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Horizontal Subbundle on Lie Algebroids

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Abstract

Providing an appropriate definition of a horizontal subbundle of a Lie algebroid will lead to construction of a better framework on Lie algebriods. In this paper, we give a new and natural definition of a horizontal subbundle using the prolongation of a Lie algebroid and then we show that any linear connection on a Lie algebroid generates a horizontal subbundle and vice versa. The same correspondence will be proved for any covariant derivative on a Lie algebroid.

Keywords: Lie algebroid, Horizontal subbundle, Double Lie algebroid.

Introduction

Lie algebroids are now a central notion in differential geometry and constitute an active domain of research. They have many applications in various parts of mathematics as well as physics. (See for example [2], [3], and [4]).

Since Lie algebroids are in fact generalized tangent bundles, it's quite normal to try to redefine notions like exterior derivative, vertical and complete lift, Lie derivative, SemiSpray and etc on Lie aglebroids, which most of them has been carried out by now.

At first in this paper, we review the basic concepts of Lie algebroids, fix our notation and address the prolongation of Lie algebroids and it's basic properties. Mostly, we use W. A. Poor's [6] notation for vector bundles and extend it to Lie algebroids based on the new basis we present for the prolongation. Then we have generalized the notion of horizontal subbundle on Lie algebroids in a very natural manner. The relations between this notion and the related concepts like covariant derivative and linear connections have been declared.

A Lie algebroid over a manifold M may be thought of as a "generalized tangent bundle" of M. Here is the definition.

Definition 1.1 A Lie algebroid over a manifold M^m is a vector bundle $\mathcal{A} \xrightarrow{\pi} M$ (of rank n) equipped with a Lie algebra structure $[\ ,\ ]$ on its space of sections and a bundle map $\rho: \mathcal{A} \longrightarrow M$ (called the anchor) which induces a Lie algebra homomorphism (also denoted ρ) from sections of $\mathcal{A}$ to vector fields on M. The identity

$$[X, fY] = f[X, Y] + (\rho(X)f)Y$$

must be satisfied for every smooth function f on M.

The standard local coordinates on $\mathcal{A}$ (as a vector bundle) have the form $(p, a) = (x^i(p), a^j)$where the x^i's are coordinates on the base manifold M and the a^j's are linear coordinates on the fibres, associated with a locall basis φ_j of sections of $\mathcal{A}$.

For a bundle chart (φ, U) where $\varphi: \pi^{-1}(U) \longrightarrow \mathbb{R}^n$, we denote the components as φ^j's ($\mathbb{R}^n$ is the standard fiber of $\mathcal{A}$.)

So the local basis of $T_p\mathcal{A}$ consists of $\{\frac{\partial}{\partial \tilde{x}^i}(p)\}_{i=1}^m$ and

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$\{\frac{\partial}{\partial \varphi^j}(p)\}_{j=1}^n$, where $\tilde{x}^i = x^i \circ \pi$.

In terms of such coordinates, the bracket and anchor have expressions

$[\varphi_i, \varphi_j] = \sum L_{ij}^k \varphi_k$

and

$$\rho(\varphi_i) = \sum \rho_i^j \frac{\partial}{\partial x^j}$$

where the L_{ij}^k and ρ_i^j are *"structural functions"* lying in $C^\infty(U)$.

We also have the relations

$$\rho([X,Y]) = [\rho(X), \rho(Y)]$$

and

$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$

for all sections $X, Y, Z \in \Gamma\mathcal{A}$.

The basic example of a Lie algebroid over M is the tangent bundle TM itself, with the identity map as anchor. Clearly, the structure functions are $L_{ij}^k = 0$ and $\rho_i^j = \delta_i^j$.

Any integrable subbundle of TM is a Lie algebroid with the inclusion as anchor and the induced bracket.

On the other hand, any Lie algebra $\mathcal{g}$ is a Lie algebroid over a point. More generally, if P is a principal G-bundle over M, then TP/G is a vector bundle over M whose sections are the G-equivariant vector fields.

A function f on M can be lifted to a function $\tilde{f}$ on vector bundle $E \longrightarrow M$ by

$\tilde{f}(a) = f(\pi(a))$, for $a \in E$.

A section of the dual bundle $\lambda: E^* \longrightarrow M$ also defines a function $\hat{\theta}$ on E by means of

$\hat{\theta}(a) =< \theta_x, a >$, for $a \in E_x$.

A function of this kind will be called a *linear function*. When θ is the differential of a function f on M, the corresponding linear function will be denoted by $\dot{f}$. Therefore $\dot{f} = \widehat{df}$.

Theorem 1.2 (*Local splitting*) Let $x_0 \in M$ be a point where ρ_{x_0} has rank q. There exists a coordinates $(x_1, \ldots, x_q, y_1, \ldots, y_{m-q})$ valid in a neighborhood U of x_0 and a basis of sections $\{\varphi_1, \ldots, \varphi_r\}$ of $\mathcal{A}$ over U, such that

$\rho(\varphi_i) = \frac{\partial}{\partial x^i}$, $(i = 1, \ldots, q)$,

$\rho(\varphi_i) = \rho_i^j \frac{\partial}{\partial y^j}$, $(i = q+1, \ldots, n)$

Double Lie algebroid $\mathcal{LA}$ to $\mathcal{A}$, also called the prolongation of $\mathcal{A}$, plays the role of $T(TM) \xrightarrow{\tau_{TM}} TM$ for $\mathcal{A}$. It is defined as follows:

Definition 1.3 The total space of the prolongation is the total space of the pull-back of $\pi_*: T\mathcal{A} \longrightarrow TM$ by the anchor map ρ

$\mathcal{LA} = \{(b, v) \in \mathcal{A} \times T\mathcal{A} \mid \rho(b) = \pi_*(v)\}$

but fibered over $\mathcal{A}$ by the mapping $pr_1: \mathcal{LA} \longrightarrow \mathcal{A}$, given by $pr_1(b, v) = \tau_{\mathcal{A}}(v)$ where $\tau_{\mathcal{A}}: T\mathcal{A} \longrightarrow \mathcal{A}$ is the tangent projection.

In fact, by showing $\mathcal{LA}$ as

$$\{(a, b, v) \in \mathcal{A} \times \mathcal{A} \times T\mathcal{A} \mid \pi(a) = \pi(b), v \in T_a\mathcal{A} \text{ and } \rho(b) = \pi_*(v)\}$$

one can easily define three projections (pr_1, pr_2 and pr_3) to $\mathcal{A}$, $\mathcal{A}$ and $T\mathcal{A}$, respectively.

An element of $L\,\mathcal{A}$ is said to be *vertical* if it is in the kernel of pr_2. Therefore it is of the form $(a, 0, v)$ with $v \in T_a\mathcal{A}$. The set of vertical elements in $\mathcal{LA}$ is a vector subbundle of $\mathcal{LA}$ and will be denoted by $V\mathcal{LA}$. One can define vertical lift as we had in the case of ordinary vector bundles. For a and b in fiber $\mathcal{A}_p$, the vertical lift of b through a is defined as

$$\mathcal{I}_a b = (a, 0, I_a b)$$

Where $I_a b = \frac{d}{dt}|_{t=0}(a + tb)$ is the vertical lift of elements of $\mathcal{A}_p$. Thus the vertical lift of a section X of $\mathcal{A}$ is denotes by X^V and

$(X^V)_a = \mathcal{I}_a X_{\pi(a)} = (a, 0, X_{\pi(a)}^v)$

where again X^v denotes the vertical lift of X to $T\mathcal{A}$.

Martinez [5] also defines a *complete lift* of section X of $\mathcal{A}$ to a unique section X^C of $\mathcal{LA}$ as

$$(X^C)_a = (X_{\pi(a)}, (X^c)_a)$$

where X^c is the ordinary complete lift on $T\mathcal{A}$ which is computed by Anastasiei [1]. In the local form we have:

$$X_{(p,a)}^C = X^k(p)\rho_k^i \frac{\partial}{\partial \tilde{x}^i}(p, a) + (\rho_r^i \frac{\partial X^k}{\partial x^i}(p) - X^s L_{sr}^k) a^r \frac{\partial}{\partial \varphi^k}(p, a)$$

Lemma 1.4 [5] The complete and vertical lift satisfy the properties

$(fX)^v = \tilde{f}X^C$ and $(fX)^C = \tilde{f}X^C + \dot{f}X^V$ for $f \in C^\infty(M)$ and $X \in \Gamma\mathcal{A}$.

Theorem 1.5 [5] There exists one and only one Lie algebroid structure on $pr_1: \mathcal{LA} \to \mathcal{A}$ such that the anchor is pr_3 and the bracket [,] satisfies the relations

$[X^V, Y^V] = 0$,

$[X^V, Y^C] = [X, Y]^V$,

$[X^C, Y^C] = [X, Y]^C$,

for $X, Y \in \Gamma\mathcal{A}$.

Results

Let $(\mathcal{A}, \pi, M, \rho)$ be a transitive Lie algebroid of rank n over a smooth manifold M^m. For $a \in \mathcal{A}_p$ if $\{\varphi_1(p), \dots, \varphi_{n-m}(p)\}$ is a local basis for $\mathrm{Ker}(\rho_p)$, we can extend it to a local basis for $\mathcal{A}_p$ say $\{ \varphi_1(p), \dots, \varphi_{n-m}(p), \dots, \varphi_n(p) \}$. On the other hand, Martinez [5] shows that $\{(a, 0, \varphi_k^V)\}_{k=1}^n$ and $\{(a, \varphi_i, \frac{\partial}{\partial \tilde{x}^i})\}_{i=1}^n$ is a local basis for $(\mathcal{LA})_a$.

So we can write the basis as

$\{(a, 0, \varphi_k^V)\}_{k=1}^n$, $\{(a, \varphi_i, 0)\}_{1=1}^{n-m}$

Definition 2.1 We call a vector subbundle $\mathcal{H}$ of a double Lie algebroid $\mathcal{LA}$ of a transitive Lie algebroid $\mathcal{A}$, *horizontal* provided that

$$\mathcal{LA} = V\mathcal{LA} \oplus \mathcal{H}$$

Theorem 2.2 For every double Lie algebroid $\mathcal{LA}$ there exists a horizontal subbundle.

Proof: Let H be a horizontal subbundle of $\mathcal{A}$. We claim that $\mathcal{H} := \mathrm{pr_3}^{-1}(H)$ has the properties of a horizontal subbundle of $\mathcal{LA}$.

For $a \in \mathcal{A}$, and any choice of $(a, b, u), (a, c, v) \in \mathcal{H}_a$ and $\lambda \in \mathbb{R} \setminus \{0\}$, we have $u - \lambda v$, $(a, b - \lambda c, u - \lambda v) \in \mathcal{H}_a$ since $u, v \in H_{\rho(a)}$. Hence $\mathcal{H}_a$ is a vector subspace of $\mathcal{LA}_a$ and so $\mathcal{H}$ is a vector subundle of $\mathcal{LA}$.

Clearly, $V\mathcal{LA} \cap \mathcal{H} = 0$. Consider the function

$$(pr_3)_{|\mathcal{H}} : \mathcal{H} \to H$$

So $\mathrm{Ker}((pr_3)_{|\mathcal{H}}) = \mathrm{Ker}(pr_3)$, also it's obvious that $\mathrm{Ker}(pr_3) = \{(a, b, 0) | b \in \mathrm{Ker}(\rho_a)\}$ has the same dimension as $\mathrm{Ker}(\rho_a)$. So

$$\dim\big(\mathrm{Ker}(pr_3)_{|\mathcal{H}}\big) = \mathrm{rank}(\mathcal{A}) - \mathrm{rank}(TM) = n - m$$

and

$$\dim(\mathcal{H}) = \dim(H) + \dim(\mathrm{Ker}(pr_3)) = m + n - m = n.$$

Thus $\mathcal{H}$ is a horizontal subbundle of $\mathcal{LA}$.

The above theorem shows that for every double Lie algebroid, there exists (at least) a horizontal subbundle. Immediately, one can define *connection map* for each horizontal subbundle of a double Lie algebroid:

Definition 2.3 We define the connection map $\mathcal{K}: \mathcal{LA} \to \mathcal{A}$ as follows:

$$\mathcal{K}(\mathcal{H}) = 0$$

$$\mathcal{K}(a, 0, I_v w) = w \quad \text{for } a, v, w \in \mathcal{A}.$$

Considering the basis given in the last section one can easily see that

$$\mathcal{K}\big(o, \varphi_j^V\big) = \varphi_j$$

$$\mathcal{K}(\varphi_i\, , 0) = \overline{N}_i^k \varphi_k$$

$$\mathcal{K}(\varphi_i\, , \tfrac{\partial}{\partial \tilde{x}^i}) = \widetilde{N}_i^k\, \varphi_k$$

where $\overline{N}_i^k, \widetilde{N}_i^k : \pi^{-1}(U) \longrightarrow \mathbb{R}$. We call these, connection coefficients.

Definition 2.4 We call a horizontal subbundle $\mathcal{H}$ of a double Lie algebroid $\mathcal{LA}$, linear if the connection coefficients are linear.

Remark Note that the above definition is independent of the choice of coordinates. In fact, M. Anastasiei ([1]) shows that changing coordinates on $\mathcal{A}$ is done based on the following invertibale matrices:

$$J = \left(\frac{\partial \widetilde{x^\iota}}{\partial x^j}\right) \; and \quad M = \left(M_b^a(x)\right)$$

and changing coordinates on $\mathcal{LA}$ is done based on the following invertible matrices:

$$J = \left(\frac{\partial \widetilde{x^\iota}}{\partial x^j}\right) \; and \quad M_x = \left(M_b^a(x)\right) \; and \; \widetilde{M} = \left(\widetilde{M}_b^a(\xi)\right)$$

where x^i and $\widetilde{x^\iota}$ are coordinates of two charts on M. So let $\{\varphi_i\}$ and $\{\psi_i\}$ be two local basis of $\mathcal{A}$ and Φ , Ψ be the corresponding bundle charts on $\mathcal{LA}$. Then for $m \in M$ and $(a_m, b_m, v_a) \in \mathcal{LA}$ we have:

$$(a, b, v) = \begin{pmatrix} (\widetilde{m}, M_m a), \\ (\widetilde{m}, M_m b), \left(\widetilde{m}, M_m a, \rho_j^i b^j \dfrac{\partial}{\partial\, \widetilde{x^\iota}}, \widetilde{M}_j^i(a) v^j \dfrac{\partial}{\partial \varphi^i}\right) \end{pmatrix}$$

where the left hand side is written with resrepct to Φ and the other side with respect to Ψ.

Thus, one can write a basis element $(a, \varphi_i, 0)$ of $\mathcal{LA}$ as

$$(a, \varphi_i, 0) = \Big((\widetilde{m}, M_m a), M_i^j(m)\psi_j, 0\Big).$$

Applying K to both sides, we have:

$$\overline{N}_i^j(a)\varphi_j = M_i^l(\widetilde{m})\overline{\mathcal{N}}_l^j(\widetilde{m}, M_m a)\varphi_j$$

where $\overline{N}$, $\overline{\mathcal{N}}$ are connection coefficients of Φ , Ψ , respectively.

So if $\mathcal{N}_l^j$'s are linear, then

$$\begin{aligned} \overline{N}_i^j(\lambda a + b)\varphi_j &= M_i^l(\widetilde{m})\overline{\mathcal{N}}_l^j\big(\widetilde{m}, M_m(\lambda a + b)\big)\varphi_j \\ &= \Big(\lambda M_i^l(\widetilde{m})\overline{\mathcal{N}}_l^j(\widetilde{m}, M_m a) + M_i^l(\widetilde{m})\overline{\mathcal{N}}_l^j(\widetilde{m}, M_m b)\Big)\varphi_j \\ &= \Big(\lambda \overline{N}_i^j(a) + \overline{N}_i^j(b)\Big)\varphi_j \end{aligned}$$

which means that $\overline{N}_i^j$'s are linear too. One can use similar proceedure to prove the independency of $\widetilde{N}_i^k$'s of the choice of coordinates.

Linearity of $\overline{N}_i^k$ and $\widetilde{N}_i^k$ allow us to write:

$$\overline{N}_i^k(p, e) = \overline{N}_i^k\left(e^j(p)\varphi_j(p)\right) = e^j(p)\overline{N}_i^k\left(\varphi_j(p)\right) = e^j(p)\overline{\Gamma}_{ij}^k(p)$$

and

$\widetilde{N}_i^k(p,e) = \widetilde{N}_i^k\left(e^j(p)\varphi_j(p)\right) =$

$e^j(p)\widetilde{N}_i^k\left(\varphi_j(p)\right) = e^j(p)\tilde{\Gamma}_{ij}^k(p)$

Where $\bar{\Gamma}_{ij}^k$, $\tilde{\Gamma}_{ij}^k: U \longrightarrow \mathbb{R}$. Similar to the ordinary case of vector bundles, one can call $\bar{\Gamma}_{ij}^k$, $\tilde{\Gamma}_{ij}^k$ *Christoffel symbols* of $\mathcal{H}$.

Based on this, and applying the same method as used for the ordinary case of vector bundles, i.e. $\nabla_X Y := \mathcal{K} \circ Y_* \circ X$, we can derive a covariant derivative of $\mathcal{H}$:

Theorem 2.5 The mapping

$$\bar{\nabla}: \Gamma\mathcal{A} \times \Gamma\mathcal{A} \longrightarrow \Gamma\mathcal{A}$$

$$\bar{\nabla}_X Y = \mathcal{K} \circ Y^C \circ X$$

is a covariant derivative (linear connection) on $\mathcal{A}$.

Proof: $\mathbb{R}$-bilinearity of $\bar{\nabla}$ is obvious. So let $f \in C^\infty(M)$ and $X, Y \in \Gamma\mathcal{A}$ then

$$\begin{aligned}
\bar{\nabla}_{fX} Y &= \mathcal{K} \circ Y^C \circ (fX)\\
&= \mathcal{K}\left(Y \circ \pi(fX),\ Y^j \rho_j^i \frac{\partial}{\partial \tilde{x}^i} + \left(\rho_r^i \frac{\partial Y^j}{\partial x^i} - Y^s L_{sr}^j\right) fX^r\ \frac{\partial}{\partial \varphi^j}\right)\\
&= \mathcal{K}\left(0{,}0 + \left(\rho_r^i \frac{\partial Y^j}{\partial x^i} - Y^s L_{sr}^j\right) fX^r\ \frac{\partial}{\partial \varphi^j}\right)\\
&\quad + \mathcal{K}((Y^t\varphi_t)\pi(fX), 0)\\
&\quad + \mathcal{K}\left(((Y^t\varphi_t)\pi(fX), Y^j \rho_j^i \frac{\partial}{\partial \tilde{x}^i}\right)\\
&= f\mathcal{K}\left(0{,}0 + \left(\rho_r^i \frac{\partial Y^j}{\partial x^i} - Y^s L_{sr}^j\right) X^r\ \frac{\partial}{\partial \varphi^j}\right)\\
&\quad + Y^t \bar{N}_t^j(fX)\varphi_k + Y^i \widetilde{N}_i^j(fX)\varphi_k\\
&= f\mathcal{K}\left(0{,}0 + \left(\rho_r^i \frac{\partial Y^j}{\partial x^i} - Y^s L_{sr}^j\right) X^r\ \frac{\partial}{\partial \varphi^j}\right)\\
&\quad + fY^t \bar{\Gamma}_{tj}^k \varphi_k + fY^i\, \tilde{\Gamma}_{ij}^k \varphi_k\\
&= f\bar{\nabla}_X Y.
\end{aligned}$$

On the Other hand

$$\begin{aligned}
\bar{\nabla}_X fY &= \mathcal{K} \circ (fY)^C \circ X\\
&= \mathcal{K} \circ \left(\tilde{f} Y^C\right) \circ X + \mathcal{K} \circ \left(\dot{f} Y^V\right) \circ X\\
&= \mathcal{K}\left(\tilde{f}(X) Y^C(X)\right) + \mathcal{K}\left(\dot{f}(X) Y^V(X)\right)\\
&= \tilde{f}(X)\bar{\nabla}_X Y + \dot{f}\ \cdot Y\\
&= (f \circ \pi \circ X)\bar{\nabla}_X Y + df\ \cdot Y\\
&= f\bar{\nabla}_X Y + (\rho(X) \cdot f)Y
\end{aligned}$$

This completes the proof.

Naturally, we expect to see that the *inverse image* of a linear horizontal subbundle of $\mathcal{A}$, is linear with our new definition; i.e.

Theorem 2.6 If H is a linear horizontal subbundle of $\mathcal{A}$, then $\mathcal{H} := {pr_3}^{-1}(H)$ is a linear horizontal subbundle of $\mathcal{LA}$.

Proof: By Theorem 2.2, $\mathcal{H}$ is a Horizontal subbundle of $\mathcal{LA}$; so we need to prove that $\mathcal{H}$ is linear.

For (H, N_i^j) and $(\mathcal{H}, \widetilde{N}_i^j, \bar{N}_t^k)$ we'll show that $N_i^j = \widetilde{N}_i^j$ and $\bar{N}_t^k = 0$.

$\mathrm{K}(\varphi_k, 0) = \bar{\mathrm{N}}_k^t\, \varphi_t$; but $(\varphi_k, 0) \in \mathcal{H}$ since $pr_3(\varphi_k, 0) = 0 \in H$. Hence $\mathcal{K}(\varphi_k, 0) = 0$ and so $\bar{N}_t^k$=0 for any choice of t and k.

Moreover, For $a \in \mathcal{A}$, $\frac{\partial}{\partial \tilde{x}^i}(a)$ is a vector in $T_a\mathcal{A}$. Thus $\frac{\partial}{\partial \tilde{x}^i}(a) = h \oplus v$ for some $h \in \mathcal{H}$ and $v \in V\mathcal{A}$. Vector v is vertical, so $K(v) = N_i^j(a)\varphi_j(\pi(a))$. By linearity of vertical lift we have

$v(a) = N_i^j(a)\varphi_j^v(\pi(a))$ i.e.,

$$\frac{\partial}{\partial \tilde{x}^i}(a) = h \oplus \left(N_i^j(a)\varphi_j^v\left(\pi(a)\right)\right).$$

Similarly,

$$\left(\varphi_i, \frac{\partial}{\partial \tilde{x}^i}\right) = \bar{h} \oplus \left(\widetilde{N}_i^j(a)\left(0, \varphi_j^v(\pi(a))\right)\right)$$

where $\bar{h} \in \mathcal{H}$.

Since $pr_3(\varphi_i, \frac{\partial}{\partial \tilde{x}^i}) = \frac{\partial}{\partial \tilde{x}^i}$ we can write:

$$\begin{aligned}
&h \oplus N_i^j(a)\varphi_j^v(\pi(a)) =\\
&pr_3\left(\bar{h} \oplus \widetilde{N}_i^j(a)\left(0, \varphi_j^v(\pi(a))\right)\right)\\
&\qquad = pr_3(\bar{h}\}) \oplus \widetilde{N}_i^j(a)(\varphi_j^v(\pi(a)))
\end{aligned}$$

So $N_i^j = \widetilde{N}_i^j$ which completes the proof.

We can say even more. Somehow the converse is also true.

Theorem 2.7 If $\mathcal{H} \subseteq \mathcal{LA}$ is a linear horizontal subbundle such that all $\bar{N}_t^k$'s are zero, then $H := \mathrm{pr}_3(\mathcal{H})$ is a linear horizontal subbundle of $T\mathcal{A}$.

Proof: Clearly (by linearity and smoothness of pr_3) H is a subbundle.

If $v \in V\mathcal{A} \cap H$ then $v \in pr_3(\mathcal{H})$. So $\exists\, a, b \in \mathcal{A}_p\ \ s.t.\ \ (a, b, v) \in \mathcal{H}_a\ \ i.e.,\ \ \rho(b) = \pi_*(v)$

Since $v \in V\mathcal{A}$, we have $\rho(b) = 0$. Hence $b \in ker(\rho)$. Thus by the assumption, $\mathcal{K}(b, 0) = b^t \bar{N}_t^k\, \varphi_k = 0$ where $b = b^t\varphi_t$. So $(a, b, 0)$ is horizontal.

On the other hand

$$\overbrace{(a, b, v)}^{\in\mathcal{H}} = (a, 0, v) \oplus \overbrace{(a, b, 0)}^{\in\mathcal{H}} \ \Rightarrow\ (0, v) \in \mathcal{H}\ \ i.e.\ \ v = 0.$$

Therefore $V\mathcal{A} \cap H = 0$.

On the other hand, $rank(H) = rank(\mathcal{H}) - \dim(ker(\rho))$ since the mapping

$$pr_{2|_{Ker(pr_3)}}: \mathrm{Ker}(pr_3) \to \mathrm{Ker}(\rho)$$

is a *bijection* on each fiber.

So $dim(H) = n - (n - m) = m$. Thus H is a horizontal subbundle of $T\mathcal{A}$.

Put $\mathbf{B} := pr_3{}^{-1}(H)$. Clearly $\mathcal{H} \leq \mathbf{B}$. Agian by Theorem 2.2 $\mathbf{B}$ is a horizontal subbundle of $\mathcal{LA}$ i.e., $\mathcal{LA} = \mathbf{B} \oplus V\mathcal{LA}$. Consequently $\mathbf{B} = \mathcal{H}$ and So by Theorem 2.6 we have $\widetilde{N}_i^j = N_i^j$. Thus linearity of $\widetilde{N}_i^j$ cause N_i^j to be linear too. Hence H is linear.

So there is a one to one correspondence between linear horizontal subbundles of $T\mathcal{A}$ and linear horizontal subbundles of $\mathcal{LA}$ that $\overline{N}_t^k = 0$.

Example 2.8 Let $\mathcal{A} = TM$. Then it's easy to see that $\mathcal{LA} = TTM$, but $X^C \neq X_*$ for $X \in \Gamma(TM) = \mathfrak{X}(M)$, since

$$X^C_{(p,a)} = X^k(p)\frac{\partial}{\partial \tilde{x}^k}(p,a) + \frac{\partial X^k}{\partial x^r}(p)a^r\frac{\partial}{\partial \varphi^k}(p,a)$$

while

$$(X_*)_{(p,a)} = a^k\frac{\partial}{\partial \tilde{x}^k}(p,a) + \frac{\partial X^k}{\partial x^r}(p)a^r\frac{\partial}{\partial \varphi^k}(p,a).$$

On the other hand, we have two linear connections on TTM now: the ordinary one which is denoted by $\nabla_U X = K \circ X_* \circ U$ and the one that comes form Theorem and will be denoted by $\overline{\nabla}_U X = \mathcal{K} \circ X^C \circ U$ for $U, X \in \Gamma TM$.

Surprisingly, these two connections are equal!

In fact on one side we have

$$\nabla_U X = K \circ X_* \circ U = K\left(U^k\frac{\partial}{\partial \tilde{x}^k} + \frac{\partial X^k}{\partial x^r}U^r\frac{\partial}{\partial \varphi^k}\right)$$

$$= U^k N_k^l(X)\varphi_l + \frac{\partial X^k}{\partial x^r}U^r\varphi_k$$

$$= U^k X^i \Gamma_{ki}^l \varphi_l + \frac{\partial X^k}{\partial x^r}U^r\varphi_k = X^i U^k U^r\frac{\partial X^k}{\partial x^r}\Gamma_{ki}^l\varphi_l$$

and on the other side

$$\overline{\nabla}_U X = \mathcal{K} \circ X^C \circ U = \mathcal{K}\left(X^k\frac{\partial}{\partial \tilde{x}^k} + \frac{\partial X^k}{\partial x^r}U^r\frac{\partial}{\partial \varphi^k}\right)$$

$$= X^k\overline{N}_k^l\varphi_l + \frac{\partial X^k}{\partial x^r}U^r\varphi_k$$

$$= X^k U^i\overline{\Gamma}_{ki}^l\varphi_l + \frac{\partial X^k}{\partial x^r}U^r\varphi_k$$

$$= X^k U^i U^r\frac{\partial X^l}{\partial x^r}\Gamma_{ik}^l\,\varphi_l\,.$$

that confirm our claim.

Note that if $\overline{\nabla}: \Gamma\mathcal{A} \times \Gamma\mathcal{A} \longrightarrow \Gamma\mathcal{A}$ is a linear connection on $\mathcal{A}$ and $U, X \in \Gamma\mathcal{A}$ and $U_p = u$ then

$$\overline{\nabla}_u X = u^i X^j\left(\overline{\nabla}_{\varphi_i}\varphi_j\right) + u^i(\rho(\varphi_i)X^j)\varphi_j$$
$$= u^i X^j\left(\dot{\Gamma}_{ij}^k\,\varphi_k\right) + u^i(\rho(\varphi_i)X^j)\varphi_j$$

Where $\dot{\Gamma}_{ij}^k$ are the *Christoffel symbols* of $\overline{\nabla}$.

Moreover,

$$\mathcal{I}_u(\overline{\nabla}_u X) = \left(u, 0, I_u(\overline{\nabla}_u X)\right)$$
$$= \left(u, 0, u^r\rho_r^s\frac{\partial X^i}{\partial x^s}\frac{\partial}{\partial \varphi^i} + u^r X^i I_u\left(\dot{\Gamma}_{ri}^k\,\varphi_k\right)\right)$$
$$= \left(u, 0, \left(\left(\rho_r^s\frac{\partial X^i}{\partial x^s} + X^k\dot{\Gamma}_{rk}^i\right)u^r\frac{\partial}{\partial \varphi^i}\right)\right).$$

Again in local coordinates we have

$$X^C(u) = \left(u, X(\pi(u)), X^c(u)\right)$$
$$= \left(u, X(\pi(u)), X^i(p)\rho_i^s\frac{\partial}{\partial \tilde{x}^s}(u) + \left(\rho_r^s\frac{\partial X^i}{\partial x^s} - X^k L_{kr}^i\right)u^r\frac{\partial}{\partial \varphi^i}\right).$$

And finally

$$X^C(u) - \mathcal{I}_u(\overline{\nabla}_u X) = \left(u, X(\pi(u)), X^i(p)\rho_i^s\frac{\partial}{\partial \tilde{x}^s} - \left(L_{kr}^i + \dot{\Gamma}_{rk}^i\right)X^k u^r\frac{\partial}{\partial \varphi^i}\right) \qquad (1)$$

Theorem 2.9 If $\overline{\nabla}: \Gamma\mathcal{A} \times \Gamma\mathcal{A} \to \Gamma\mathcal{A}$ is a linear connection on $\mathcal{A}$ and If we put $\mathcal{H}_a := \{X^C(a) - \mathcal{I}_a(\overline{\nabla}_a X) \mid X \in \Gamma A\}$ for $p \in M$ and $a \in \mathcal{A}_p$ then $\mathcal{H} := \bigcup_{a\in\mathcal{A}}\mathcal{H}_a$ is a linear horizontal subbundle of $\mathcal{LA}$.

Proof: Vertical and complete lift and also $\overline{\nabla}$ are all linear, so

$$(X^C(a) - \mathcal{I}_a\,\overline{\nabla}_a X) - \lambda(Y^C(a) - \mathcal{I}_a\,\overline{\nabla}_a Y)$$
$$= (X - \lambda Y)^C(a) - \mathcal{I}_a\left(\overline{\nabla}_a(X - \lambda Y)\right)$$

for arbitrary $X, Y \in \Gamma\mathcal{A}$ and $\lambda \in \mathbb{R}$. Hence $\mathcal{H}_a$ is a vector subbundle of $(\mathcal{LA})_a$.

Moreover, if $(X^C(a) - \mathcal{I}_a\,\overline{\nabla}_a X) \in (V\mathcal{LA})_a$ then

$$pr_2(X^C(a) - \mathcal{I}_a\,\overline{\nabla}_a X) = 0.$$

Hence $pr_2(X^C(a)) = 0$ since $\mathcal{I}_a\,\overline{\nabla}_a X$ is vertical. So $pr_2\left(X(\pi(a)), X^c(a)\right) = 0$, *i.e.*, $X(\pi(a)) = 0$ which means, $\forall k\ X^k(\pi(a)) = 0$.

So $(X^C(a) - \mathcal{I}_a\,\overline{\nabla}_a X) - \lambda(Y^C(a) - \mathcal{I}_a\,\overline{\nabla}_a Y) = 0$ i.e.,

$\mathcal{H}_a \cap (V\mathcal{LA})_a = 0$

To complete the proof, we must show that one can write any arbitary element $(u, b, v) \in \mathcal{LA}$ as a summation of a vertical element and one of the type

$X^C(a) - \mathcal{I}_a\,\overline{\nabla}_a X$

for some $X \in \Gamma\mathcal{A}$.

Considering the mentioned basis for $\mathcal{LA}$, the problem is clearly solved for the case of vertical elements of the basis.

So let $(\varphi_t, 0)$ be a basis member that $\rho(\varphi_t) = 0$, putting

$Y = \left(0, \left(L_{kr}^i - \dot{\Gamma}_{rk}^i\right)u^r\delta_t^k\frac{\partial}{\partial \varphi^i}\right)$ and $X := \varphi_t$, one can easily check that:

$$(\varphi_t, 0)_a = Y_a + (X_a^C - \mathcal{I}_a\,\overline{\nabla}_a X)$$

since Y is vertical and $X^C - \mathcal{J}_a \overline{\nabla}_a X = \left(\varphi_t, -(L^i_{kr} - \dot{\Gamma}^i_{rk})u^r \delta^k_t \frac{\partial}{\partial \varphi^i}\right)$.

Now let $(\varphi_s, \frac{\partial}{\partial \tilde{x}^s})$ be another member of the basis of the form $\rho(\varphi_s) = \frac{\partial}{\partial \tilde{x}^s}$. This time put $X = \varphi_s$ and $Y = (L^i_{kr} + \dot{\Gamma}^i_{rk})u^r \delta^k_s \frac{\partial}{\partial \varphi^i}$ (which is vertical).

Hence

$$(X^C - \mathcal{J}_a \overline{\nabla}_a X)_a = \left(\varphi_s(p), \frac{\partial}{\partial \tilde{x}^s}(a) - Y(a)\right).$$

So

$$\left(\varphi_s, \frac{\partial}{\partial \tilde{x}^s}\right) = Y_a + (X^C - \mathcal{J}_a \overline{\nabla}_a X)_a .$$

This completes the proof.

So our definition of horizontal subbundle looks really compatible with our expectations.

In order to observe the application of the new definition of horizontal subbundle on Lie algebroids we need to review some usefull facts about transitive Lie algebroids. We use [4] for this purpose.

Again let $(\mathcal{A}, \pi, M, \rho, [,])$ be a transitive Lie algebroid of rank n over a smooth manifold M^m and let $L := ker(\rho)$. Clearly L is a Lie algebra bundle and the short exact sequence $0 \to L \to \mathcal{A} \to TM \to 0$ of vector bundles always splits. One can define a linear TM-connection on L for any splitting λ of the above sequence:

$$\nabla^\lambda : \mathfrak{X}(M) \times \Gamma L \to \Gamma L$$
$$\nabla^\lambda_U S = [\lambda(U), S]$$

called *adjoint connection* and satisfies

$$\nabla^\lambda_U [S_1, S_2] = \left[\nabla^\lambda_U S_1, S_2\right] + \left[S_1, \nabla^\lambda_U S_2\right].$$

The curvature of ∇^λ is denoted by R^λ and the 2-form $\Omega^\lambda \in A^2(M, L)$ is defined such that

$$2\Omega^\lambda(U, V) = [\lambda(U), \lambda(V)] - \lambda([U, V])$$

and relates to the curvature tensor of ∇^λ by

$$R^\lambda(U, V)(s) = \left[2\Omega^\lambda(U, V), s\right]$$

Consequently, the Lie bracket on $\mathcal{A}$ with respect to the decomposition $\mathcal{A} = \lambda(TM) \oplus L$ is written as follows:

$$\begin{aligned}[\lambda(U) + S_1, \lambda(V) + S_2] &= \lambda([U, V]) + \nabla^\lambda_U S_2 - \nabla^\lambda_V S_1 + [S_1, S_2] \\ &\quad + 2\Omega(U, V).\end{aligned}$$

Conversely, if L is a bundle of Lie algebras, and ∇ is a connection on L that preserves its Lie bracket and the curvature of ∇ is in the form $[2\Omega(U, V), S]$ for $S \in \Gamma L$ and some $\Omega^\lambda \in A^2(M, L)$, then one can make $TM \oplus L$ into a transitive Lie algebroid by defining a Lie bracket on $\Gamma(TM \oplus L)$ as follows:

$$[U + S_1, V + S_2] = [U, V] + \nabla_U S_2 - \nabla_V S_1 + [S_1, S_2] + 2\Omega(U, V).$$

Thus we have the following theorem:

Theorem 2.10 [4] All transitive Lie algebroids have the above structure.

Having these in mind, we can have a Lie algebroid structure on $TM \oplus L(TM)$ where $L(TM)$ is the bundle of all vector bundle morphisms on TM.

Let ∇ be a torsion-free connection on M. Denote also, the induced connection on $L(TM)$, by ∇; the curvature tensors of ∇ on M and $L(TM)$ by R and R' respectively. If we consider the Lie bracket $[T, S] := T \circ S - S \circ T$ then It is easy to check that $L(TM)$ is a bundle of Lie algebras and ∇ preserves its Lie bracket, i.e.,

$$\nabla_U [T, S] = [\nabla_U T, S] + [T, \nabla_U S].$$

Moreover, a simple calculation shows that $R'(U, V)(T) = [R(U, V), T]$ where $U, V \in \mathfrak{X}(M)$ and $S, T \in \Gamma L(TM)$.

So we can define a Lie algebroid structure on $TM \oplus L(TM)$ by the following Lie bracket on $\Gamma(TM \oplus L(TM))$:

$$\begin{aligned}[U + T, V + S] = [U, V] + \nabla_U S - \nabla_V T + [T, S] \\ + 2\Omega(U, V)\end{aligned}$$

and projection as the anchor map.

Now, $\mathcal{A} = TM \oplus L(TM)$ is a Lie algebroid. We define

$$\begin{aligned}\overline{\nabla}_{U+T}(V + S) := 2\nabla_U(V + S) - \nabla_V(U + T) \\ - [U + T, V + S].\end{aligned}$$

One can check that $\overline{\nabla}$ is a linear $\mathcal{A}$ −connection. So by Theorem the set

$\mathcal{H}_a := \{ X^C(a) - \mathcal{J}_a \overline{\nabla}_a X \mid X \in \Gamma A\}$ for $p \in M$ and $a \in \mathcal{A}_p$

is the horizontal subbundle of $TM \oplus L(TM)$, i.e., by (1) we should calculate

$$X^i \rho^s_i \frac{\partial}{\partial \tilde{x}^s} - \left(L^i_{kr} + \dot{\Gamma}^i_{rk}\right) X^k u^r \frac{\partial}{\partial \varphi^i}.$$

ρ is the projection, so if $\{e_k\}^n_{k=1}$ is a local basis for $\mathcal{A}$ then

$$\rho^i_k = 0 \quad k > m \qquad and \qquad \rho^i_k = 1 \quad k \leq m.$$

Clearly, It's not easy to calculate the coefficients L^i_{kr} but the tricky point in choosing the linear $\mathcal{A}$-connection allows us to omit them. In fact while calculating the Christoffel symbols of $\overline{\nabla}$, the bracket coefficients appear again with the opposit sign. So by Theorem 2.9 it remains just to evaluate $2\nabla_U(V + S) - \nabla_V(U + T)$ on a local basis, which can be done by a straightforward calculation.

Note that $\{\partial_i\} \cup \{ T^{ls}\}$ is a local basis for $\mathcal{A}$ where T^{ls}

takes ∂_l to ∂_s and annihilate the other basis members. This can make the above calculation, yet more convenient.

Discussion

A transitive Lie algebroid is in fact a tangent bundle plus a bundle of Lie algebra. On the matter of connections, we knew what happened for the tangent bundle part but not for the other part or the whole thing. Here we showed that with a natural definition of a horizontal subbundle, many relations –which we had on tangent bundles- remain unchanged in the new framework. Moreover, the paper shows that the bundle of Lie algebras (commonly denoted by L) plays the main role in determining the structure of the Lie algebraic, from the connection theory point of view.

Besides, the new natural and compatible definition, leads to construction of a better framework to deal with other important notions relating to connections like Sprays, Lagrangians and Hamiltonian and etc.

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