=\widetilde{P}_m^{m+1}(\xi)+i\widetilde{Q}_m^{m+1}(\xi);\quad Z_m=\widetilde{R}_m^{m}(\xi),$$
$$\vartheta_i^{\mathrm{I}}=-2\beta^{\mathrm{I}}\alpha_2^{\mathrm{I}}(\lambda_i^{\mathrm{I}}+\beta^{\mathrm{I}})+\alpha_2^{\mathrm{I}}(\lambda_i^{\mathrm{I}}+\beta^{\mathrm{I}})^2-\xi^2(1+\alpha_1^{\mathrm{I}})$$
$$\vartheta_i^{\mathrm{II}}=-2\beta^{\mathrm{II}}\alpha_2^{\mathrm{II}}(\lambda_i^{\mathrm{II}}+\beta^{\mathrm{II}})+\alpha_2^{\mathrm{II}}(\lambda_i^{\mathrm{II}}+\beta^{\mathrm{II}})^2-\xi^2(1+\alpha_1^{\mathrm{II}})$$
$$\varphi_i^{\mathrm{I}}=2\beta^{\mathrm{I}}\alpha_2^{\mathrm{I}}-\alpha_3^{\mathrm{I}}(\lambda_i^{\mathrm{I}}+\beta^{\mathrm{I}})$$
$$\varphi_i^{\mathrm{II}}=2\beta^{\mathrm{II}}\alpha_2^{\mathrm{II}}-\alpha_3^{\mathrm{II}}(\lambda_i^{\mathrm{II}}+\beta^{\mathrm{II}}) \tag{31}$$
$$\eta_i^{\mathrm{I}}=\left[(\alpha_3^{\mathrm{I}}-\alpha_2^{\mathrm{I}})(\lambda_i^{\mathrm{I}}+\beta^{\mathrm{I}})^2+\xi^2(1+\alpha_1^{\mathrm{I}})\right]C_{44}^{\mathrm{I}}$$
$$\eta_i^{\mathrm{II}}=\left[(\alpha_3^{\mathrm{II}}-\alpha_2^{\mathrm{II}})(\lambda_i^{\mathrm{II}}+\beta^{\mathrm{II}})^2+\xi^2(1+\alpha_1^{\mathrm{II}})\right]C_{44}^{\mathrm{II}}$$
$$\upsilon_i^{\mathrm{I}}=-C_{33}^{\mathrm{I}}(\lambda_i^{\mathrm{I}}+\beta^{\mathrm{I}})\vartheta_i^{\mathrm{I}}+C_{13}^{\mathrm{I}}\xi^2\varphi_i^{\mathrm{I}}$$
$$\upsilon_i^{\mathrm{II}}=-C_{33}^{\mathrm{II}}(\lambda_i^{\mathrm{II}}+\beta^{\mathrm{II}})\vartheta_i^{\mathrm{II}}+C_{13}^{\mathrm{II}}\xi^2\varphi_i^{\mathrm{II}},\qquad i=1,2$$

In expressions (28-31), C_{ij}: are the piecewise constant elastic moduli which are given by Eq. (32). in addition, l_i and k_i: are functions defined as:

$$C_{ij}=\begin{cases}C_{ij}^{\mathrm{I}}, z<0\\ \\ C_{ij}^{\mathrm{II}}, z>0\end{cases} \tag{32}$$

$$k_i^{\mathrm{I}}=\vartheta_i^{\mathrm{I}}\left(\eta_2^{\mathrm{II}}\varphi_1^{\mathrm{II}}-\eta_1^{\mathrm{II}}\varphi_2^{\mathrm{II}}\right)+\varphi_i^{\mathrm{I}}\left(\eta_2^{\mathrm{II}}\vartheta_1^{\mathrm{II}}-\eta_1^{\mathrm{II}}\vartheta_2^{\mathrm{II}}\right)+\eta_i^{\mathrm{I}}\left(\varphi_2^{\mathrm{II}}\vartheta_1^{\mathrm{II}}-\varphi_1^{\mathrm{II}}\vartheta_2^{\mathrm{II}}\right)$$
$$l_i^{\mathrm{I}}=\vartheta_i^{\mathrm{I}}\left(\upsilon_1^{\mathrm{II}}\varphi_2^{\mathrm{II}}-\upsilon_2^{\mathrm{II}}\varphi_1^{\mathrm{II}}\right)+\varphi_i^{\mathrm{I}}\left(\upsilon_1^{\mathrm{II}}\vartheta_2^{\mathrm{II}}-\upsilon_2^{\mathrm{II}}\vartheta_1^{\mathrm{II}}\right)-\upsilon_i^{\mathrm{I}}\left(\varphi_1^{\mathrm{II}}\vartheta_2^{\mathrm{II}}-\varphi_2^{\mathrm{II}}\vartheta_1^{\mathrm{II}}\right)$$
$$k_i^{\mathrm{II}}=\vartheta_i^{\mathrm{II}}\left(\eta_2^{\mathrm{I}}\varphi_1^{\mathrm{I}}-\eta_1^{\mathrm{I}}\varphi_2^{\mathrm{I}}\right)+\varphi_i^{\mathrm{II}}\left(\eta_2^{\mathrm{I}}\vartheta_1^{\mathrm{I}}-\eta_1^{\mathrm{I}}\vartheta_2^{\mathrm{I}}\right)+\eta_i^{\mathrm{II}}\left(\varphi_2^{\mathrm{I}}\vartheta_1^{\mathrm{I}}-\varphi_1^{\mathrm{I}}\vartheta_2^{\mathrm{I}}\right) \tag{33}$$
$$l_i^{\mathrm{II}}=\vartheta_i^{\mathrm{II}}\left(\upsilon_1^{\mathrm{I}}\varphi_2^{\mathrm{I}}-\upsilon_2^{\mathrm{I}}\varphi_1^{\mathrm{I}}\right)+\varphi_i^{\mathrm{II}}\left(\upsilon_1^{\mathrm{I}}\vartheta_2^{\mathrm{I}}-\upsilon_2^{\mathrm{I}}\vartheta_1^{\mathrm{I}}\right)-\upsilon_i^{\mathrm{II}}\left(\varphi_1^{\mathrm{I}}\vartheta_2^{\mathrm{I}}-\varphi_2^{\mathrm{I}}\vartheta_1^{\mathrm{I}}\right),\qquad i=1,2$$

Also,

$$
\begin{aligned}
S(\xi) = &\left(\eta_2^{\mathrm{II}}\varphi_1^{\mathrm{II}} - \eta_1^{\mathrm{II}}\varphi_2^{\mathrm{II}}\right)\left(\upsilon_1^{\mathrm{I}}\vartheta_2^{\mathrm{I}} - \upsilon_2^{\mathrm{I}}\vartheta_1^{\mathrm{I}}\right) + \left(\eta_2^{\mathrm{I}}\varphi_1^{\mathrm{I}} - \eta_1^{\mathrm{I}}\varphi_2^{\mathrm{I}}\right)\left(\upsilon_1^{\mathrm{II}}\vartheta_2^{\mathrm{II}} - \upsilon_2^{\mathrm{II}}\vartheta_1^{\mathrm{II}}\right) \\
&+ \left(\eta_2^{\mathrm{II}}\upsilon_1^{\mathrm{II}} - \eta_1^{\mathrm{II}}\upsilon_2^{\mathrm{II}}\right)\left(\vartheta_1^{\mathrm{I}}\varphi_2^{\mathrm{I}} - \vartheta_2^{\mathrm{I}}\varphi_1^{\mathrm{I}}\right) + \left(\eta_1^{\mathrm{II}}\vartheta_2^{\mathrm{II}} - \eta_2^{\mathrm{II}}\vartheta_1^{\mathrm{II}}\right)\left(\varphi_1^{\mathrm{I}}\upsilon_2^{\mathrm{I}} - \varphi_2^{\mathrm{I}}\upsilon_1^{\mathrm{I}}\right) \\
&+ \left(\eta_1^{\mathrm{I}}\vartheta_2^{\mathrm{I}} - \eta_2^{\mathrm{I}}\vartheta_1^{\mathrm{I}}\right)\left(\varphi_1^{\mathrm{II}}\upsilon_2^{\mathrm{II}} - \varphi_2^{\mathrm{II}}\upsilon_1^{\mathrm{II}}\right) + \left(\eta_2^{\mathrm{I}}\upsilon_1^{\mathrm{I}} - \eta_1^{\mathrm{I}}\upsilon_2^{\mathrm{I}}\right)\left(\varphi_2^{\mathrm{II}}\vartheta_1^{\mathrm{II}} - \varphi_1^{\mathrm{II}}\vartheta_2^{\mathrm{II}}\right)
\end{aligned}
\tag{34}
$$

Upon setting $\beta^{\mathrm{I}} = \beta^{\mathrm{II}} = 0$, $S(\xi)$ leads exactly to the same result obtained in Khojasteh et al. (2008a) for the homogeneous transversely isotropic bi-materials in the static condition.

On substituting the inverted Fourier components of the displacements and stresses into the corresponding angular Eigen function expansion, the desired formal solution to the general exponentially graded bi-material problem under consideration can be obtained.

POINT-LOAD GREEN'S FUNCTION

In the previous sections, the general solution has been formulated for an arbitrary source distributed on the plane $z = 0$. In order to obtain the point-load Green's functions, which are useful for the integral formulations of boundary value problems, one may define the distributed traction source may be defined as (Khojasteh et al., 2008a):

$$
\begin{aligned}
f_{\mathrm{h}}(r,\theta,z) &= F_{\mathrm{h}}\frac{\delta(r)}{2\pi r}\delta(z)e_h, \\
f_v(r,\theta,z) &= F_v\frac{\delta(r)}{2\pi r}\delta(z)e_z
\end{aligned}
\tag{35}
$$

in the horizontal and vertical directions, respectively. Where δ: is the one-dimensional Dirac delta function, e_h: is the unit horizontal vector in the $\theta = \theta_0$ direction given by (Figure 2).

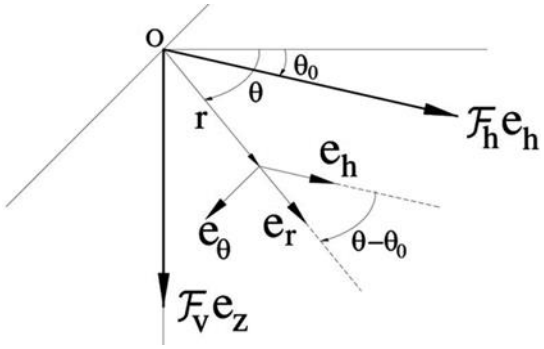

Fig. 2. Vertical and horizontal point-load configurations

$$
e_h = e_r\cos(\theta-\theta_0) - e_\theta\sin(\theta-\theta_0) \tag{36}
$$

e_r, e_θ, and e_z: are the unit vector in the radial, angular, and vertical directions, respectively; and F_h and F_v: are the point-load magnitudes. By virtue of the angular expansions of the stress discontinuities across the plane $z = 0$ and the orthogonality of the angular Eigen functions $\left\{e^{im\theta}\right\}_{m=-\infty}^{\infty}$, the expression of Eq. (37) can be found for point-load in Eq. (35).

$$
\begin{aligned}
&P_{\pm1}(r) = \pm F_h e^{\mp i\theta_0}\frac{\delta(r)}{4\pi r}, \\
&P_m(r) = 0, \qquad m \neq \pm 1, \\
&Q_{\pm1}(r) = \pm F_h e^{\mp i\theta_0}\frac{\delta(r)}{4\pi r}, \\
&Q_m(r) = 0, \qquad m \neq \pm 1, \\
&R_0(r) = F_v\frac{\delta(r)}{2\pi r}, \\
&R_m(r) = 0, \qquad m \neq 0
\end{aligned}
\tag{37}
$$

Subsequently, the transformed loading coefficients X_m, Y_m and Z_m can be expressed as (Khojaste et al., 2008b):

$$X_1 = \frac{F_h}{2\pi} e^{-i\theta_0}, \quad X_m = 0, \quad m \neq 1,$$
$$Y_{-1} = \frac{F_h}{2\pi} e^{i\theta_0}, \quad Y_m = 0, \quad m \neq -1, \quad (38)$$
$$Z_0 = \frac{F_v}{2\pi}, \quad Z_m = 0, \quad m \neq 0$$

Upon inverting the transformed Eqs. (28) and (29) and using Eq. (38), the displacement and stress point-load Green's functions may be written as:

$$\hat{u}_r^*(r,\theta,z;s) = \frac{-1}{4\pi C_{44}}\{2F_v\int_0^\infty \gamma_3 \xi J_1(r\xi)d\xi - F_h\cos(\theta-\theta_0) \times\left(-\int_0^\infty(\gamma_1+\gamma_2)\xi J_0(r\xi)d\xi - \int_0^\infty(\gamma_1-\gamma_2)\xi J_2(r\xi)d\xi\right)\},$$

$$\hat{u}_\theta^*(r,\theta,z;s) = \frac{-1}{4\pi C_{44}}\{F_h\cos(\theta-\theta_0)\times\left(-\int_0^\infty(\gamma_1+\gamma_2)\xi J_0(r\xi)d\xi - \int_0^\infty(\gamma_1-\gamma_2)\xi J_2(r\xi)d\xi\right)\},$$

$$\hat{u}_z^*(r,\theta,z;s) = \frac{1}{2\pi C_{44}}\{F_v\int_0^\infty \Omega_2 \xi J_0(r\xi)d\xi + F_h\cos(\theta-\theta_0)\times\int_0^\infty \Omega_1 \xi J_1(r\xi)d\xi\},$$

$$\hat{\tau}_{zz}^*(r,\theta,z;s) = \frac{e^{2\beta|z|}}{2\pi C_{44}}\{F_v\int_0^\infty\left(C_{33}\frac{d\Omega_2}{dz} - C_{13}\xi\gamma_3\right)\xi J_0(r\xi)d\xi + F_h\cos(\theta-\theta_0)\times\left(\int_0^\infty(C_{33}\frac{d\Omega_1}{dz} - C_{13}\xi\gamma_1)\xi J_1(r\xi)d\xi\right)\},$$

$$\hat{\tau}_{zr}^*(r,\theta,z;s) = \frac{-e^{2\beta|z|}}{4\pi}\{2F_v\int_0^\infty\left(\xi\Omega_2 + \frac{d\gamma_3}{dz}\right)\xi J_1(r\xi)d\xi - F_h\cos(\theta-\theta_0) \times\left[\int_0^\infty(\xi\Omega_1 + \frac{d\gamma_1}{dz} + \frac{d\gamma_2}{dz})\xi J_0(r\xi)d\xi - \int_0^\infty(\xi\Omega_1 + \frac{d\gamma_1}{dz} - \frac{d\gamma_2}{dz})\xi J_2(r\xi)d\xi\right]\}, \quad (39)$$

$$\hat{\tau}_{z\theta}^*(r,\theta,z;s) = \frac{-e^{2\beta|z|}}{4\pi}\{F_h\sin(\theta-\theta_0) \times\left[\int_0^\infty(\xi\Omega_1 + \frac{d\gamma_1}{dz} + \frac{d\gamma_2}{dz})\xi J_0(r\xi)d\xi - \int_0^\infty(\xi\Omega_1 + \frac{d\gamma_1}{dz} - \frac{d\gamma_2}{dz})\xi J_2(r\xi)d\xi\right]\},$$

$$\hat{\tau}_{rr}^*(r,\theta,z;s) + \frac{2C_{66}}{r}\{\hat{u}_r^* + i(\hat{u}_{\theta_1}^* e^{i\theta} - \hat{u}_{\theta_{-1}}^* e^{-i\theta})\} = \frac{e^{2\beta|z|}}{2\pi C_{44}}\{F_v\int_0^\infty\left(C_{13}\frac{d\Omega_2}{dz} - C_{11}\xi\gamma_3\right)\xi J_0(r\xi)d\xi + F_h\cos(\theta-\theta_0)\times\left(\int_0^\infty(C_{13}\frac{d\Omega_1}{dz} - C_{11}\xi\gamma_1)\xi J_1(r\xi)d\xi\right)\},$$

$$\hat{\tau}_{\theta\theta}^*(r,\theta,z;s) - \frac{2C_{66}}{r}\{\hat{u}_r^* + i(\hat{u}_{\theta_1}^* e^{i\theta} - \hat{u}_{\theta_{-1}}^* e^{-i\theta})\} = \frac{e^{2\beta|z|}}{2\pi C_{44}}\{F_v\int_0^\infty\left(C_{13}\frac{d\Omega_2}{dz} - C_{12}\xi\gamma_3\right)\xi J_0(r\xi)d\xi + F_h\cos(\theta-\theta_0)\times\left(\int_0^\infty(C_{13}\frac{d\Omega_1}{dz} - C_{12}\xi\gamma_1)\xi J_1(r\xi)d\xi\right)\},$$

$$\hat{\tau}_{r\theta}^*(r,\theta,z;s) + \frac{2C_{66}}{r}\{\hat{u}_\theta^* - i(\hat{u}_{r_1}^* e^{i\theta} - \hat{u}_{r_{-1}}^* e^{-i\theta})\} = \frac{e^{2\beta|z|}}{2\pi\alpha_2}\{F_h\sin(\theta-\theta_0)\times\int_0^\infty \gamma_2\xi^2\xi J_1(r\xi)d\xi\}.$$

In the aforementioned expressions, the symbols "$\hat{u}_i^*$" and "$\hat{\tau}_{ik}^*$" $(i,k=r,\theta,z)$ denote, respectively, the displacement and stress Green's functions, with the superscript "*" denoting the direction of the point-load upon appropriate specifications of F_h, F_v, and θ_0 in Eq. (35).

SPECIAL CASES

In this section, the examination of two degenerate cases is of relevance: (i) when the exponential variation of the material properties for both of half-spaces is zero, i.e. $\beta^{\mathrm{I}}=\beta^{\mathrm{II}}=0$, and (ii) when the modulus of the upper medium $z<0$ is zero, such degenerate forms of the general formulation correspond to the homogeneous bi-materials and the exponentially graded half-space solutions, respectively.

Homogeneous Transversely Isotropic Bi-Materials

Upon setting $\beta^{\mathrm{I}}=\beta^{\mathrm{II}}=0$, the corresponding solutions corresponds exactly to that of Khojasteh et al. (2008a) for the homogeneous transversely isotropic bi-materials in the static condition.

Exponentially Graded Half-Space

Adopting $C_{ij}^{\mathrm{I}}\to 0$, $\beta^{\mathrm{I}}\to 0$, $C_{ij}^{\mathrm{II}}=C_{ij}$, and $\beta^{\mathrm{II}}=\beta$, degenerates the Kernel functions (28) to the following expressions for the exponentially graded half-space problem.

$$\gamma_1(z,\xi)=\frac{1}{I(\xi)}\left(\phi_1\upsilon_2 e^{-(\lambda_1+\beta)|z|}-\phi_2\upsilon_1 e^{-(\lambda_2+\beta)|z|}\right)$$

$$\gamma_2(z,\xi)=-\frac{1}{C_{44}(\lambda_3+\beta)}e^{-(\lambda_3+\beta)z}$$

$$\gamma_3(z,\xi)=-\frac{1}{I(\xi)}\left(\phi_1\eta_2 e^{-(\lambda_1+\beta)|z|}-\phi_2\eta_1 e^{-(\lambda_2+\beta)|z|}\right)$$

$$\Omega_1(z,\xi)=-\frac{1}{I(\xi)}\left(\vartheta_1\upsilon_2 e^{-(\lambda_1+\beta)|z|}-\vartheta_2\upsilon_1 e^{-(\lambda_2+\beta)|z|}\right)$$

$$\Omega_2(z,\xi)=\frac{1}{I(\xi)}\left(\vartheta_1\eta_2 e^{-(\lambda_1+\beta)|z|}-\vartheta_2\eta_1 e^{-(\lambda_2+\beta)|z|}\right) \tag{40}$$

where

$$I(\xi)=\eta_2\upsilon_1-\eta_1\upsilon_2 \tag{41}$$

The substitution of the Kernel functions (40) into Eqs. (38) and (39) yields the static point-load Green's functions for exponentially graded half-space. In this case, vertical displacements arising from vertical point-load are exactly the same as the results given in Eskandari and Shodja (2010).

NUMERICAL EVALUATION

In the previous section, the point-load Green's functions were expressed in terms of one-dimensional semi-infinite integrals. As the integrations generally cannot be carried out in exact closed-forms (Apsel and Luco, 1983; Pak and Guzina, 2002; Rajapakse and Wang, 1993; Rahimian et al., 2007; Khojasteh et al., 2011), a numerical quadrature technique is usually adopted in such evaluations. In order to accurately evaluate integrals accurately, it of importance to pay attention to the oscillatory nature of the integrands because of the presence of Bessel functions. In the present work, an adaptive quadrature rule demonstrated in Rahimian et al. (2007) has been incorporated and successfully used. Several numerical examples were carried out

to compare the present solution with existing numerical solutions, with satisfactory results.

In the first step for the numerical verification, the vertical displacement Green's functions were determined in the static case for the homogeneous bi-materials and compared with the static result given in Khojasteh et al. (2008a) and shown in Figure 3. Properties of materials have been given in Table 2. From the results it was observed that the solutions were identical. In addition, Figure 4 represents the numerical solution for the case of vertical displacement due to the vertical point-load at the surface for the exponentially graded half-space, with the inhomogeneity factor $\beta = 0.5$ and the solutions are shown to correspond with the solution in Eskandari and Shodja (2010), whose material properties are $C_{11} = 41.3$, $C_{12} = 14.7$, $C_{13} = 10.1$, $C_{33} = 36.2$, $C_{44} = 10.0$. For the case of isotropic solution, the vertical displacement of an incompressible isotropic solid along the z-axis and along the r-axis at the interface with the initial shear modulus $G_0 = 3.33$ are depicted in Figures 5 and 6 and corresponded with the results obtained by Selvadurai and Katebi (2013), with both results having excellent agreement. The material constants for an isotropic medium can be reduced to $C_{11} = C_{33} = \lambda + 2\mu$, $C_{12} = C_{13} = \lambda$, $C_{44} = C_{66} = \mu$, where λ and μ are the Lame's constants of the isotropic solid.

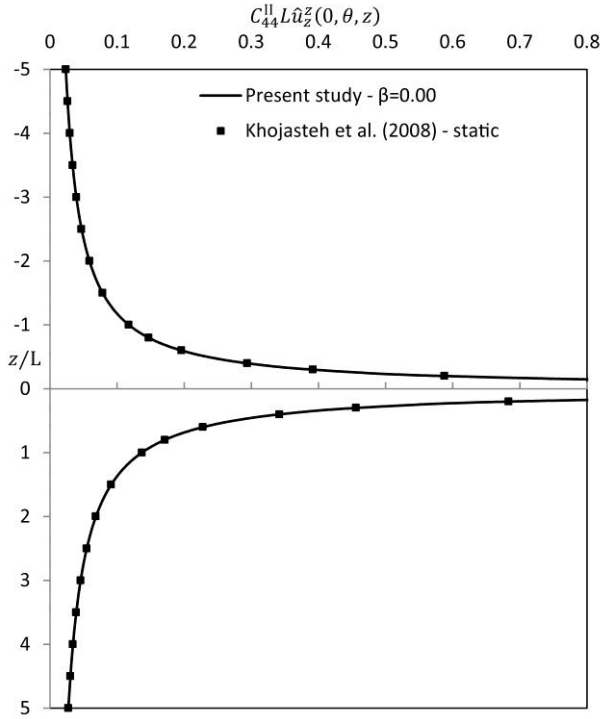

Fig. 3. Displacement Green's function $\hat{u}_z^z$ along z-axis obtained in this study compared with result reported by Khojaste et al. (2008) for static case $\omega_0 = 0$ and $\beta = 0$

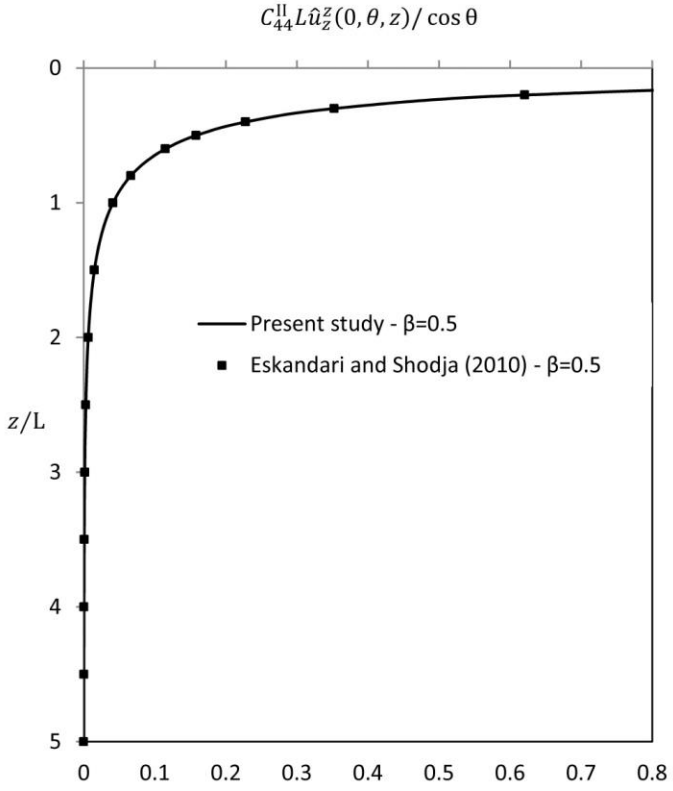

Fig. 4. Displacement Green's function $\hat{u}_z^z$ along z -axis obtained in this study compared with result reported by Eskandari and Shodja (2010) for $\beta = 0.5$

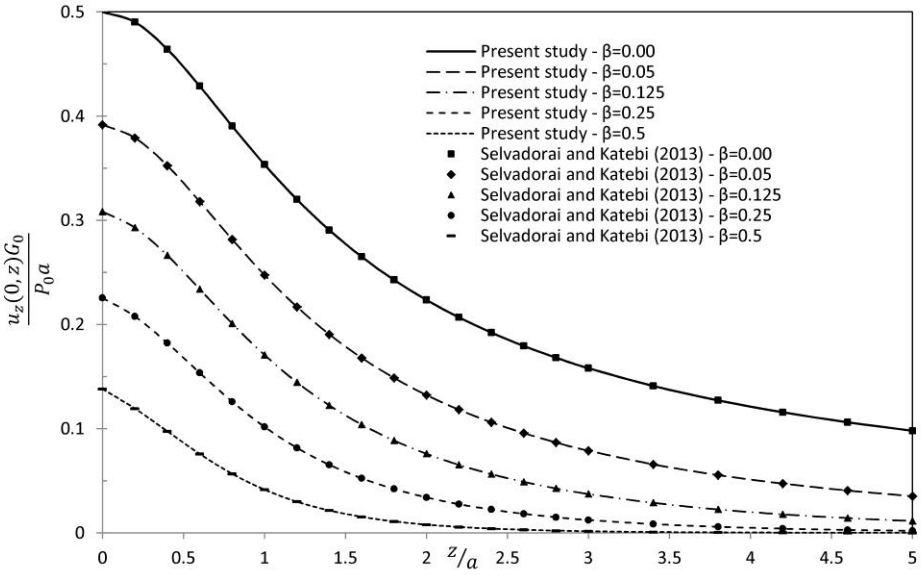

Fig. 5. Displacement Green's function u_z along z -axis obtained in this study compared with result reported by Selvadurai and Katebi (2013) for static case due to uniform vertical surface load

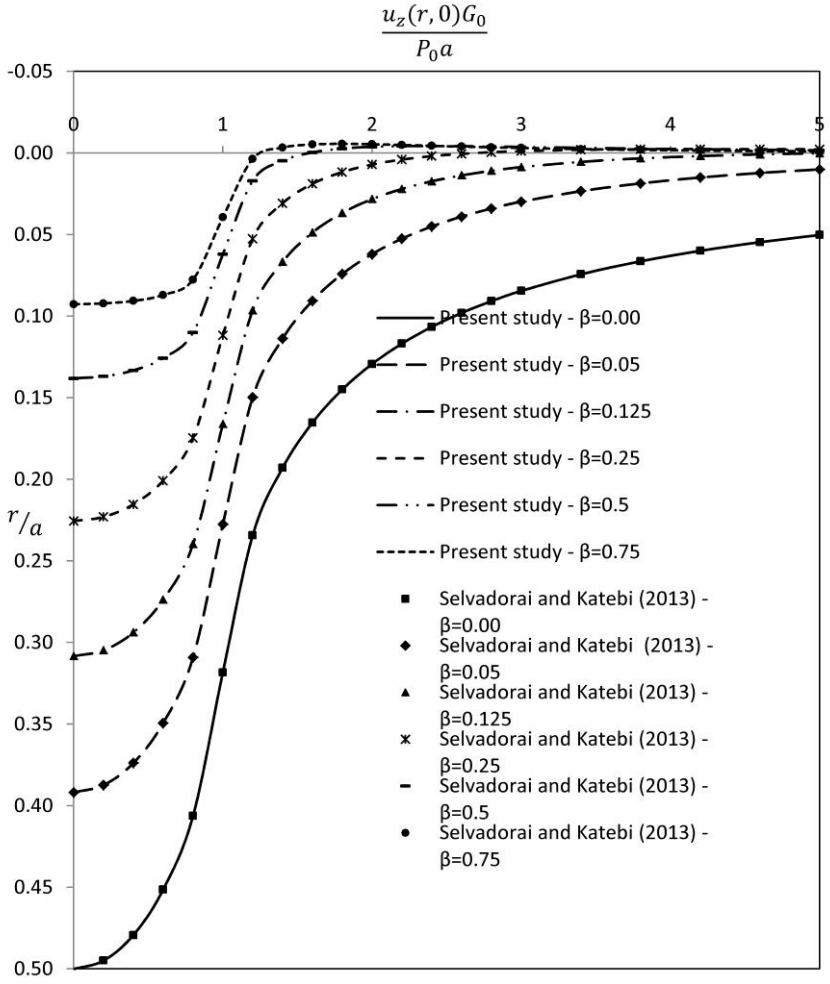

Fig. 6. Displacement Green's function u_z at the surface along r -axis obtained in this study compared with result reported by Selvadurai and Katebi (2013) for static case due to uniform vertical surface load

To illustrate the results obtained in the previous sections, some typical point-load Green's functions are presented in Figures 7 to 12 for two exponentially graded transversely isotropic materials, with a total of three characteristic cases. The values of the engineering elastic constants for the considered transversely isotropic materials are given in Table 1, where E_h and E_v are the Young's moduli with respect to directions lying in the plane of isotropy and perpendicular to it; ν_h and ν_{hv} are Poisson ratios which characterize the effect of the horizontal strain on its orthogonal counterpart and the vertical strain (i.e., the z -direction strain), respectively; ν_{vh} is the Poisson ratio which characterizes the effect of the vertical strain on horizontal strains; and $f/2$ is the shear modulus for the planes normal to the plane of isotropy (Khojasteh et al., 2008c). Upon converting to the elasticity moduli C_{ij} and choosing $E_{v_2} = 100$ Gpa, the pertinent elastic constants C_{ij} can be stated as those given in Table 2.

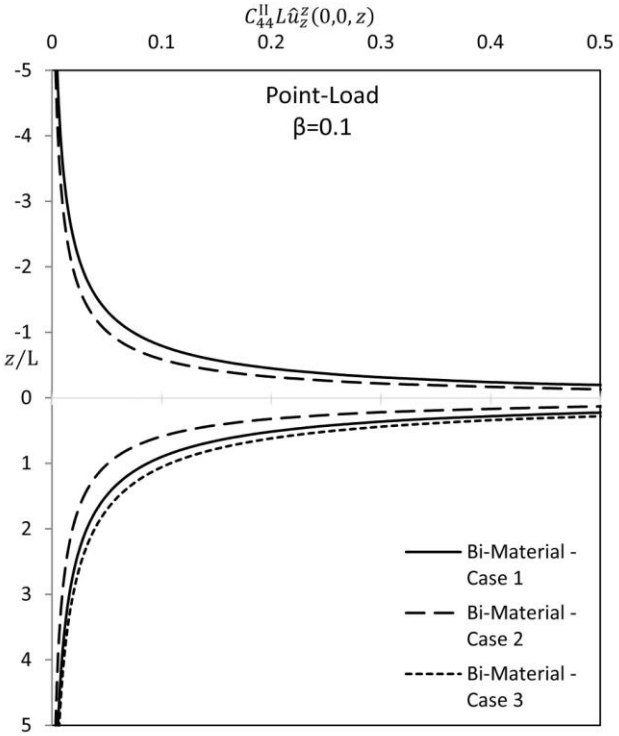

Fig. 7. Displacement Green's function $\hat{u}_z^z$ along z -axis ($\beta = 0.1$)

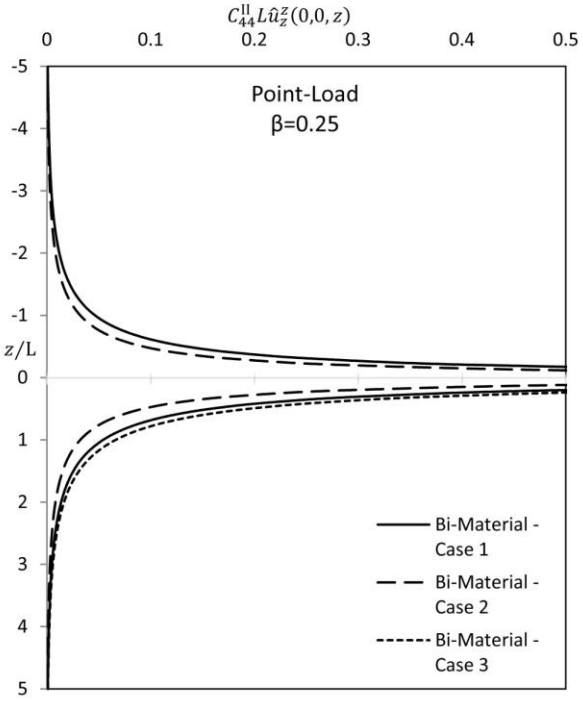

Fig. 8. Displacement Green's function $\hat{u}_z^z$ along z -axis ($\beta = 0.25$)

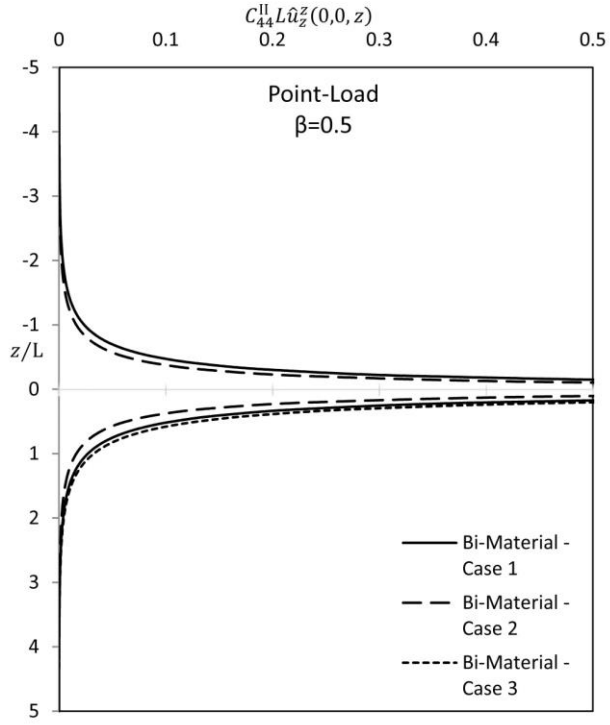

Fig. 9. Displacement Green's function $\hat{u}_z^z$ along z -axis ($\beta = 0.5$)

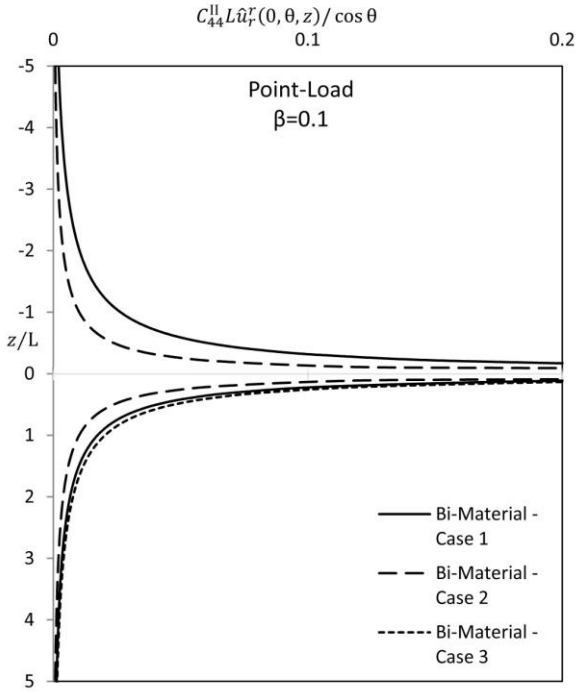

Fig. 10. Displacement Green's function $\hat{u}_r^r$ along z -axis ($\beta = 0.1$)

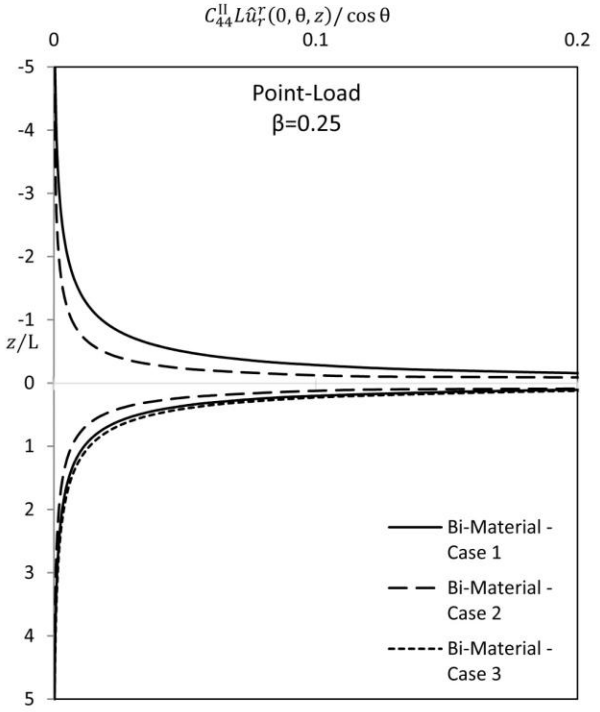

Fig. 11. Displacement Green's function $\hat{u}_r^r$ along z -axis ($\beta = 0.25$)

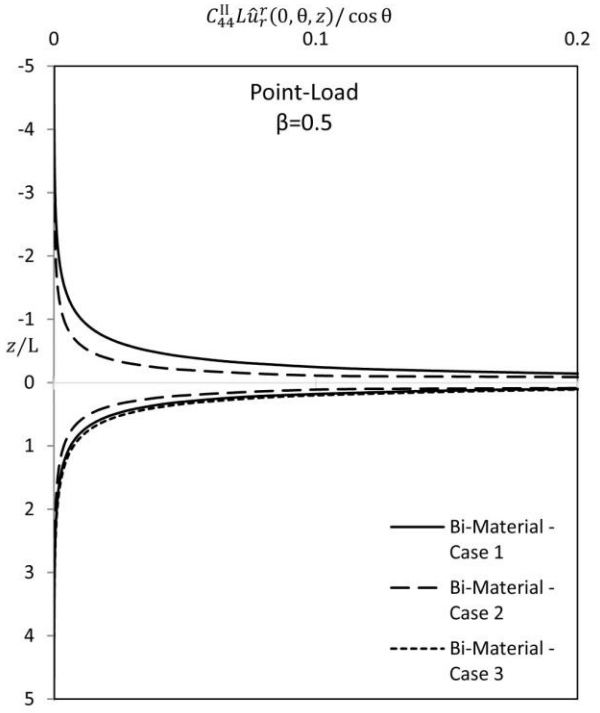

Fig. 12. Displacement Green's function $\hat{u}_r^r$ along z -axis ($\beta = 0.5$)

Table 1. Engineering constants of transversely isotropic materials

Material No.	$\frac{E_h}{E_v}$	$\frac{f}{E_v}$	ν_h	ν_{hv}	ν_{vh}	$\frac{E_{v_1}}{E_{v_2}}$
Material 1	1.5	0.9	0.25	0.3	0.2	0.25
Material 2	3.0	1.0	0.1	0.9	0.3	-

Table 2. Elastic constants of transversely isotropic materials

Material No.	C_{11} **(Gpa)**	C_{12} **(Gpa)**	C_{13} **(Gpa)**	C_{33} **(Gpa)**	C_{44} **(Gpa)**
Material 1	44.8	14.8	11.9	29.8	11.3
Material 2	553	280	250	250	50

The three cases considered here are:

Case 1. Exponentially graded bi-material with material 1 in medium I and material 2 in medium II (stiffer lower half-space).

Case 2. Exponentially graded bi-material with material 2 in medium I and medium II (two equal half-space).

Case 3. Exponentially graded bi-material with material 2 in medium II and $C_{ij}^{\mathrm{I}}, \beta^{\mathrm{I}} \to 0$ in medium I (no upper half-space).

The source point is taken to be the origin with coordinates (0,0,0). It is of necessity to point out that all numerical results presented here are dimensionless, where L represents the unit of length. Figures 7 to 9 depict the displacement Green's function $\hat{u}_z^z$ due to the unit point-load in the z-direction for $\beta = 0.1,0.25,0.50$. Also, the displacement Green's function $\hat{u}_r^r$ due to the unit point-load in the r-direction are delineated in Figures 10 to 12 for $\beta = 0.1,0.25,0.50$.

Furthermore, in order to provide further insight into the problem, the distributions of $\hat{u}_z^z$ at the interface and along the r-axis are shown in Figures 13 to 15 for $\beta = 0.1,0.25,0.50$. As expected, in case 2 when a stiffer bi-material than in the other two cases (see Table 2), results in lowest values of displacement Green's functions.

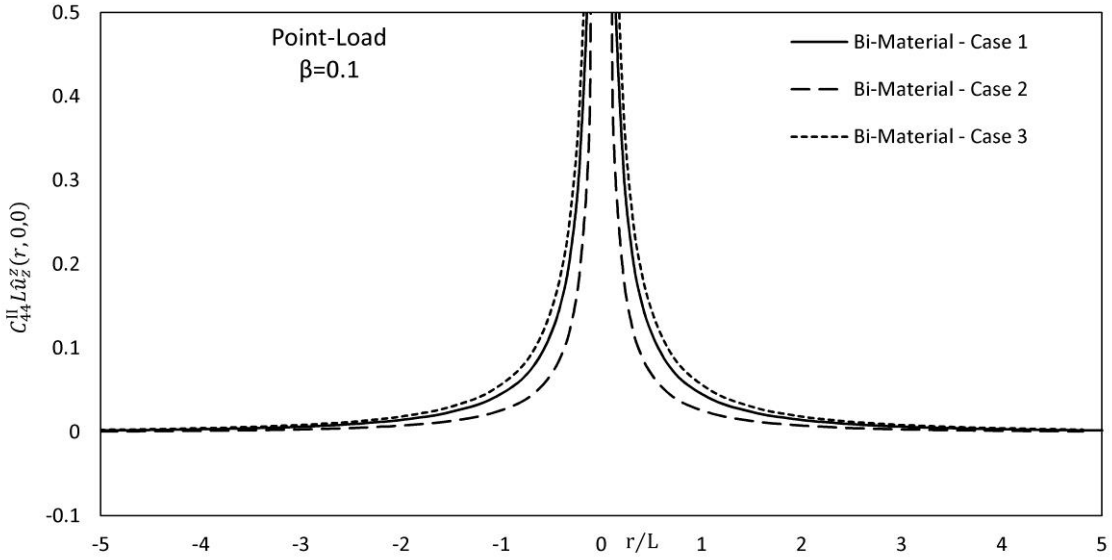

Fig. 13. Displacement Green's function $\hat{u}_z^z$ at the interface along r-axis ($\beta = 0.1$)

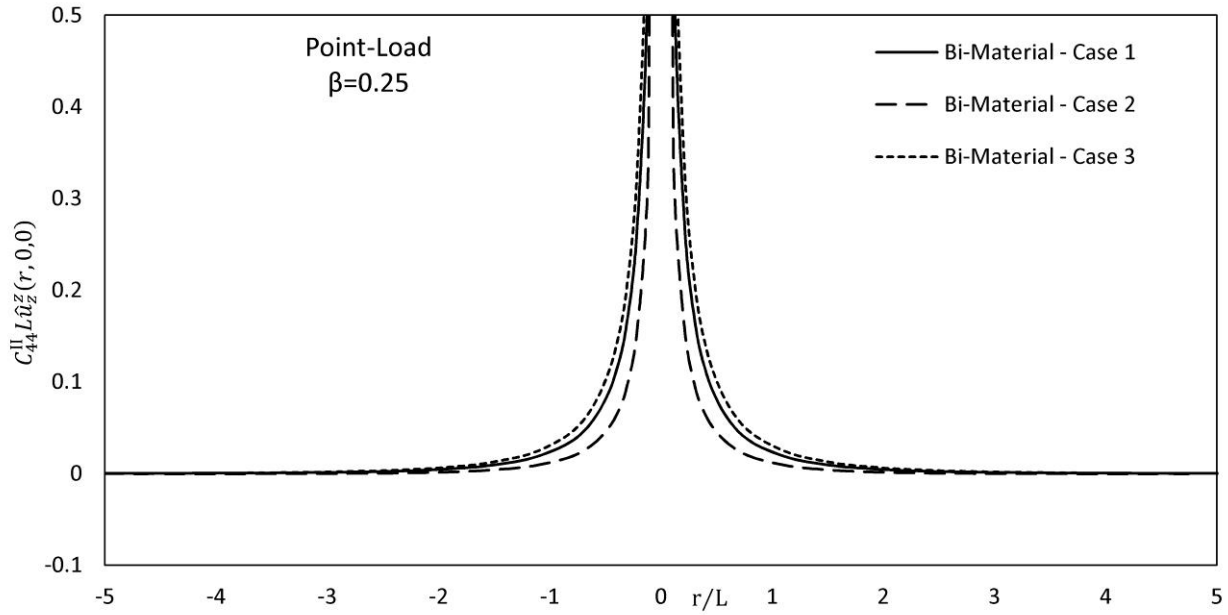

Fig. 14. Displacement Green's function $\hat{u}_z^z$ at the interface along r -axis ($\beta = 0.25$)

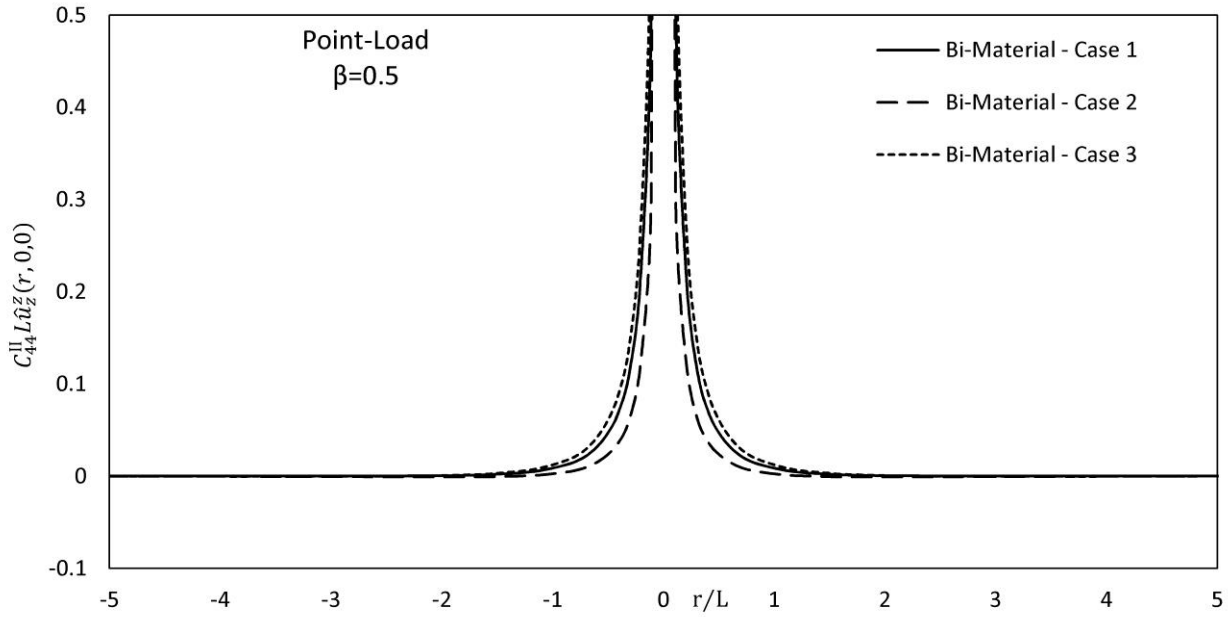

Fig. 15. Displacement Green's function $\hat{u}_z^z$ at the interface along r -axis ($\beta = 0.5$)

The displacement Green's functions are singular at the interface $z = 0$ and tend to zero with increasing depth. Figures 16 to 19 display the effect of inhomogeneity. As observed, the higher β factor reduced the displacement Green's functions due to higher stiffness of the medium.

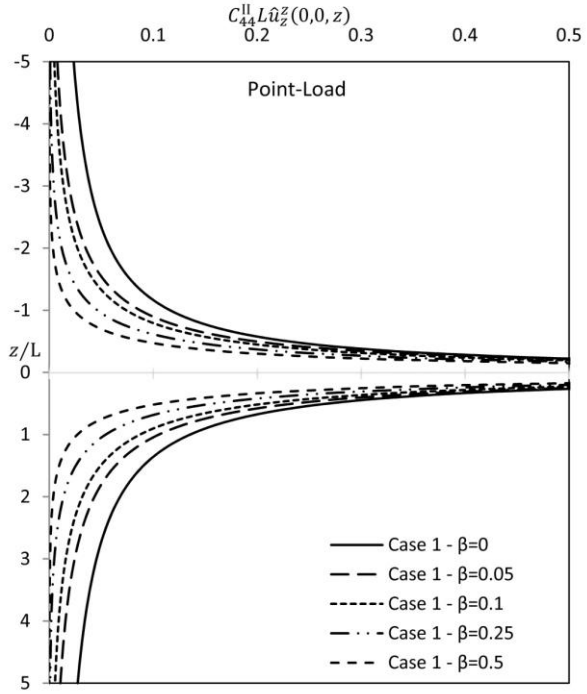

Fig. 16. Displacement Green's function $\hat{u}_z^z$ along z -axis (Case 1)

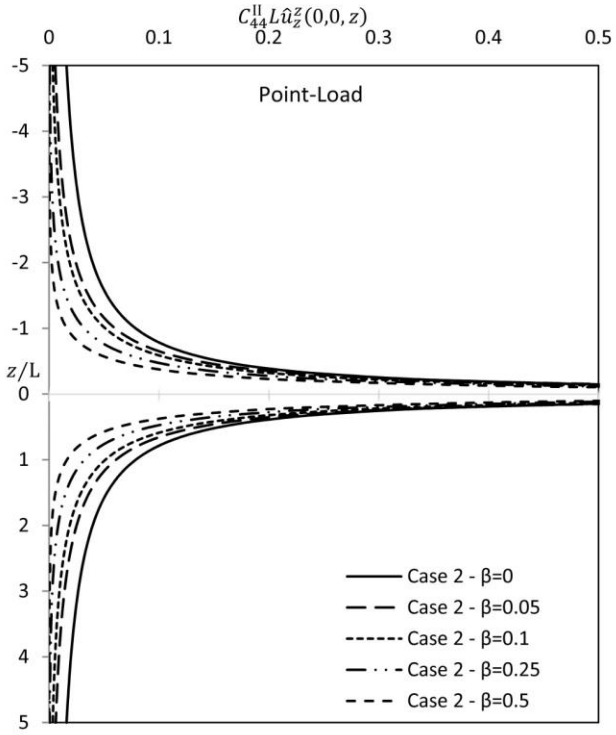

Fig. 17. Displacement Green's function $\hat{u}_z^z$ along z -axis (Case 2)

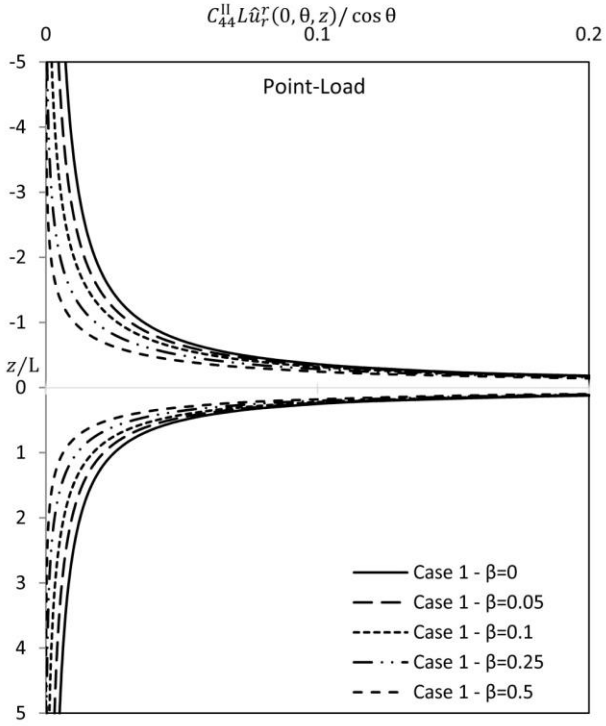

Fig. 18. Displacement Green's function $\hat{u}_r^r$ along z -axis (Case 1)

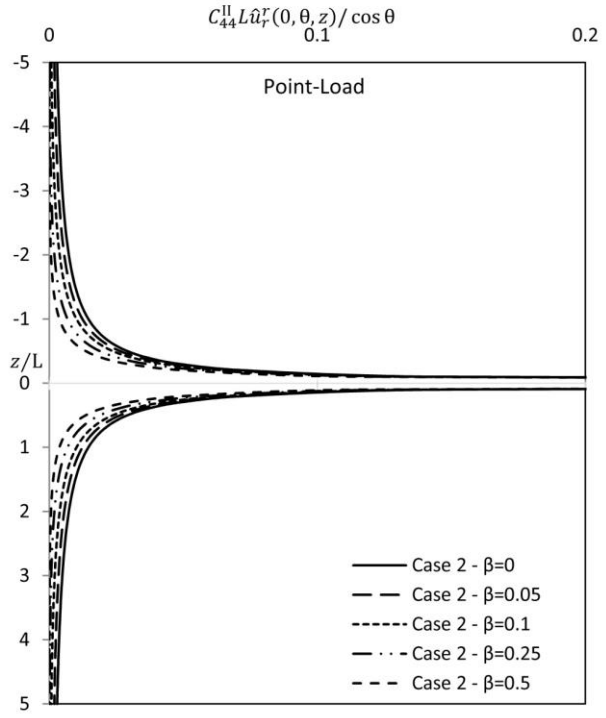

Fig. 19. Displacement Green's function $\hat{u}_r^r$ along z -axis (Case 2)

The distribution of the stress Green's functions $\hat{\tau}_{zz}^{z}$ due to the unit point-load in the z-direction are shown in Figures 20 to 22 for $\beta = 0.1,0.25,0.50$. Similar to the displacement Green's functions, the stress Green's functions are singular at the interface $z = 0$. It is noteworthy here that in the determination of $\hat{\tau}_{zz}^{z}$, the elastic constant C_{33} is the dominant component, with a value for the lower half-space (medium II) which is about seven times larger than that for the upper one (medium I) in the bi-material case 1 (Table 2). For this reason, for a given β in case 1, the value of $\hat{\tau}_{zz}^{z}$ for the lower half-space is generally higher than that for the upper one. The effect of this neighboring medium on the stress distribution in either half-space is most pronounced near the material interface. Similar to the displacement Green's functions, the stress Green's functions tend to zero with increasing depth and intensifies with increasing β. The reason for this increase is existence of the factor $e^{\beta z}$ in the stress Green's functions. In consistency with the symmetry of the problem, all Green's functions for the full-space configuration (case 2) are symmetric with respect to the plane $z = 0$.

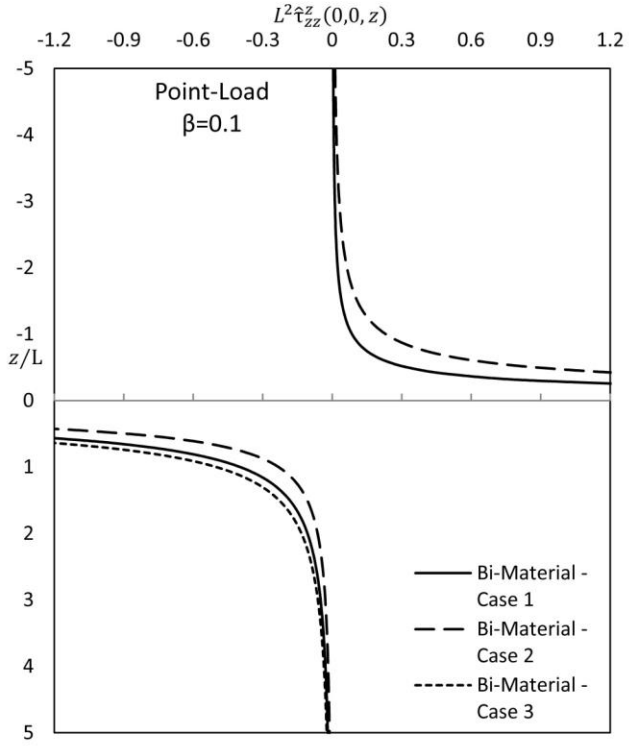

Fig. 20. Stress Green's function $\hat{\tau}_{z}^{z}$ along z-axis ($\beta = 0.1$)

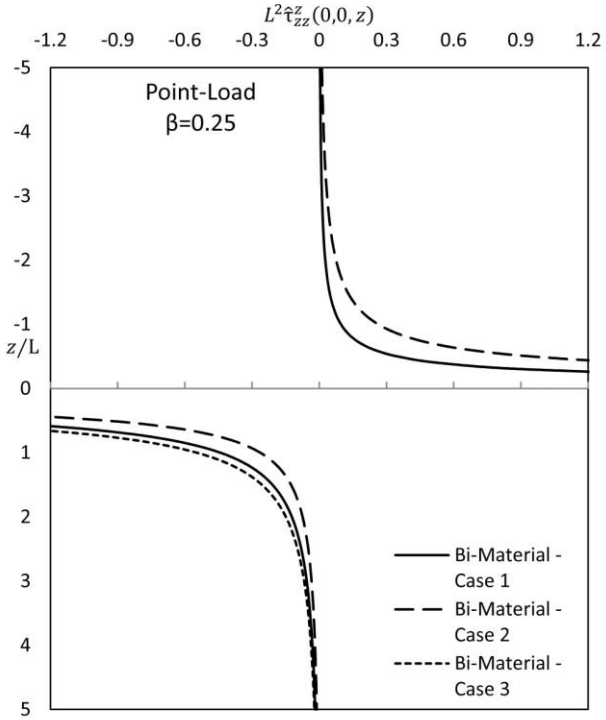

Fig. 21. Stress Green's function $\hat{\tau}_z^z$ along z -axis ($\beta = 0.25$)

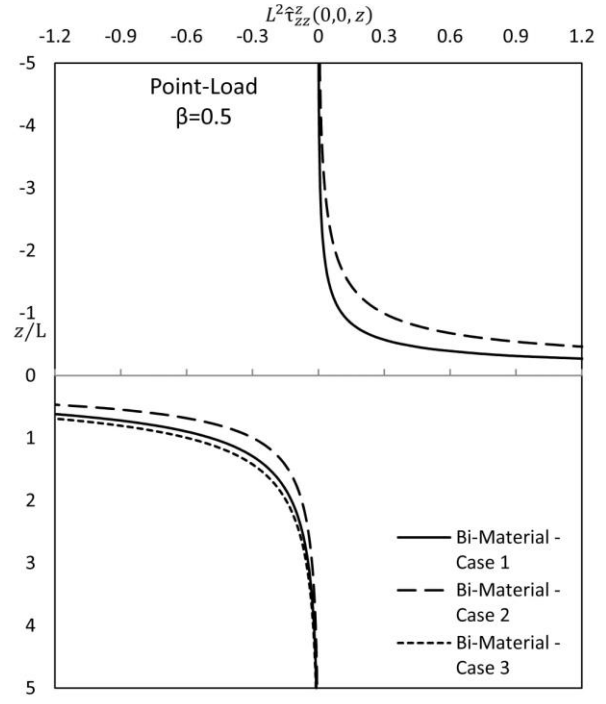

Fig. 22. Stress Green's function $\hat{\tau}_z^z$ along z -axis ($\beta = 0.5$)

CONCLUSIONS

The three-dimensional static Green's functions of an exponentially graded transversely isotropic bi-material elastic full-space due to the point-load are derived by means of integral transforms and the method of displacement potential. They are expressed in the form of explicit line-integral representations which are essential for the efficient boundary element formulations of the related elastoestatic problems and are used in developing better evaluation of composites and anisotropic media. It is shown that the present exponentially graded transversely isotropic bi-material Green's functions can be analytically and numerically degenerated to the special cases such as the static solution for exponentially graded half-space and the homogeneous full-space bi-material media. Numerical examples have also been presented to elucidate the influence of the degree of inhomogeneity of the material.

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