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A New Modification of the Reconstruction of Variational Iteration Method for Solving Multi-order Fractional Differential Equations

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Abstract

Fractional calculus has been used to model the physical and engineering processes that have found to be best described by fractional differential equations. For that reason, we need a reliable and efficient technique for the solution of fractional differential equations. The aim of this paper is to present an analytical approximation solution for linear and nonlinear multi-order fractional differential equations (FDEs). The fractional derivatives are described in the Caputo sense. In this work, the Reconstruction of Variational Iteration Method (RVIM) technique has been successfully used to solve two types of multi-order fractional differential equations, linear and nonlinear. For this purpose, we convert FDE in to a counterpart system and then using proposed method to solve the result system. Advantage of the RVIM, is simplicity of the computations and convergent successive approximations without any restrictive assumptions. Illustrative examples are included to demonstrate the validity and applicability of the presented technique.

Keywords: Multi-order fractional differential equations; Reconstruction of Variational Iteration Method; Caputo fractional derivative.

Introduction

It is known that many phenomena in several branches of science can be described very successfully by models using mathematical tools from fractional calculus. Methods of solutions of problems for fractional differential equations have been studied extensively by many researchers (see, e.g., [1–3] and the references given therein). The analytic results on the existence and uniqueness of solutions to the FDEs have been investigated by many authors; among them, [4]. In general, most of FDEs do not have exact analytic solutions, so approximation and numerical techniques must be used. Finding accurate and efficient methods for solving FDEs has become an active research undertaking. Therefore several methods for the approximate solutions to classical differential equations are extended to solve differential equations of fractional order numerically. These methods include: Adomian decomposition method [5], homotopy perturbation method [6], homotopy analysis method [7], variational iteration method [8], generalized differential transform

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method [9], finite difference method [10], operational matrix of B-spline functions [11], operational matrix of Legendre polynomials [12,13], operational matrix of Chebyshev polynomials [14], Legendre collocation method [15], pseudo-spectral method [16,25], Legendre multi wavelet collocation method [17] and other methods [18,19]. The variational iteration method (VIM) was first introduced by He in 1999 [20,26,27]. In this method, the solution is approximated at first iteration by using the initial conditions. A correction functional is established by the general Lagrange multiplier which can be identified optimally via the variational theory. Although a number of useful attempts have been made to solve fractional equations via the VIM, the problem has not yet been completely resolved, i. e. , most of the previous work avoid the term of fractional derivative, handle them as restricted variation and they cannot identify the fractional Lagrange multipliers explicitly in the correction function. Hesameddini and Latifizadeh proposed a new alternative approach based on variational iteration formulations and Laplace transform. This method used for solving ordinary differential equation [21] and fractional differential equation [22].

In this work, we extend the reconstruction of variational iteration method to solve multi-order fractional differential equations. Indeed, the method is based on converting a given problem into a system of FDEs and then solving the resultant system by using the reconstruction of variational iteration method. The aim of this work is to present an alternative approach based on RVIM to find the solution of linear and nonlinear system of fractional differential equations. The efficiency and accuracy of RVIM is demonstrated through several test examples. The paper has been organized as follows. In Section 2, we introduce some mathematical preliminaries of the fractional calculus theory. In Section 3 we describe how to convert a multi-order FDE into a system of FDE. Section 4 is devoted to applying the RVIM method for solving multi-order linear and nonlinear FDEs. Some numerical experiments are presented in Section 5. Finally, we conclude the paper with some remarks.

Preliminaries

In this section, we present some notations, definitions and preliminary facts of the fractional calculus theory which will be used further in this work.

Definition 1. Let $\mathrm{C}[\mathrm{a},\mathrm{b}]$denotes the space of all continuous functions defined on $[\mathrm{a},\mathrm{b}]$ and $\mathrm{C}^n[\mathrm{a},\mathrm{b}]$denotes a class of all real valued functions defined on $[\mathrm{a},\mathrm{b}]$ which have continuous n-th order derivative.

Definition 2. Let $f\in C[a,b]$ and $\alpha\geq 0$, then the expression

$$J^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau,$$

is called the Riemman-Liouville integral of order α.

Definition 3. The fractional derivative of f(x) in the Caputo sense is defined as;

$$D_*^{\alpha}f(t)=\begin{cases} J^{m-1}f^{(m)}(t)=\dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t (t-\tau)^{m-\alpha-1}f^{(m)}(\tau)d\tau\,, & m-1<\alpha<m\,, \\ \dfrac{d^m}{dx^m}f(t), \quad \alpha=m\,, & \text{for } m\in N and f\in \mathrm{C}^n[\mathrm{a},\mathrm{b}]. \end{cases}$$

Note that

$$J^{\alpha}J^{\beta}f(t)=J^{\alpha+\beta}f(t),\quad \alpha,\beta\geq 0, \tag{1}$$

$$J^{\alpha}t^{\gamma}=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}t^{\gamma+\alpha},\quad \alpha>0,\gamma>-1,t>0. \tag{2}$$

Definition 4. Given a function $f(t)$ defined for all$t\geq 0$, the Laplace transformof $f(t)$ is the function $F(s)$defined as follows:

$$F(s)=l\{f(t;s)\}=\int_0^{\infty} e^{-st}f(t)dt,$$

for all values of$s$ for which the improper integral converges.

Definition 5. A function f is said to be of exponential order as $t\to+\infty$, if there exist nonnegative Constants M,c and Tsuch that:

$$|f(t)|\leq Me^{cT} for t\geq T. \tag{3}$$

Definition 6. A function $f(t)$ is said to be piecewise continuous on the bounded interval $a\leq t\leq b$ provided that $[a,b]$ can be subdivided into finitely many abutting subintervals in such way that:

1. f is continuous in the interior of each of these subintervals: and
2. $f(t)$ has a finite limit as t approaches each endpoint of each subinterval from the interior.

Theorem 1. (Existence of the Laplace transforms) If a function f is piecewise continuous for $t \geq 0$ and is of exponential order as $t \to +\infty$ then its Laplace transform F(s) exists. More precisely, if f is piecewise continuous and satisfies the condition (3), then $F(s)$ exists for all $s > c$.

Definition 7. Let functions $f(t)$ and $g(t)$ be defined for $t \geq 0$, then the convolution of them is denoted by $(f * g)(t)$, and is defined as the following integral;

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau \, .$$

In other words, if $l\{f(t)\} = F(s)$, $l\{g(t)\} = G(s)$, then $l\{(f*g)(t)\} = F(s)G(s)$. Or equivalently, $l\left\{\int_0^t f(\tau)g(t-\tau)d\tau\right\} = F(s)G(s)$. Therefore, the inverse Laplace transform will be defined as; $l^{-1}\{F(s)G(s)\} = \int_0^t f(\tau)g(t-\tau)d\tau$.

Definition 8. The Laplace transform of the Caputo fractional derivative $D_*^{\alpha} f(t)$ is given by:

$$l\{D_*^{\alpha} f(t); s\} = s^{\alpha}F(s) - \sum_{k=0}^{m-1} s^{(\alpha-k-1)} f^{(k)}(0^+),$$

where $F(s) = l\{f(t); s\}, m-1 \leq \alpha < m$.

Converting a multi-order FDE into a system of FDE

Consider the multi-order FDE:

$$D_*^{\alpha} y(t) = f\left(t, y(t), D_*^{\beta_1} y(t), \dots, D_*^{\beta_n} y(t)\right), y^{(k)}(0) = c_k, k = 0,1,\dots,m, \quad (4)$$

where $m < \alpha \leq m+1, 0 < \beta_1 < \beta_2 < \cdots \leq \beta_n < \alpha$ and D_*^{α} denotes Caputo fractional derivative of order α. Daftardar-Gejji et al. [5] convert this equation to a system of FDE, as follows:

Set $y_1 = y$ and define

$D_*^{\beta_1} y_1 = y_2$. Now we consider two cases:

Case (i) If $m-1 \leq \beta_1 < \beta_2 \leq m$, then define, $D_*^{\beta_2-\beta_1} y_2 = y_3$.

Case(ii) For $m-1 \leq \beta_1 < m \leq \beta_2$. If $\beta_1 = m-1$, then define $D_*^{\beta_2-\beta_1} y_2 = y_3$, and if $m-1 < \beta_1 < m \leq \beta_2$, then define $D_*^{m-\beta_1} y_2 = y_3$.

Continuing similarly, the initial value problem (4), will be converted in to a system of FDE.

Application of Reconstruction of Variational Iteration Method (RVIM) for a System of FDE

As described in section 3, we can present the multi-order Eq. (4), as a system of fractional differential equations as follows:

$$D_*^{\alpha_i} y_i(t) = y_{i+1}(t) \, , \quad i = 1,2,\dots,n-1,$$
$$D_*^{\alpha_n} y_i(t) = f(t, y_1, y_2, \dots, y_n),$$
$$y_i^{(k)} = c_k^i \quad 0 \leq k \leq m_i, \qquad m_i < \alpha_i \leq m_i + 1, \quad 1 \leq i \leq n.$$

Eq. (5), can be rewritten in the following form:

$$D_*^{\alpha_i} y_i(t) = N_i(t, y_1, y_2, \dots, y_n), \qquad i = 1,\dots,n \, , \quad (6)$$

where N_iś are linear/nonlinear functionsof$t, y_1, y_2, \dots, y_n$ and $D_*^{\alpha_i}$ is the fractional derivative of order α_i in the sense of Caputo, subjected to the initial conditions:

$$y_i^{(k)} = c_k^i m_i < \alpha_i \leq m_i + 1, \quad 1 \leq i \leq n. \quad (7)$$

Therefore, the approximate solution can be obtained as follows:

$$y_i(t) = \lim_{n\to\infty} y_i^n(t), \quad i = 1, \dots n, \quad (8)$$

where y_i^n indicates the n-th approximation of y_i andy_i^0 is $y_i(0) + ty'_i(0) + \dots + \frac{t^n y_i^{(n)}(0)}{n!}$, where $y_i^{(k)}(0)$, $k = 0,1,\dots,n$ are substituted by the initial condition of the main problem.

Taking Laplace transform of both sides of Eq. (6), in the usual way and using the homogenous initial conditions (i. e. artificial initial conditions equal to zero), the result can be obtained asfollows:

$$s^{\alpha_i} l\{y_i(t)\} = l\{N_i(t, y_1, y_2, \dots, y_n)\}. \quad (9)$$

Now by applying the inverse Laplace transform to both sides of Eq. (9), and using the convolution theorem, the following relation can be resulted:

$$y_i(t) = \int_0^t \frac{(t-\tau)^{\alpha_i-1}}{\Gamma(\alpha_i)} N_i(\tau, y_1(\tau), y_2(\tau), \dots, y_n(\tau))d\tau.$$

Therefore,

$$y_i^{n+1}(t) = y_i^0(t) + \int_0^t \frac{(t-\tau)^{\alpha_i-1}}{\Gamma(\alpha_i)} N_i(\tau, y_1^n(\tau), y_2^n(\tau), .., y_n^n(\tau))d\tau \, , i = 1,\dots,n. \quad (10)$$

After identifying the initial approximation of y_i^0 , the remaining approximations $y_1^n(t), n > 0$ can be obtained, so that each term is determined by the previous term and the approximation of iteration formula will be entirely evaluated. Consequently, the solution may be written as:

$$y_i(t) = \lim_{n\to\infty} y_i^n(t), \qquad i = 1, \dots n.$$

Results

To illustrate the effectiveness of the proposed method, we consider various types of multi-order FDEs. At first, we transform multi-order FDE to a system of FDE and then the resultant system will be solved by using RVIM.

Example 1. As a first example, we consider the following initial value problem in the case of the inhomogeneous Bagley–Torvik equation [23]

$$D_*^2 y(t) + D_*^{1.5} y(t) + y(t) = 1 + t, \quad y(0) = 1, y'(0) = 1, x \in [0,1]. \tag{11}$$

In order to apply the present method, at first we convert this equation into the following system of FDE:

$$D_*^{1.5} y_1(t) = y_2(t), \quad D_*^{0.5} y_2(t) = -y_2(t) - y_1(t) + 1 + t, \tag{12}$$

Subjectedto the initial conditions:

$$y_1(0) = 1, y_1{}'(0) = 1, y_2(0) = 0. \tag{13}$$

Applying the Laplace transform to Eq. (12), the result is as follows:

$$l\{y_1(t)\} = \frac{1}{s^{1.5}} l\{y_2(t)\}, \quad l\{y_2(t)\} = \frac{1}{s^{0.5}} l\{-y_2(t) - y_1(t) + 1 + t\}. \tag{14}$$

Making use of the inverse Laplace transform to both sides of Eq. (14), result in:

$$y_1(t) = \frac{1}{\Gamma(1.5)} \int_0^t (t-\tau)^{1.5-1} [y_2(\tau)] d\tau,$$

$$y_2(t) = \frac{1}{\Gamma(0.5)} \int_0^t (t-\tau)^{0.5-1} [-y_2(\tau) - y_1(\tau) + 1 + \tau] d\tau. \tag{15}$$

Therefore, approximate solution of Eq. (12), can be readily obtained as:

$$y_1^{n+1}(t) = y_1^0(t) + \frac{1}{\Gamma(1.5)} \int_0^t (t-\tau)^{1.5-1} [y_2^n(\tau)] d\tau,$$

$$y_2^{n+1}(t) = y_2^0(t) + \frac{1}{\Gamma(0.5)} \int_0^t (t-\tau)^{0.5-1} [-y_2^n(\tau) - y_1^n(\tau) + 1 + \tau] d\tau, \tag{16}$$

where $y_1^0(t) = 1 + t$, $y_2^0(t) = 0$.

According to (16), at first iteration, the following set of relationsis resulted:

$$y_1^1(t) = 1 + t, \qquad y_2^1(t) = 0.$$

Therefore, the solution is $y(t) = y_1(t) = 1 + t$. This is coinciding with its exact solution given in [24]. In comparison with the procedure in [24] to solve this example, one can see the importance of our numerical scheme insolving multi-order fractional differential equations.

Example 2. Consider the following initial value problem,

$$D_*^3 y(t) + D_*^{2.5} y(t) + y^2(t) = t^4, \quad y(0) = y'(0) = 0, y''(0) = 2. \tag{17}$$

If we choose $y(t) = y_1$ and $D_*^{2.5} y_1 = y_2$, then Eq. (17), will be converted to the following system of nonlinear FDE:

$$D_*^{2.5} y_1(t) = y_2(t), \quad D_*^{0.5} y_2(t) = t^4 - y_2(t) - (y_1(t))^2, \tag{18}$$

subjected to the initial conditions:

$$y_1(0) = y_1{}'(0) = 0, y_1{}''(0) = 2, y_2(0) = 0. \tag{19}$$

Applying the Laplace transform to Eq. (18), the result is as follows:

$$l\{y_1(t)\} = \frac{1}{s^{2.5}} l\{-y_2(t)\},$$
$$l\{y_2(t)\} = \frac{1}{s^{0.5}} l\{t^4 - y_2(t) - (y_1(t))^2\}. \quad (20)$$

Benefiting the inverse Laplace transform to both sides of Eq. (20), one obtains:

$$y_1(t) = \frac{1}{\Gamma(2.5)} \int_0^t (t-\tau)^{2.5-1}[-y_2(\tau)]d\tau,$$
$$y_2(t) = \frac{1}{\Gamma(0.5)} \int_0^t (t-\tau)^{0.5-1}[\tau^4 - y_2(\tau) - (y_1(\tau))^2]d\tau. \quad (21)$$

Therefore, approximate solution of Eq. (21), can be readily obtained as:

$$y_1^{n+1}(t) = y_1^0(t) + \frac{1}{\Gamma(2.5)} \int_0^t (t-\tau)^{2.5-1}[-y_2^n(\tau)]d\tau,$$
$$y_2^{n+1}(t) = y_2^0(t) + \frac{1}{\Gamma(0.5)} \int_0^t (t-\tau)^{0.5-1}[\tau^4 - y_2^n(\tau) - y_1^n(\tau))^2]d\tau, \quad (22)$$

where $y_1^0(t) = t^2$, $y_2^0(t) = 0$ and y_i^n indicates the n-th approximation of y_i for $i = 1,2$.

By using the definition of Riemman-Liouville integral of order α, we can rewrite Eq. (22), in the following form:

$$y_1^{n+1}(t) = y_1^0(t) + J^{2.5}(-y_2^n(t)),$$
$$y_2^{n+1}(t) = y_2^0(t) + J^{0.5}(t^4 - y_2^n(t) - y_1^n(t))^2). \quad (23)$$

According to Eq. (23),at first iteration we have:

$$y_1^1(t) = t^2 \ , \qquad y_2^1(t) = 0\ .$$

Therefore, our solution is$y(t) = y_1(t) = t^2$. This is coinciding with the exact solution of this equation.

In comparison with the Adomian decomposition method [7], our method gives the exact solution faster than that mentioned method and it is one of the advantages of this current work.

Example 3. Lastly we consider the following nonlinear FDE with the exact solution $y(t) = t^2$:

$$t^{\frac{7}{2}} D_*^{\frac{3}{2}} y(t) + y''(t) + y^2(t) = \left(1 + \frac{4}{\sqrt{\pi}}\right) t^4 + 2, y(0) = y'(0) = 0, x\epsilon[0,1]. \quad (24)$$

If we choose $y(t) = y_1$ and $D_*^{\frac{3}{2}} y_1 = y_2$, then Eq. (24), can be converted to the following system of nonlinear FDE:

$$D_*^{\frac{3}{2}} y_1(t) = y_2(t),$$
$$D_*^{0.5} y_2(t) = -t^{\frac{7}{2}} y_2(t) - (y_1(t))^2 + \left(1 + \frac{4}{\sqrt{\pi}}\right) t^4 , \quad (25)$$

subjected to the initial conditions:

$$y_1(0) = y_1{}'(0) = 0 \ ,$$
$$y_2(0) = 0. \quad (26)$$

Applying the Laplace transform to Eq. (25), the result is as follows:

$$l\{y_1(t)\} = \frac{1}{s^{1.5}} l\{y_2(t)\},$$
$$l\{y_2(t)\} = \frac{1}{s^{0.5}} l\left\{-t^{\frac{7}{2}} y_2(t) - (y_1(t))^2 + \left(1 + \frac{4}{\sqrt{\pi}}\right) t^4\right\}. \quad (27)$$

Benefiting the inverse Laplace transform to both sides of Eq. (27), one obtains:

$$y_1(t) = \frac{1}{\Gamma(1.5)} \int_0^t (t-\tau)^{0.5}[y_2(\tau)]d\tau ,$$
$$y_2(t) = \frac{1}{\Gamma(0.5)} \int_0^t (t-\tau)^{0.5-1} \left[-\tau^{\frac{7}{2}} y_2(\tau) - (y_1(\tau))^2 + \left(1 + \frac{4}{\sqrt{\pi}}\right) \tau^4\right] d\tau . \quad (28)$$

Therefore, the approximate solution of Eq. (25), can be readily obtained as:

$$y_1^{n+1}(t) = y_1^0(t) + \frac{1}{\Gamma(1.5)} \int_0^t (t-\tau)^{1.5-1}[y_2^n(\tau)]d\tau,$$
$$y_2^{n+1}(t) = y_2^0(t) +$$

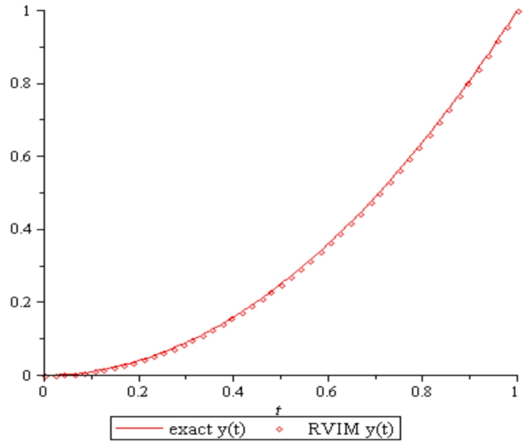

Figure 1. Comparison between theexact solutionand approximate solution (by RVIM method) for Example (3).

$$+\frac{1}{\Gamma(0.5)}\int_0^t (t-\tau)^{0.5-1}\Big[-\tau^{\frac{7}{2}}y_2^n(\tau) - (y_1^n(\tau))^2 + \left(1+\frac{4}{\sqrt{\pi}}\right)\tau^4\Big]\,d\tau\,, \quad (29)$$

where $y_1^0(t)=0$, $y_2^0(t)=0$ and y_i^n indicates the n-th approximation of y_i for $i=1,2$.

According to the definition of Riemman-Liouville integral of order α, Eq. (29), can be rewrite in the following form:

$$y_1^{n+1}(t) = y_1^0(t) + J^{1.5}(y_2^n(t)),$$

$$y_2^{n+1}(t) = y_2^0(t) + J^{0.5}\left(-t^{\frac{7}{2}}y_2^n(t) - (y_1^n(t))^2 + \left(1+\frac{4}{\sqrt{\pi}}\right)t^4\right). \quad (30)$$

According to Eq. (30), after some simplification and substitution, the following sets of equations are resulted:

$$y_1^1(t) = 0,$$

$$y_2^1(t) = \left(1+\frac{4}{\sqrt{\pi}}\right)\frac{\Gamma(5)}{\Gamma(5.5)}t^{4.5} + \frac{2}{\Gamma(1.5)}t^{0.5},$$

$$y_1^2(t) = t^2 + \frac{1}{30}\left(1+\frac{4}{\sqrt{\pi}}\right)t^6,$$

$$y_2^2(t) = -\left(1+\frac{4}{\sqrt{\pi}}\right)\frac{\Gamma(5)}{\Gamma(5.5)}\frac{\Gamma(9)}{\Gamma(9.5)}t^{8.5} + \frac{\Gamma(5)}{\Gamma(5.5)}t^{4.5} + \frac{2}{\Gamma(1.5)}t^{0.5},$$

$$y_1^3(t) = t^2 + \frac{1}{30}t^6 - \frac{1}{90}\left(1+\frac{4}{\sqrt{\pi}}\right)\frac{\Gamma(5)}{\Gamma(5.5)}t^{10},$$

$$y_2^3(t) = \frac{2}{\Gamma(1.5)}t^{0.5} + \left(\frac{1}{15}\left(1+\frac{4}{\sqrt{\pi}}\right) - \frac{\Gamma(5)}{\Gamma(5.5)}\right)\frac{\Gamma(9)}{\Gamma(9.5)}t^{8.5} + \left(1+\frac{4}{\sqrt{\pi}}\right)\left(\frac{\Gamma(5)\Gamma(9)}{\Gamma(5.5)\Gamma(9.5)} - \frac{1}{900}\left(1+\frac{4}{\sqrt{\pi}}\right)\right)\frac{\Gamma(13)}{\Gamma(13.5)},$$

$$y_1^4(t) = t^2 - 0.002682208t^{10} + 0.002708415t^{14},\vdots$$

Therefore, the approximate solution after four

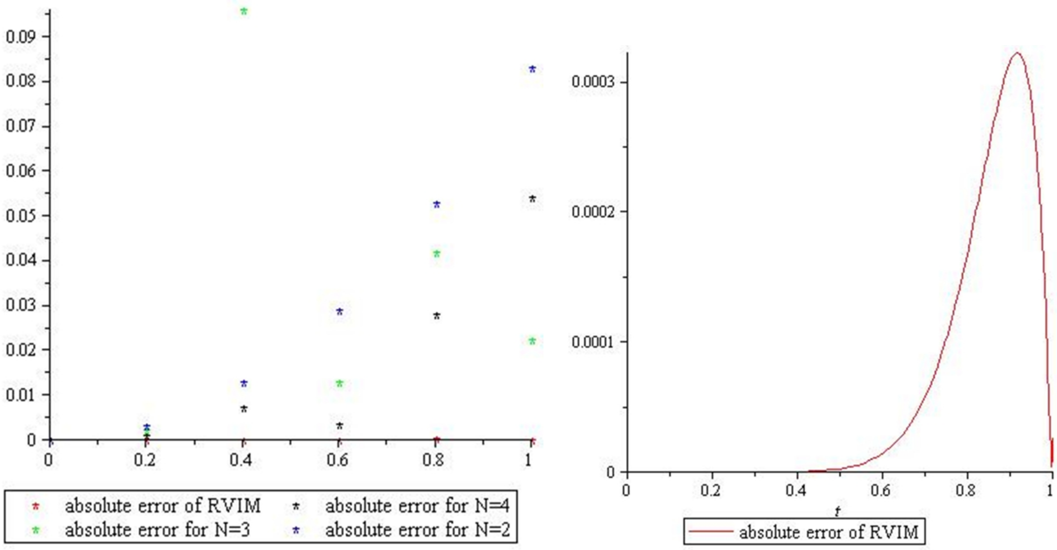

Figure 2. Comparison between the absolute error arises from RVIM and Ref. [24] with various choices of N for Example (3).

Table 1. Absolute error of example (3). Our results are compared to Ref. [24]with various choices of N.

t	N=2	N=3	N=4	RVIM (4 iteration)
0.0	0.0	0.0	0.0	0.0
0.2	3.3e-3	2.2e-3	8.8e-4	2.7e-10
0.4	1.3e-2	9.6e-3	7.1e-3	2.7e-7
0.6	2.9e-2	1.3e-2	3.5e-3	1.4e-5
0.8	5.3e-2	4.2e-2	2.8e-2	1.6e-4
1	8.3e-2	6.7e-2	6.7e-2	2.6e-5

iterations is $y(t) = y_1(t) = t^2 - 0.002682208t^{10} + 0.002708415t^{14}$. Atabakzadeh et al. [24] applied the chebyshev operational matrix method for solving this FDE. Numerical results with comparison to Ref. [24] are given in Table 1 and Fig. 2. on the interval $[0,1]$. The accuracy of our results with respect to the results given in Ref. [24] is obvious.

Fig. 1.shows the approximate solution of Eq. (24). One can see that our approximate solution is in a good agreement with its exact solution.

In spite of the method using in [24], that for large N need a large computation process and this may create a computational error, our method is very effective to use and give very good approximate solution.

Conclusion

In this article, we have discussed an analytical approximation method to solve some classes of FDE. Our method is based on converting a given problem into a system of FDE and then solving the resultant systems by a technique so-called reconstruction of variational iteration method (RVIM). This work emphasized to our belief that the method is a reliable technique to handle linear and nonlinear systems of fractional differential equations. The obtained results are compared to the exact solution and also with the solution that were obtained by other numerical methods in literature. It is worth mentioning that in a few iterations, we can obtain good results. Moreover, the method presented rapidly convergent successive approximations without any restrictive assumptions or transformation which may change the physical behavior of the problem. Evidently, the RVIM reduced the size of calculation and also the iteration was direct and straightforward. The proposed technique is easy to implement, efficient and yields accurate results. Generally, the proposed method is promising and applicable to board classes of linear and nonlinear systems in the theory of fractional calculus.

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