

Government and Central Bank Interaction under Uncertainty: A Differential Games Approach

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Received: 2016/02/08

Accepted: 2016/06/13

Abstract

Today, debt stabilization in an uncertain environment is an important issue. In particular, the question how fiscal and monetary authorities should deal with this uncertainty is of much importance. Especially for some developing countries such as Iran, in which on average 60 percent of government revenues comes from oil, and consequently uncertainty about oil prices has a large effect on budget planning, this is a significant question. For this reason, we extend in this paper the well-known debt stabilization game introduced by Tabellini (1986). We incorporate deterministic noise into that framework. Also we solve this extended game under a Non-cooperative, Cooperative and Stackelberg setting assuming a feedback information structure. The main result shows that under all three regimes, more active policies are used to track debt to its equilibrium level and the smaller this equilibrium level becomes, the more fiscal and monetary authorities are concerned about noise. Furthermore, the best-response policy configuration if policy-makers are confronted with uncertainty seems to depend on the level of anticipated uncertainty.

Keywords: Fiscal and Monetary Policy Interaction, Differential Game, Dynamic System, Uncertainty.

JEL Classification: E61, E62, E52, C7, C6.

1. Introduction

In recent years, many countries experienced an increase in government debt and deficit. The risk of a default on Greek debt formed the first serious crisis and raised the issue of debt stability. The rising of government debt in an

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environment of financial instability and low economic growth has increased the need for considering debt stability. Therefore, fiscal and monetary interactions in problems of fiscal deficit and debt stabilization have been the focus of intense attention both institutionally and academically. In many countries, government and central bank are two policy decision-making institutions, which manage independently fiscal and monetary policy, respectively. In reaching binding agreements, both authorities cannot be always successful and typically a conflict arises about whether fiscal or monetary instruments should be adjusted to stabilize government debt.

While talking about stability, we do not mean budgets have to be balanced at all times. We mean that government's solvency is threatened when government deficits become excessive and debt increases (Collignon, 2012). The increase of debt may force a government to change policy and cut planned expenditures in order to repay debt obligations. So, the fiscal authority can reduce fiscal deficit by reducing government expenditure or increasing taxes (in rich oil countries by increasing oil revenues) and hence reduce the accumulation of debt. On the other hand, the monetary authority can finance government budget by money printing or buying government debt in the bond market. In the context of dynamic games, there is an interaction between fiscal and monetary authorities concerning debt stabilization.

In some developing countries, fiscal and monetary policy-makers are interdependent. There, government deficit and debt accumulation are important for monetary policy and it can also affect monetary authorities' ability to control inflation. In recent decades, several authors considered debt stabilization from a different perspective (Collignon, 2012; Neck and Sturm, 2008; Neaime, 2010). In his pioneering study (1986), Tabellini considers the interaction between fiscal and monetary policy-makers in the framework of dynamic linear quadratic games. After then, several authors extended his model in different ways. Van Aarle et al. (1997), considers government debt stabilization within a framework of cooperative and non-cooperative games in a two-country setting. Bartolomeo and Gioacchino (2008) analyzed the interaction between policy-makers in a two-stage game. They found that debt in a Nash regime is lower than in Stackelberg regimes, and the central bank cannot guarantee monetary stability under fiscal leadership. Engwerda et al. (2013) introduced an endogenous risk premium in the Tabellini model, and using a nonlinear differential game setting, they show that in the cooperative and non-cooperative case, the equilibrium of debt depends on the strength of the risk premium parameter.

While making policy decisions, there is no full knowledge about past,

current and future economic data. Hence including uncertainty in fiscal and monetary policies interaction is inevitable and essential. Uncertainty about monetary and fiscal instruments such as inflation, exchange rate, real interest rate, expenditure, etc. has a significant role in stabilization policies (see Lane, 2003).

Also in some developing countries such as Iran, with a large oil sector in which the share of oil revenues in government budget is about 60 percent, uncertainty is an important issue. The presence of large oil shocks and large volatilities in oil revenues can change government's planned policy and leads to uncertainty about future government revenues. The negative oil shocks can increase budget deficit and debt accumulation. On the other hand, in the Iranian case, there is a game between government and central bank in which fiscal policy acts as a leader and monetary policy as a follower, and the fiscal leadership might be associated with the existence of a fiscal dominance. Under this condition, whenever government is faced with deficits, the central bank will try to offset it by creation of money. The result of this process is inflation. Also, in recent years, US and UN sanctions on the Iran nuclear program implied a big challenge for the Iranian economy and policy-makers. This issue increased uncertainty about oil production, oil prices and exchange rates.

There are quite a number of studies that consider the role uncertainty plays in policy-makers interaction. Most of the early studies consider a static (game) framework with stochastic uncertainty. They try to show the effects of different kind of uncertainty and volatility on fiscal and monetary planning. Brainard (1967) shows that uncertainty can make policy-makers more prudent and it can change response to policy actions. Bartolomeo et al. (2009) consider the fiscal and monetary interaction model of Dixit and Lambertini (2003) under multiplicative uncertainty. They show that when policy-makers face multiplicative uncertainty, the symbiosis assumption does not hold and time consistency problems arise. An increase in uncertainty reduces real output and raises inflation. Lane (2003) incorporates uncertainty into Brainard's (1967) model, showing that various forms of uncertainty affect interaction between fiscal and monetary policy-makers in EMU. Dupuis and Hostland (2001) illustrate effects of parameter uncertainty on fiscal planning. Their results show that the fiscal policy trade-off can be more difficult by introducing parameter uncertainty and increase in forecast errors. Generally speaking, most of these studies predict a more prudent behavior by policy-makers if uncertainty increases.

Now, in most macro-economic situations, the current value of variables has an impact on their future values as well. Especially in considering debt, a

dynamic approach is more appropriate. Therefore, more recent studies have considered the impact of uncertainty in dynamic models. For instance, within a simple theoretical one state two control linear-quadratic framework, Mercado and Kendrick (2000) showed that the response of a control variable to uncertainty depends on the ratio of the assigned weight on each control variable to the variance of the parameter that multiplies the control. In particular they found, different from the static studies, that control variables will be used more intensely if future uncertainty increases.

This last observation is confirmed in a simulation study by Soderstrom (2002). He also found that within a dynamic model developed by Svensson (1997), uncertainty about parameters in this macroeconomic model could lead to more aggressive action by policy-makers. He showed that for a central bank facing persistence of inflation uncertainty, it is optimal to act more aggressively to shocks. In the framework of a DSGE model, Giuli (2010) considered central bank behavior under deterministic uncertainty. Using robust control techniques (see e.g. Hansen and Sargent, 2004), he showed that model uncertainty causes monetary authority overreacts to a cost-push shock by injecting money. In an empirical study, Klomp and Haan (2009) found positive effects of fiscal and monetary policy uncertainty and government instability on economic volatility.

In this paper, we study the impact of uncertainty within a dynamic model with more than one player. Like in Hansen and Sargent (2004) uncertainty enters the model as a fictitious "evil agent". We study the impact of uncertainty, and more in particular the expectations about it, by fiscal and monetary policy-makers on debt, deficits and policies. For this purpose, we extend Tabellini's base model in two ways. First, we consider this model in a deterministic noise setting, and assume that players have their own expectation about this deterministic noise. Second, in contrast to other works such as Van Aarle et al. (1995, 1997) and Bartolomeo and Gioacchino (2008), we assume that players know the current state of the system at any point of time and determine Cooperative, Non-cooperative and Stackelberg linear feedback strategies in this uncertain setting. Using the concept of soft-constrained equilibria (see e.g. Engwerda, 2005) we derive these equilibria for this extended Tabellini model, under a Non-cooperative, a Cooperative and Stackelberg setting respectively. Then, after finding the equilibrium actions, we simulate the model for Iran's economy and analyze the outcomes.

The paper is organized as follows. Section (2) considers the uncertainty in macroeconomic variables in Iran's economy. Section (3) provides a brief introduction in the theory of differential games. The base model studied in this paper and equilibrium strategies for the Cooperative, Non-cooperative

and Stackelberg game setting are considered in section (4). Simulations with this model are considered in section (5). Section (6) compares some main observations from section (5) and makes some preliminary conjectures. Furthermore it contains some directions for future research.

2. Uncertainty in Macroeconomic Variables

In this section we consider empirical evidence for the occurrence of macroeconomic volatility during the period 1972-2012 in Iran's economy. Figure (1) shows the volatility of inflation, exchange rate, oil and real rate of interest that are obtained from the EGARCH¹ model. As is well known, in the early 1980s and the late 1980s, during the war with Iraq, Iran observed a large increase in uncertainty concerning its oil revenues. Also volatility in inflation was relatively high from the mid-1970s to the late 1990s. But the magnitude of fluctuations in inflation in the most recent period was low. Also the real rate of interest was quite volatile during the period under review. The variation in exchange rate was typically more stable than oil, inflation and real interest rate. We only observe some higher fluctuations during 1978-1980.

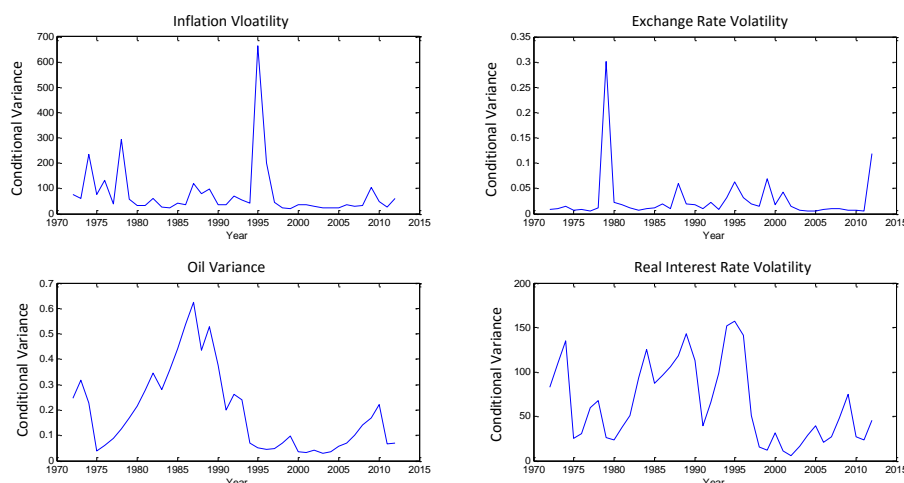


Fig. 1: Macroeconomic Variables Uncertainty

Table 1 shows the important role of oil revenues in government budget in Iran. According to the table, the Share of oil and tax revenues to government income during the period 1973-2012 is more than 56 and 32 percent, respectively. Also oil income has a significant role on deficits, as the average of government deficit to GDP with and without oil income over the past four

1. Exponential GARCH (Generalized Autoregressive Conditional Heteroscedasticity) model was proposed by Nelson (1991)

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decades is 4.3 and 16.6 respectively. However, despite the huge oil revenues, the government expenditure was always higher than its revenues, which means that deficits and the accumulation of debt are the rule in the Iran fiscal policy. During the war with Iraq (1981-1989) the issuing of base money and government debt reached its highest level, because in this period, the oil revenues decreased.

Table 1: Data

| | 1973-1980 | 1981-1990 | 1991-2000 | 2001-2012 |
|--|-----------|-----------|-----------|-----------|
| Share of oil revenues to government revenues | 73.10 | 49.90 | 59.10 | 48.07 |
| Share of Tax revenues to government revenues | 21.7 | 37.8 | 30.9 | 36.2 |
| Government deficit to GDP ratio (Without Oil Revenue) | 32.80 | 15.20 | 11.80 | 10.90 |
| Government deficit to GDP ratio (With Oil Revenue) | 7.73 | 6.61 | 1 | 3.0 |
| Base Money to GDP ratio | 16.8 | 34.7 | 17.5 | 12.48 |
| Government Debt to GDP ratio | 17.80 | 44.04 | 21.25 | 6.58 |

3. Differential Game

In this paper, we will use the theory of differential games under uncertainty for considering the interaction between fiscal and monetary policy-makers. Before considering the base model, we give a short introduction about game theory and discuss one of its branches, differential game theory, in more detail. Game theory is a method used to analyze situations, involving two or more decision-makers or players (such as individuals, governments, central banks, political parties, firms, regulatory agencies, households, etc.) in which the outcome depends on the actions of all players. Publication of the book "Theory of Games and Economic Behavior" by Von Neumann and Morgenstern (1944) is generally seen as the initial point of modern game theory. The theory was introduced as a separate research field by John Nash's fundamental works during 1950-1953. Within game theory one can distinguish two important issues: The first is the difference between Static games and Dynamic games. Simultaneous-move or Static games are games in which decisions are made once and for all, actions are implemented at the same time by all players and their moves are unseen by the other players. On the other hand, Sequential-move or Dynamic games are games that the order in which the decisions are made is important. In other words, one player moves first and the other players see the first player's move and respond to it. Furthermore, usually these types of games involve more than one action of

the involved players. The second important issue concerns the distinction between cooperative and non-cooperative games. In the cooperative case the players can communicate and make binding agreements with others. Also in this case, the players seek optimal joint actions. On the other hand, in the non-cooperative case, there are no binding agreements and each player in this game pursues his own interests which are usually conflicting with others (See Osborn & Rubinsten, 1994; Carmichael, 2005; Bauso, 2014).

Differential games are a subclass of dynamic games and introduced by Isaacs (1965) where he studied in particular so-called pursuit-evasion problems. As Engwerda (2005) expressed, optimal control theory is an important instrument for solving dynamic games and therefore is one of the roots of dynamic game theory. Hence it is clear that there is a connection between optimal control theory and differential games. Or, as Sethi and Thompson (2000) noted, differential game theory represents a generalization of optimal control theory in cases where there is more than one player (see Table 2 for more details).

Table 2: Differential Game Theory

| | One player | More than one player |
|----------------|--------------------------|---|
| Static | Mathematical programming | Static game theory |
| Dynamic | Optimal control theory | Differential game theory (Dynamic game theory) |

Source: Basar and Olsdar, 1999, p 2

In differential games one often considers the case that players use either one of the next two strategies: a so-called Open-loop strategy, or a Feedback strategy. Open-loop strategies are often referred to as pre-commitment strategies. They base their actions in principle solely on the initial state of the system. In this case each player maximizes his payoff and takes the current and future actions of opponents as given. Final actions are then obtained as those where no player has an incentive to deviate from. In this information framework, it is assumed that once the game runs, the players cannot change their actions anymore. Hence open-loop strategies are not robust to perturbations. In the feedback information case, it is assumed that players have at every point in time access to the current state of the system and they base their actions on that state. As a consequence they are able to respond to any disturbance in an optimal way. Hence Feedback strategies are robust for deviations and players can react to disturbances during the evolution of the game and adapt their actions accordingly (Engwerda, 2005 and Van Long, 2010).

Also, one can distinguish several equilibrium concepts in dynamic games

such as e.g. the Nash equilibrium and Stackelberg equilibrium. In the Nash equilibrium, no player can unilaterally change his strategy and get a better payoff. So it's a best response to the other player's strategies. In the Stackelberg equilibrium, there are two types of players, known as leader and followers, in which the leader declares his strategy first and the followers react to it.

4. The Base Model

In this section we present a simple model for stabilization of debt assuming two players affect debt by their policies, i.e. a fiscal and monetary authority. We assume that both authorities operate in an uncertain environment. For this reason, we extend Tabellini's model (1986) to incorporate uncertainty about policy interactions. We consider a single country in which government sets fiscal policy and central bank sets monetary policy. We assume that both players are interested in government debt stabilization and for realizing their objectives, they control their own policy instruments. Then, we first formulate the relationship between government and central bank in debt creation. For this purpose, we employ the government budget constraint which has been used by several authors as e.g. Sargent and Wallace (1981), Fisher and Easterly (1990), Togo (2007) and Ley (2010):

$$D_t = (1 + i_t)D_{t-1} + F_t - \Delta M_t \quad (1)$$

Here, D_t is the stock of government debt, F_t is the primary fiscal deficit (difference between government expenditure and tax revenues), i_t is the nominal interest rate and ΔM_t is the change of base money. Equation (1) shows that the government deficit must be financed by issuing of currency or new debt or a combination of them. It should be noted that the average share of oil revenue in government budget is more than 50 percent in Iran's economy. Therefore, oil income has an important role on deficit and debt targets. Hence, in equation (1), fiscal deficit is the difference of government expenditure and income from tax and oil revenues. Dividing equation (1) by nominal income, the dynamic government budget constraint is given by:

$$\dot{d}(t) = (r - g)d(t) + f(t) - m(t) \quad d(0) = d_0 \quad (2)$$

In which, d denotes government debt scaled to nominal income, f is the primary fiscal deficit scaled to nominal income, and m is the growth rate of base money scaled to nominal income. Furthermore, r and g are real interest rate and growth rate of real income (per capita) respectively, and they are assumed to be exogenous and independent of time (see, Van Aarle et al., 1997; Engwerda et al., 2013). To express the uncertainty present in the various variables and parameters in this model, we include an additive

disturbance term into the government budget constraint (2). Instead of equation (2) we consider equation (3) below,

$$\dot{d}(t) = (r - g)d(t) + f(t) - m(t) + w(t), \quad d(0) = d_0 \quad (3)$$

where $w(t)$ is a factor that represents unknown disturbances affecting the budget constraint. As already mentioned, this unknown disturbance can be government and central bank uncertainty about oil income, inflation, exchange rate, economic growth, etc. Also, we assume that debt can not grow forever. Or, in game theoretic phrasing, the no Ponzi game condition holds. The inter-temporal loss function of fiscal and monetary authorities under uncertainty are (4) and (5) respectively.

$$L_F = \frac{1}{2} \int_0^{\infty} e^{-\rho t} \{ (f(t) - \bar{f})^2 + \varphi(m(t) - \bar{m})^2 + \theta(d(t) - \bar{d})^2 - v_f w^2(t) \} dt \quad (4)$$

$$L_M = \frac{1}{2} \int_0^{\infty} e^{-\rho t} \{ (m(t) - \bar{m})^2 + \eta(f(t) - \bar{f})^2 + \tau(d(t) - \bar{d})^2 - v_m w^2(t) \} dt \quad (5)$$

The two policy-makers are trying to minimize their loss functions subject to the budget constraint (3). The parameter ρ in both loss functions indicates the discount rate and \bar{f} , \bar{m} and \bar{d} are the target levels for deficit, monetary base and debt respectively. Parameters θ and τ denote the relative weight assigned to debt by fiscal and monetary authorities. As Van Aarle et al. (1995; 1997) said, the more the fiscal policy-maker than the central bank cares about debt, we observe a stronger central bank and a weaker government and the burden of debt stabilization shifts to the government. The more τ converges to zero, the more conservative the central bank is. We assume that the government cares also about monetary growth and central bank cares also about fiscal deficits. Parameters φ and η indicate the relative weight assigned to monetary growth and deficit by the two policy-makers respectively.

Notice that the noise $w(t)$ actually occurs when time evolves. We assume that both players base their actions on the assumption that from their point of view, in future a worst-case scenario of the disturbance would occur. For that reason, the noise quadratically is included in the loss functions with a minus sign and it is assumed that a third player called "nature" tries to maximize both loss functions w.r.t. this variable $w(t)$. Parameters v_f and v_m are risk-sensitivity. They express how risk-sensitive policy-makers are w.r.t. this worst-case realization of noise. If $v_i \rightarrow \infty$ for $i = f, m$, then the optimal response functions converge to those of the "noise-free" case.

4.1 Non-Cooperative Case

In this section we consider the differential game defined by equations (3-5) in which the government and central bank act non-cooperatively (both players move simultaneously) and they have a feedback information structure about the game. Linear differential games with a feedback information structure are well-known in literature and the resulting strategies are strongly time consistent. That is, if one would reconsider the game at any point in time starting at any state, the previously determined strategies are still optimal from each player's point of view. So, from an optimal response to disturbances point of view they are very robust. The disadvantage is that perfect knowledge of the state of the system is needed at any point in time for the implementation of the actions (see Engwerda, 2005; Basar and Olsder, 1999). Now, assuming all players use a linear feedback control policy where every player considers a worst-case realization of noise for himself, we obtain from Engwerda (2005), Theorem 9.18, next result (see Appendix A for details).

Theorem 4.1. *Consider the differential game (3-5). Assume some technical conditions are satisfied (See Appendix A for details). Then the set of soft-constrained Nash equilibrium policies are given by:*

$$f^e(t) = \bar{f} - k_{11}(d^e(t) - \bar{d}) - k_{12} \quad (6)$$

$$m^e(t) = \bar{m} + k_{21}(d^e(t) - \bar{d}) + k_{22} \quad (7)$$

Here debt, $d^e(t)$, is the solution of next differential equation and depends on the realization of the disturbance $w(t)$, i.e:

$$\dot{d}^e(t) = \alpha d^e(t) + \beta + w(t), \quad d(0) = d_0 \quad (8)$$

Where $\alpha = r - g - k_{11} - k_{21}$ and $\beta = \bar{f} - \bar{m} + (k_{11} + k_{21})\bar{d} - k_{12} - k_{22}$. Furthermore, k_{ij} , $i, j = 1, 2$ are obtained from the stabilizing solutions of the Riccati equations (23) and (24).

4.2 Cooperative Case

In this section, we assume that two players can enter into binding agreements and act cooperatively. As Tabellini (1986) expressed, the cooperative game between fiscal and monetary authorities can be explained by a central coordination institute such as legislature that determines the guidelines for how to cope with future fiscal deficits, money creation and debt. In this cooperative scenario, let $\omega \in (0, 1)$ denote the relative weight attached to the

loss function of fiscal and $(1 - \omega)$ to that of monetary authorities respectively. In cases where the two policy-makers have an equal bargaining strength, ω is equal 0.5. The set of Pareto efficient solutions can be obtained by solving for all $\omega \in (0,1)$ the optimal control problem, subject to dynamic constraint (3).

$$\min_{f,m} \max_w \omega L_F + (1 - \omega)L_M - vW^2(t) \quad (9)$$

Assuming some technical conditions are satisfied, we obtain from Engwerda (2005), Corollary 9.11, next result (see Appendix B for details).

Theorem 4.2. *Consider the differential game (3-5). The soft-constrained cooperative fiscal and monetary policies are given by:*

$$f^e(t) = \bar{f} - \frac{1}{\omega + (1 - \omega)\eta} (k_{11}(d^e(t) - \bar{d}) + k_{12}) \quad (10)$$

$$m^e(t) = \bar{m} + \frac{1}{\varphi\omega + (1 - \omega)} (k_{11}(d^e(t) - \bar{d}) + k_{12}) \quad (11)$$

Here debt, $d^e(t)$ is the solution of next differential equation and depends on the realization of the disturbance $w(t)$, i.e:

$$\dot{d}^e(t) = \alpha d^e(t) + \beta + w(t), \quad d(0) = d_0 \quad (12)$$

Where, with $\bar{\omega} = \frac{1}{\omega + (1 - \omega)\eta} + \frac{1}{\varphi\omega + (1 - \omega)}$, $\alpha = r - g - \bar{\omega}k_{11}$ and $\beta = \bar{f} - \bar{m} + \bar{\omega}k_{11}\bar{d} - \bar{\omega}k_{12}$. Furthermore, $k_{ij}, j = 1,2$ are obtained from the stabilizing solution of equation (30).

4.3 Stackelberg Case

In this section, we consider the solution of the differential game assuming a sequential decision-making process. The solution concept we use is known as the Stackelberg equilibrium. In this game, the players have asymmetric roles. We distinguish a leader and a follower. The leader announces his policy and then the follower optimizes his own under the leader's that he reacts to. In this current study we assume that government acts as leader and central bank acts as follower. This regime seems an appropriate one to use for Iran's economy. In Appendix C next result is derived under the assumption that the policy announced by the fiscal authority is given by:

$$f(t) = \bar{f} + \alpha_f(d(t) - \bar{d}) + \beta_f \quad (13)$$

Theorem 4.3. Consider the differential game (3-5). Assume that fiscal authority is the leader in the game and announces its policy (13). Then, provided some technical conditions are met (see Appendix C for details), the set of soft-constrained Stackelberg equilibrium policies are given by:

$$m^e(t) = \bar{m} + k_{11}(d^e(t) - \bar{d}) + k_{12} \quad (14)$$

$$f^e(t) = \bar{f} + \alpha_f(d^e(t) - \bar{d}) + \beta_f \quad (15)$$

Here debt, $d^e(t)$ is the solution of next differential equation and depends on the realization of the disturbance $w(t)$, i.e.

$$\dot{d}^e(t) = \alpha d^e(t) + \beta + w(t); \quad d(0) = d_0 \quad (16)$$

Where $\alpha = r - g + \alpha_f - k_{11}$ and $\beta = \bar{f} - \bar{m} + (k_{11} - \alpha)\bar{d} + \beta_f - k_{12}$. Furthermore, k_{ij} , $i, j = 1, 2$ are obtained as the stabilizing solution of the Riccati equation (36) and parameters α_f and β_f are obtained as the solution of the minimization problem (43).

5. Simulation

To compare the different types of strategic interactions, we performed a numerical simulation study. As the benchmark case, we used next parameters tabulated in Table 3.

Table 3. Parameters

| g | r | ρ | \bar{d} | d_0 | \bar{f} | \bar{m} | θ | τ | φ | η |
|-------------|-------|--------|-----------|-------|-----------|-----------|----------|--------|-----------|--------|
| 0.02 | -0.03 | 0.07 | 0.1 | 0.4 | 0.02 | 0.01 | 0.05 | 0.05 | 0.05 | 0.05 |

We tried to choose the parameter values such that they are representative for Iran's economy. For example, the values for g and r are the average of real income growth and real interest rate during the sample period respectively. Also ρ is estimated by Abdoli (2009) for the Iranian economy. \bar{f} , \bar{d} and \bar{m} are the levels of government deficit, debt and monetary base that are targeted by policy-makers and we assume these are lower than their average rate during 1972-2012. Also, the initial debt d_0 is at its highest level. Furthermore we assumed in this benchmark case that the government and central bank assign an equal weight for deviation of debt from its target and rivals actions. Finally recall that, v_i , $i = f, m$ are parameters expressing the magnitude of the noise expected by policy-makers. Too large v_i , $i = f$ and m means that the policy-makers don't vastly worry about uncertainty in future; conversely, if v_i , $i = f, m$ become small (close to zero), the policy-makers expect a large impact of future uncertainty.

5.1 Simulation for Non-Cooperative Case

Solving differential equation (8) shows that in the non-cooperative game equilibrium debt is given by:

$$d^e(t) = -\frac{\beta}{\alpha} + e^{\alpha t} \left(d_0 + \frac{\beta}{\alpha} + \int_0^t e^{-\alpha s} w(s) ds \right) \quad (17)$$

In this equation, the parameters α and β depend on v_i for $i = f, m$. When v_i goes to infinity, the values of α and β converge to values representing the noise-free case. For the noise-free case the Riccati equations (23) and (24) have the unique stabilizing solution:

$$K_1^{No\ noise} = K_2^{No\ noise} = \begin{bmatrix} 0.1072 & 0.0015 \\ 0.0015 & 1.6273e - 04 \end{bmatrix}.$$

The corresponding values of $\alpha^{no\ noise}$ and $\beta^{no\ noise}$ are -0.2544 and 0.0284 respectively. When the risk-sensitivity parameters of policy-makers are $v_i = 2$ for $i = f, m$, the solution to the Riccati equations is:

$$K_1^{Noise} = K_2^{Noise} = \begin{bmatrix} 0.1155 & 0.0018 \\ 0.0018 & 1.9346e - 04 \end{bmatrix}.$$

With corresponding values for $\alpha^{Noise} = -0.2710$ and $\beta^{Noise} = 0.0296$.

In Figure (2) we plotted the realization of debt if actually no noise occurs during the planning horizon. We plotted both above mentioned scenarios: The case when policy-makers expect a priori no noise, and the case they expect noise and cope with this by choosing risk-aversion parameters $v_i = 2$ for $i = f, m$ respectively. The graphs show that the effect of including noise into their expectations is twofold. On the one hand, policy-makers set a more stringent debt target, and on the other hand, they try more actively to reach this target as fast as possible. In this case, fiscal policy copes with uncertainty by increasing of fiscal surpluses (for a country such as Iran by increasing its oil income) and monetary policy by increasing the base money (creation of money). This brings on that on the short term debt decreases faster under a policy motivated by noise expectations than when policy-makers don't expect any noise.

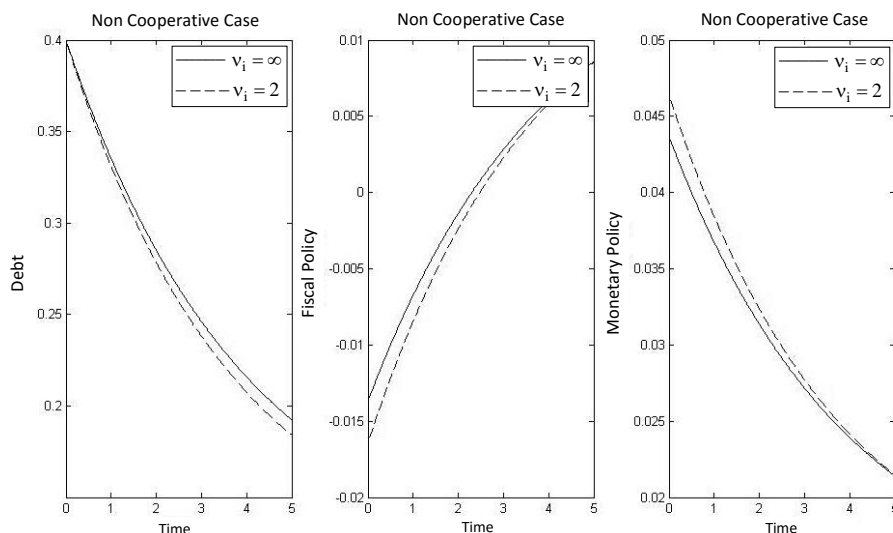


Fig. 2: The Realization of Debt if actually No Noise Occurs During the Planning Horizon

Figure (3) visualizes the convergence speed of debt towards its equilibrium in both scenarios (i.e. $d^e(t) - \left(\frac{-\beta}{\alpha}\right)$ for $w(t) = 0$).

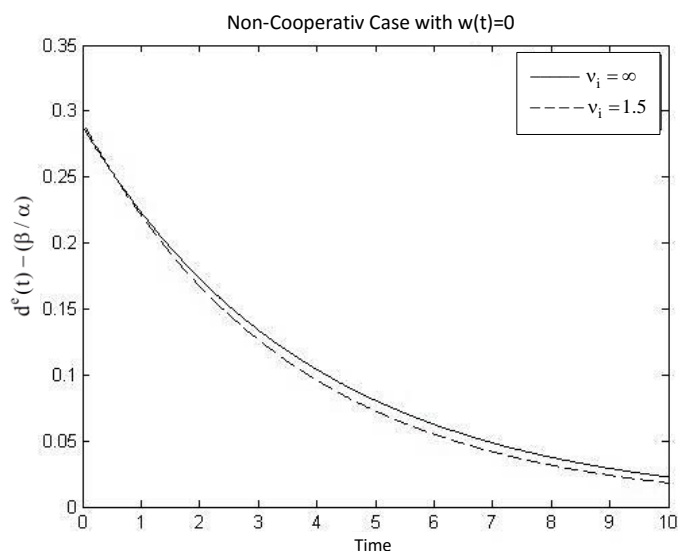


Fig. 3: Convergence Speed of Debt towards its Equilibrium

We also calculated for other noise scenarios equilibrium debt, $\frac{-\beta}{\alpha}$, and convergence speed α . In all scenarios we assumed that the risk-sensitivity parameters set by the players coincide. Table 4 shows some different

scenarios corresponding values of debt and convergence speed, whereas Figure (4) visualizes the relationship between debt and the risk-sensitivity parameter. Both the table and figure confirm the two effects we already observed above: equilibrium debt becomes smaller and convergence speed faster if players are more concerned about noise.

Table 4: Convergence Speed and Debt Equilibrium by Different Risk-Sensitivity Parameter

| v_i | α | β | $-\left(\frac{\beta}{\alpha}\right)$ |
|-------|----------|---------|--------------------------------------|
| 1 | -0.2924 | 0.0310 | 0.1060 |
| 1.25 | -0.2831 | 0.0304 | 0.1073 |
| 1.5 | -0.2775 | 0.0300 | 0.1082 |
| 1.75 | -0.2737 | 0.0298 | 0.1087 |
| 2 | -0.2710 | 0.0296 | 0.1091 |
| 2.5 | -0.2673 | 0.0293 | 0.1097 |
| 5 | -0.2606 | 0.0289 | 0.1107 |
| 10 | -0.2575 | 0.0286 | 0.1112 |

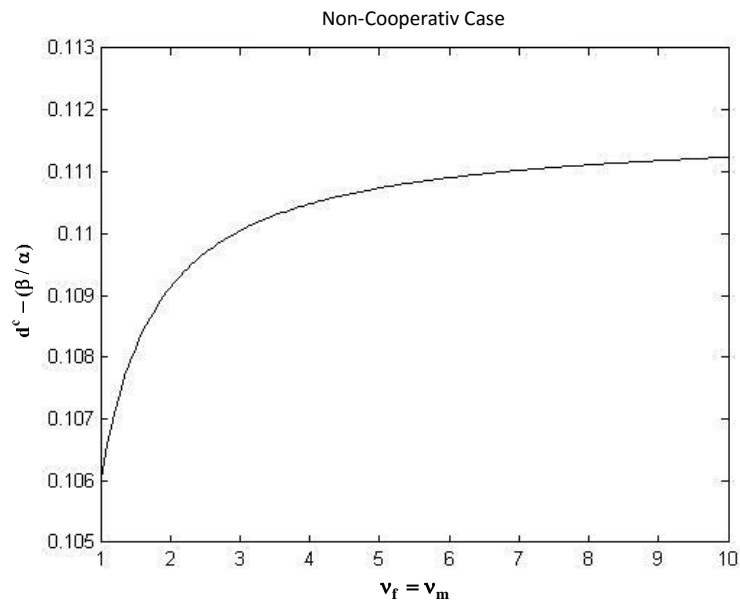


Fig. 4: Relationship between Debt equilibrium and the Risk-Sensitivity Parameter

Below in Figure (5), we illustrate the dynamic adjustment of debt if a noise signal that converges exponentially to zero, corrupts the debt equation. As to be expected, a similar behavior as for the noise free case occurs.

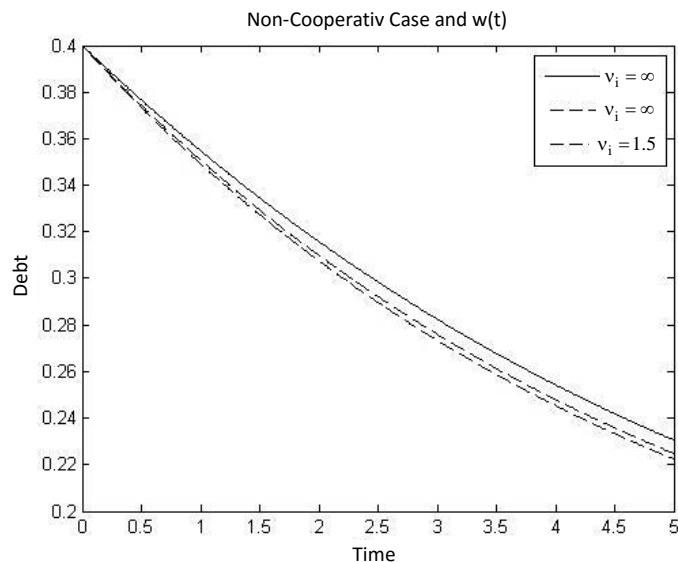


Fig. 5: The Dynamic Adjustment of Debt

Another experiment we performed is to see how debt will react on a "conjunctural disturbance" (or "bell-shaped" disturbance). That is, we considered a disturbance $w(t) = w_{max} * (0.5t - \frac{1}{16} * t^2)$ on the time interval $[0,8]$. This noise signal is zero at the boundaries of the time interval and reaches its maximum amplitude w_{max} at $t = 4$. Figure (6) illustrates different values of the risk-sensitivity parameters (which are again assumed to coincide for both players), the resulting debt trajectories.

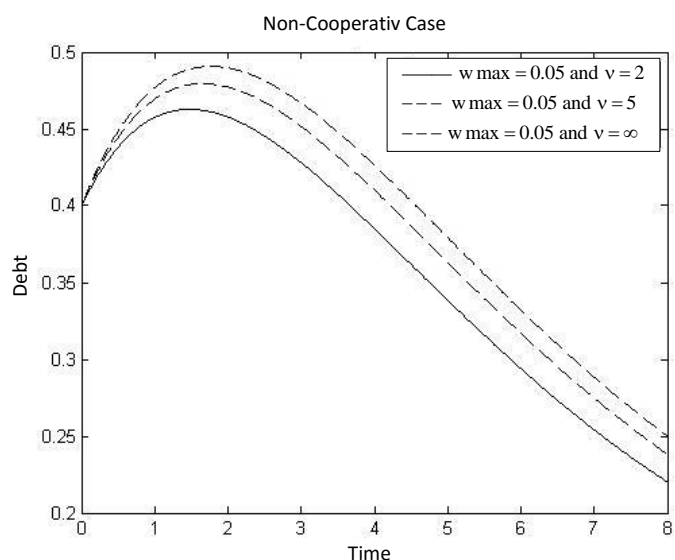


Fig. 6: Conjunctural Disturbance

As initial debt already has been reduced significantly during the first part of the time interval, worst-case realization of debt occurs earlier than actual worst-case disturbance occurs at $t = 4$. Furthermore, this point of time shifts more to the nearby future the more players are concerned with uncertainty, as they use more active control policies to reduce debt. We also see that as a consequence of this more active control policy, the maximum amplitude of debt is lower the more players are concerned with uncertainty. To see how debt responds to a worst-case noise signal under different noise expectation scenarios, we considered again a setting where the expectations of monetary and fiscal players about noise are the same. In Figure (7) we plotted the debt response when the worst-case noise signal occurs for risk sensitivity parameter $v_i = 2$ and $v_i = 1.5$ respectively. We see that apparently the worst-case signal has a larger impact the more both players are concerned by noise that may enter the system. In that case, debt is higher under $v_i = 1.5$ than $v_i = 2$.

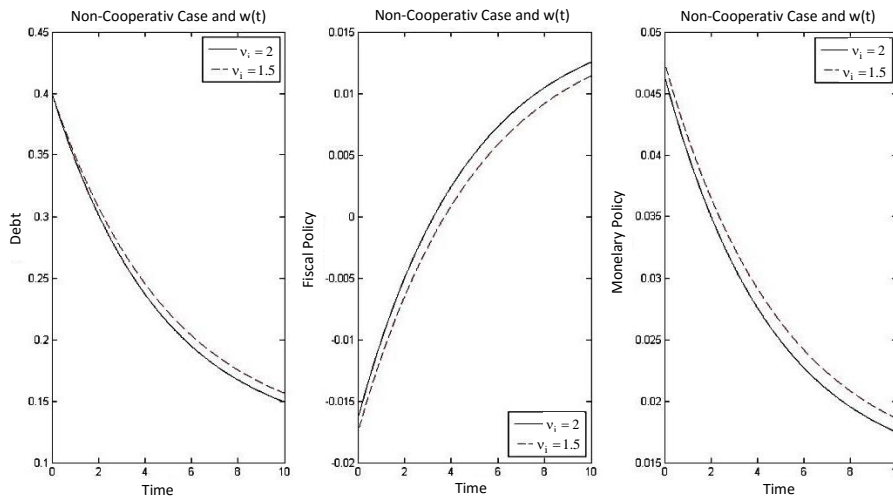


Fig. 7: Debt Responds to a Worst-Case Noise Signal

In fact, in this scenario where risk parameters for both players coincide and the worst-case noise signal occurs, costs for both players can be easily calculated. The integrals (4) and (5) equal $J_i = x_0^T K_i x_0$ for $i = f, m$ (where $K_1 = K_2$ in this case, see Engwerda, 2005). Figure (8) shows the worst-case cost as function of v_i for $i = f, m$. It shows that the more players take risk into account in their decision-making, the higher worst-case cost takes place. For $v_i = \infty, i = f, m$, the cost converge to those of the 'noise-free' case. Maybe superfluous to note, but of course this graph just provides an upper bound for the worst-case cost (4) if noise-expectations of players coincide and the model is "correctly specified".

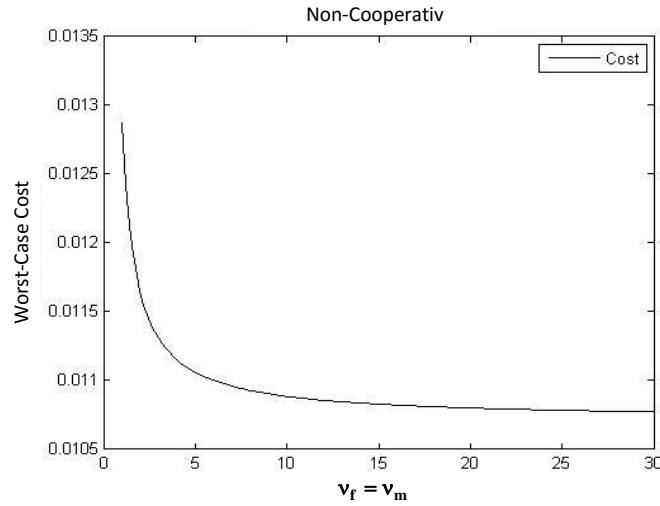


Fig. 8: Worst-Case Cost

5.2. Simulation for Cooperative Case

Next we simulate the model for a cooperative setting. Assuming both fiscal and monetary policy-makers have an equal say in the decision-making process, we consider $\omega = 0.5$ in the social cost function (9). For the case policy-makers don't incorporate noise expectations in their decision making (i.e. $v_i = \infty$), Riccati equation (30) has the unique stabilizing solution:

$$K^{No\ noise} = \begin{bmatrix} 0.0965 & 0.0012 \\ 0.0012 & 1.2785e - 04 \end{bmatrix}$$

Which results in a debt trajectory described by (12), where $\alpha^{No\ noise} = -0.4078$ and $\beta^{No\ noise} = 0.0422$. In case the policy-makers do expect disturbances will affect debt in the future, and they incorporate this expectation by considering a risk-sensitivity parameter $v = 2$, the solution to the Riccati equation (30) is:

$$K^{Noise} = \begin{bmatrix} 0.1023 & 0.0014 \\ 0.0014 & 1.4605e - 04 \end{bmatrix}$$

With corresponding values $\alpha^{Noise} = -0.4298$ and $\beta^{Noise} = 0.0438$.

We see that convergence speed almost doubles compared to the non-cooperative case. A similar result was also obtained by Tabellini (1986) for the noise-free case. That indeed, policy-makers are much more active in trying to control debt than in the non-cooperative case. This is visualized in panels b and c of Figure 9 below.

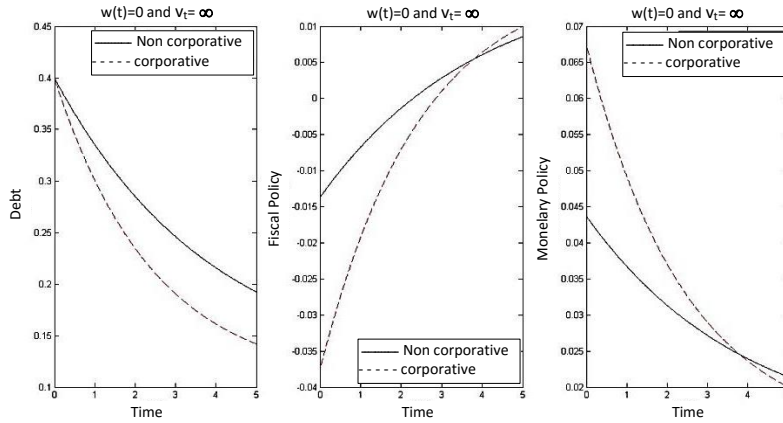


Fig. 9: The Realization of Debt if actually no Noise Occurs During the Planning Horizon in Cooperative and Non Cooperative Settings

To visualize the impact on policies including noise expectations, we plotted in Figure 10 the corresponding debt and policy trajectories if no-noise is incorporated and a risk-sensitivity parameter $v = 2$ is assumed respectively. Assuming no disturbance actually will disturb the system (i.e. $w(t) = 0$), we see, first of all (as already noticed above too), that debt converges faster to its equilibrium value in the "noisy-case" than in the "no-noise" case. Furthermore, we see that trajectories now lie much closer to each other than in the non-cooperative case. So, the effect of taking noise into account in the decision-making has a smaller impact on policies and resulting debt than in the non-cooperative setting.

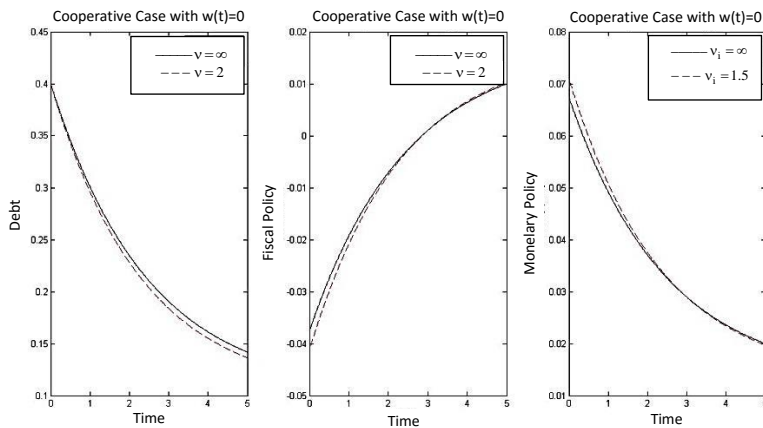


Fig. 10: The Realization of Debt if actually No Noise Occurs During the Planning Horizon

To see the impact of noise expectations in this cooperative setting, we calculated also for different risk-sensitivity parameters the resulting

equilibrium debt, $-\frac{\beta}{\alpha}$ and converge speed α . Table 5 shows a number of parameter choices corresponding values of debt and convergence speed.

Table 5: Convergence Speed and Debt Equilibrium by Different Risk-Sensitivity Parameter

| v_i | α | β | $-\left(\frac{\beta}{\alpha}\right)$ |
|-------------|----------|---------|--------------------------------------|
| 1 | -0.4566 | 0.0457 | 0.1000 |
| 1.25 | -0.4452 | 0.0449 | 0.1008 |
| 1.5 | -0.4381 | 0.0444 | 0.1013 |
| 1.75 | -0.4333 | 0.0440 | 0.1016 |
| 2 | -0.4298 | 0.0438 | 0.1018 |
| 2.5 | -0.4251 | 0.0434 | 0.1022 |
| 5 | -0.4161 | 0.0428 | 0.1028 |
| 10 | -0.4119 | 0.0425 | 0.1031 |

The table illustrates on the one hand a result we already noticed above, i.e. compared to the non-cooperative setting, convergence speed almost doubles. On the other hand, we see that equilibrium values in the cooperative and non-cooperative setting do not differ that much. So, apparently the main difference between a cooperative and a non-cooperative mode of play seems to be that in a cooperative setting more active short-term policies are used to track debt to its target level.

In Figure 11 we plotted under both settings equilibrium debt as a function of the risk-sensitivity parameter. From the figure we observe two effects: First, the difference between the extreme levels of equilibrium debt that are attained in the non-cooperative setting is larger than in the cooperative setting. Second, for small values of the risk-sensitivity parameter, a small change in has more impact on debt in the non-cooperative setting than in the cooperative one.

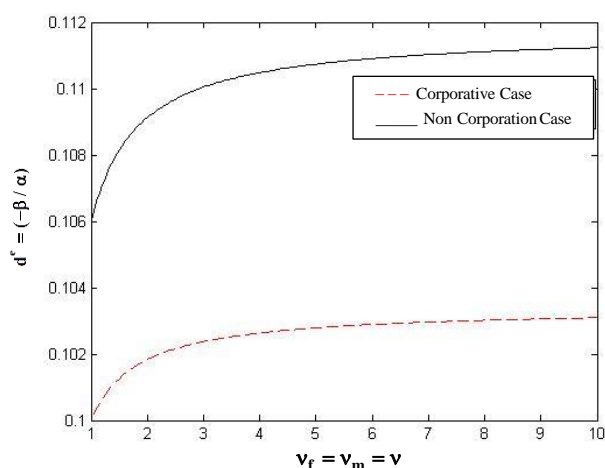


Fig. 11: Debt as a Function of the Risk-Sensitivity Parameter in two Settings

Figure 12 shows that differences in debt and policy trajectories under different worst-case scenarios are negligible compared to those we observed in the non-cooperative case (see Figure 7).

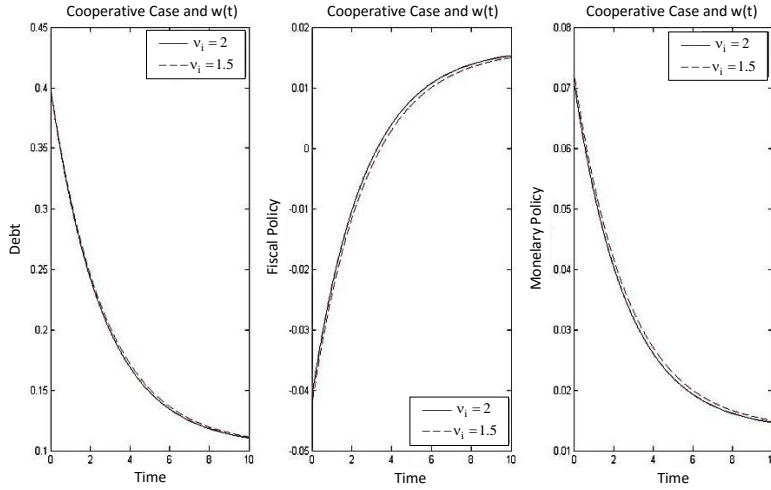


Fig. 12: Debt Responds to a Worst-Case Noise Signal

Furthermore, we plotted in Figure 13 the corresponding cost (which equal $x_0^T K x_0$). Compared to the non-cooperative case we see that for every choice of the risk-sensitivity parameter, cost is approximately 0.0012 lower.

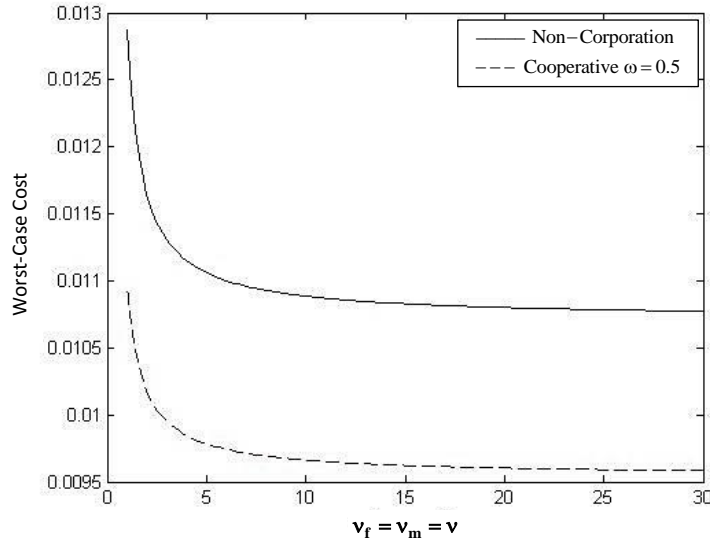


Fig. 13: Worst-Case Cost

As a final experiment we considered again how debt will react to the conjunctural disturbance we introduced in section 5.1. Figure 14 illustrates this case. Like in the non-cooperative case, we see that debt converges quicker the more policy-makers take disturbance expectations into account. Furthermore, we see that in this cooperative scenario, differences between the debt trajectories are smaller than under the non-cooperative one. Also, due to the more active control policies, the "bubble" we observed in the non-cooperative debt trajectories, doesn't appear in the current scenario anymore.

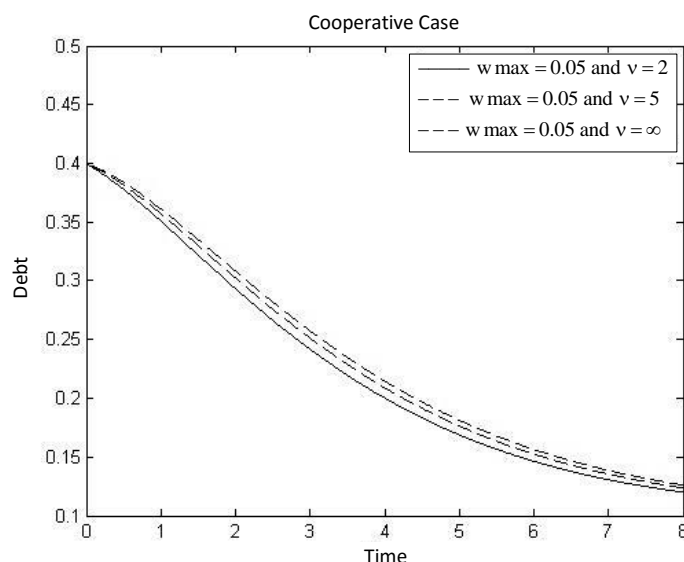


Fig. 14: Conjunctural Disturbance

5.3 Simulation for Stackelberg Case

In this section, we simulate the Stackelberg game, with government as leader and central bank as follower. Like in the previous subsections, we assume that in case players take noise into account in their decision-making, the risk-sensitivity parameters chosen by both players coincide. According to Appendix, calculation of the equilibrium actions is in this case numerically somewhat more involved. For that reason, the simulation study is at some points less detailed than in the previous subsections.

Our first study concerns the no-noise expectations case ($v = \infty$) again. Recall from Theorem 4.3 that convergence speed and equilibrium debt depend on the fiscal policy parameters α_f and β_f introduced in (15). For this no-noise case we calculated next values for these parameters: $\alpha_f = -0.0150$ and $\beta_f = -0.005$ respectively. This results then in values $\alpha^{No-noise} = -0.2061$ and $\beta^{No-noise} = 0.0210$.

While policy-makers use a risk-sensitivity parameter of $v = 2$, we obtained $\alpha_f = 0.1250$ and $\beta_f = -0.005$, yielding values $\alpha^{Noise} = -0.349$ and $\beta^{Noise} = 0.0339$ respectively.

In Figure 15, we plotted the realization of debt if actually no noise (i.e. $w(t) = 0$) occurs. We see that the effect of including noise expectations is much larger than we observed in both the non-cooperative and cooperative settings. Due to a much more active policy performed by both, the fiscal (an initial increase of its policy by more than 5 times) and the monetary policy-maker (an initial increase of its policy be 2.5 times) debt converges much faster to zero if noise expectations are included.

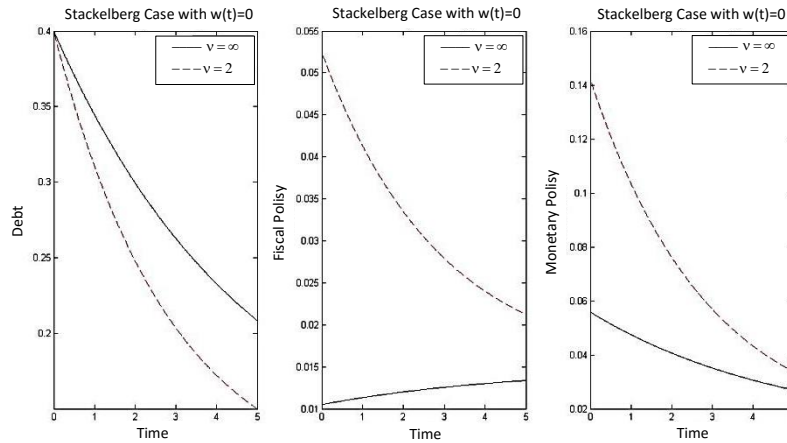


Fig. 15: The Realization of Debt if actually No Noise Occurs During the Planning Horizon

This increase in convergence speed of debt towards its equilibrium is also illustrated in next Figure 16.

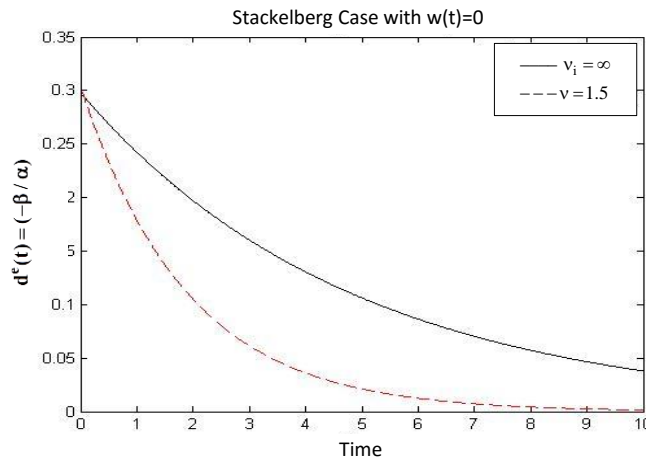


Fig. 16: Convergence Speed of Debt Towards its Equilibrium

Figures 18 and 17 compare equilibrium debt and policy trajectories under the Stackelberg scenario with those we obtained for the non-cooperative and cooperative setting respectively. Figure 17 compares the results for the no-noise expectations case, whereas Figure 18 compares results when players use a risk-sensitivity parameter of 2. Figure 17, panel b, shows that in the Stackelberg setting, initial fiscal policy is much closer to its target value \bar{f} compared to the other two settings. As a consequence, this policy has almost no changes over time. From panel c, we see that this policy is sustainable due to a more active policy of the monetary authority over time. In contrast to the other two settings that policies of both authorities are initially high but converge to the equilibrium value faster.

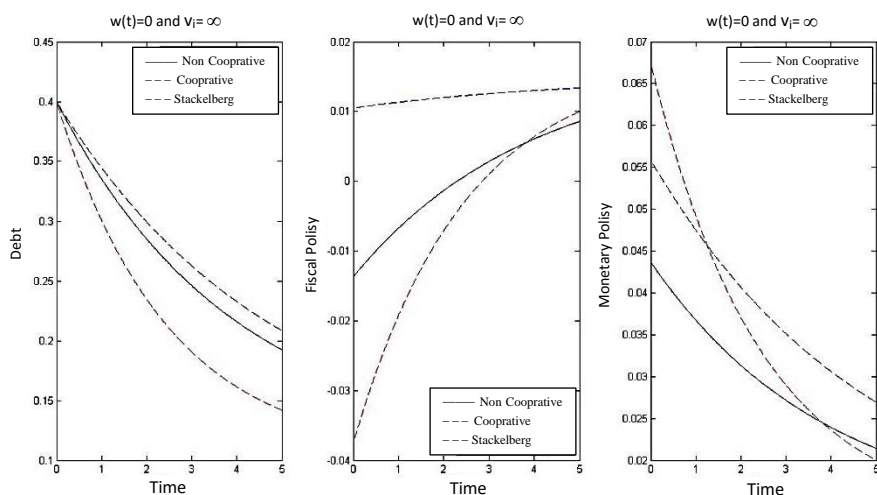


Fig. 17: Equilibrium Debt under three Settings for No-Noise Expectations Case

From Figure 18, we see that the incorporation of noise expectations makes that the fiscal player becomes more active too. Notice from panel b that in the Stackelberg setting, the sign of initial fiscal policies still differs from that of the other two settings, and that this policy converges faster to its target value than in the other two settings. This again, at the expense of monetary policy which convergence towards its target value, is much slower under the Stackelberg setting (panel c) than under the other two settings.

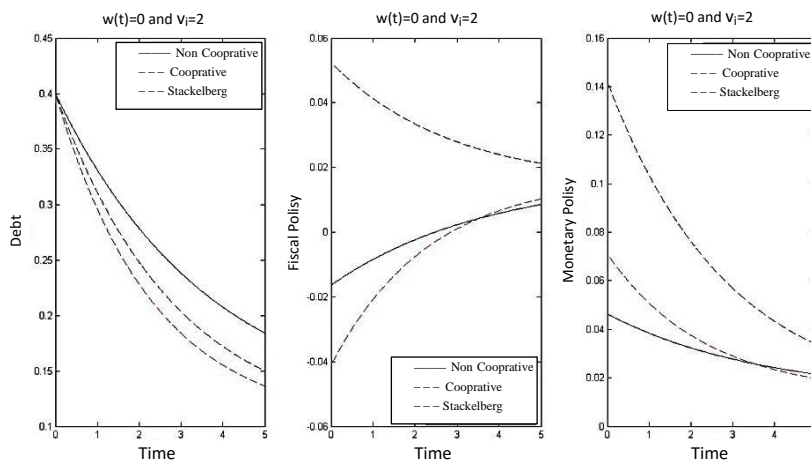


Fig. 18: Equilibrium Debt under three Settings for Noise Expectations Case

In Table 6 we show how convergence speed and equilibrium debt depend on the risk-sensitivity parameter v . We see a similar behavior as in the previous two settings. That is, when convergence speed increases, the more players take noise expectations into account in their policy-making. Also, equilibrium debt decreases then. Furthermore, this decrease is not that large and of the same order of magnitude as in the previous two cases. A striking difference between this Stackelberg scenario and the previous two scenarios is that convergence speed increases by a factor 4 when v is decreased from 10 to 1. In the previous two scenarios, this factor was approximately 1.1. A possible explanation for this might be that like in the other two settings, fiscal authorities react to uncertainty by pursuing a more active control in the short run. The monetary authorities, however, also respond in a similar way. Whatever the announced policy of the fiscal authorities is, they will also respond with a more active control policy the more uncertain they are about the future. So, the overall effect will be that short-run policy actions enforce each other, implying a fast decrease of debt.

Table 6: Convergence Speed and Debt Equilibrium by Different Risk-Sensitivity Parameter

| v_i | α | β | $-\left(\frac{\beta}{\alpha}\right)$ |
|-------------|----------|---------|--------------------------------------|
| 1 | -0.83 | 0.0713 | 0.0859 |
| 1.25 | -0.6966 | 0.0655 | 0.0941 |
| 1.5 | -0.5320 | 0.0509 | 0.0957 |
| 1.75 | -0.4284 | 0.0413 | 0.0965 |
| 2 | -0.3490 | 0.0339 | 0.0971 |
| 2.5 | -0.2619 | 0.0258 | 0.0986 |
| 5 | -0.2133 | 0.0216 | 0.1013 |
| 10 | -0.2080 | 0.0212 | 0.1018 |

Finally, we also simulated in this Stackelberg scenario how equilibrium debt responds on a conjunctural disturbance. Again, for different values of the risk-sensitivity parameter, we plotted in Figure 19 the closed-loop response. The findings are in line with above observations and the previous scenarios. As convergence speed is smaller in this scenario than in the other two when no-noise expectations are included, initial debt is less stabilized. As a consequence the resulting worst-case debt amplitude is higher and closer to the point of time when actually the disturbance is largest. The reverse conclusion applies when expected noise is seriously accounted for in the decision-making process, as in that case convergence speed in the Stackelberg scenario is higher than that under the other two scenarios.

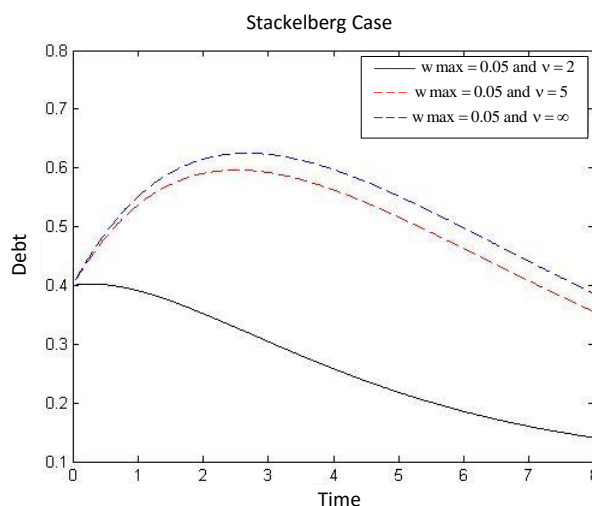


Fig. 19: Conjunctural Disturbance

6. Conclusion

During the past decades, the increase of debt and budget deficits has become a serious policy problem in many countries. This issue gets even more important when policy-makers face uncertainty about the state of the economy, and one of the most important challenges for economic decision-makers is how to deal with such uncertainty. In some developing countries such as Iran, in which energy revenues are the main source of government budget financing, and creation of money for financing budget deficit and debt is traditional, uncertainty about energy revenues and fluctuations in it has an important effect on economic activities and government budget planning. Observations from the past suggest that in the Iranian economy, the Stackelberg game between fiscal and monetary policy-makers is played and in this game, government acts as leader and central bank acts as follower.

Hence in this paper, we consider debt stabilization issues and interaction between fiscal and monetary policy-makers within a framework of uncertain linear quadratic differential games. As far as the authors know, such an approach has not been used previously in economic literature to study uncertainty in a multi-player context. We extended Tabellini's model (1986) by adding deterministic noise to the debt equation. Furthermore, we assume that players have their own expectation about this deterministic noise, and they use linear feedback strategies. We solve the model assuming a Non-cooperative, Cooperative and Stackelberg setting respectively. Using calibrated parameters for the Iranian economy, we performed a simulation study. Our results show that in all three settings, taking into account disturbance expectations in the debt stabilization model leads to a more active control pursued by players in the short-run and thus, leading to a faster debt decrease. Furthermore, equilibrium debt becomes smaller in all three settings.

The simulation results also show that Tabellini's conclusion that the Cooperative equilibrium always has a higher adjustment speed and a lower steady state value of public debt than the Non-cooperative equilibrium, extends to this setting, whatever the noise expectations of both players are. Among all three settings, the Stackelberg equilibrium seems to be the most sensitive for the risk-sensitivity parameter. If players are very risk-sensitive, converge speed towards its equilibrium is faster than both in the Cooperative and Non-cooperative. On the contrary to the case, in which players are very risk-insensitive, the result is reversed. This last result was also found by Van Aarle (1995) in a similar noise-free setting. So, from this point of view, including deterministic noise may impact their conclusions concerning the optimal policy configuration for tackling debt.

Furthermore, we observe that worst-case cost in the Non-cooperative setting is always higher than in the Cooperative one. On the other hand, we observe that the gap between equilibrium debts under these two settings becomes smaller when players get more risk-sensitive.

The above observations hint into the direction that from the perspective of policy-making for the Iranian economy, the "optimal" policy setting depends on how uncertain policy-makers are about the future. If they do not expect too much disturbances, it seems best for the central bank and government to interact within a cooperative setting. In case there is much uncertainty, a fiscal leadership seems more appropriate to stabilize debt.

Clearly, though above observation may be in line with current Iranian policy-making, it is a very preliminary one. For direct future work, it would be interesting to see whether the above observation can be substantiated

more analytically and/or by a sensitivity study. In particular it might be interesting to see which impact the use of different risk-sensitivity parameters by policy-makers, and weights assumed in the performance criteria of government and central bank have on this observation. Other, more elaborative extensions are to include in this framework the relationship of debt with output and financial markets.

Appendix A: Non-Cooperative Case

Let $\tilde{f}(t) := (f(t) - \bar{f})e^{-\frac{1}{2}\rho t}$, $\tilde{m}(t) := (m(t) - \bar{m})e^{-\frac{1}{2}\rho t}$, $\tilde{d}(t) := (d(t) - \bar{d})e^{-\frac{1}{2}\rho t}$, $\tilde{w}(t) := w(t)e^{-\frac{1}{2}\rho t}$, where $w(\cdot)$ is an arbitrary square integrable function on $[0, \infty)$, and

$$x := \begin{bmatrix} \tilde{d}(t) \\ e^{-\frac{1}{2}\rho t} \end{bmatrix} \rightarrow \dot{x} := \begin{bmatrix} \dot{\tilde{d}}(t) \\ -\frac{1}{2}e^{-\frac{1}{2}\rho t} \end{bmatrix};$$

Using this notation, equations (3-5) can then be rewritten as follows:

$$\min_f \max_w L_F = \frac{1}{2} \int_0^\infty \{x^T(t)Q_1x(t) + \tilde{f}^2(t) + \varphi\tilde{m}^2(t) - v_f\tilde{w}^2(t)\}dt \quad (18)$$

$$\min_m \max_w L_M = \frac{1}{2} \int_0^\infty \{x^T(t)Q_2x(t) + \tilde{m}^2(t) + \eta\tilde{f}^2(t) - v_m\tilde{w}^2(t)\}dt \quad (19)$$

Subject to:

$$\dot{x}(t) = Ax(t) + B_1\tilde{f}(t) + B_2\tilde{m}(t) + E\tilde{w}(t) \quad (20)$$

Where

$$A = \begin{bmatrix} (r - g - \frac{1}{2}\rho) & ((r - g)\bar{d} + \bar{f} - \bar{m}) \\ 0 & -\frac{1}{2}\rho \end{bmatrix}; B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; B_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}; E = \begin{bmatrix} 1 \\ 0 \end{bmatrix};$$

$$Q_1 = \begin{bmatrix} \theta & 0 \\ 0 & 0 \end{bmatrix}; Q_2 = \begin{bmatrix} \tau & 0 \\ 0 & 0 \end{bmatrix}; R_{11} = 1; R_{12} = \varphi; R_{21} = \eta; R_{22} = 1; v_f = V_1; v_m = V_2$$

Next, introduce

$$\begin{aligned} S_1 &= B_1R_{11}^{-1}B_1^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; & S_2 &= B_2R_{22}^{-1}B_2^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \\ S_{12} &= B_1R_{11}^{-1}R_{21}R_{22}^{-1}B_2^T = \begin{bmatrix} \eta & 0 \\ 0 & 0 \end{bmatrix}; & S_{21} &= B_2R_{22}^{-1}R_{12}R_{11}^{-1}B_1^T = \begin{bmatrix} \varphi & 0 \\ 0 & 0 \end{bmatrix}; \\ M_1 &= EV_1^{-1}E^T = \begin{bmatrix} 1 & 0 \\ v_f & 0 \\ 0 & 0 \end{bmatrix}; & M_2 &= EV_2^{-1}E^T = \begin{bmatrix} 1 & 0 \\ v_m & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Additionally, we define:

$$K_1 = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{13} \end{bmatrix}; K_2 = \begin{bmatrix} k_{21} & k_{22} \\ k_{22} & k_{23} \end{bmatrix}.$$

The soft-constrained Nash equilibrium policies for problems (3-5) are (see Engwerda, 2005)

$$\tilde{f}(t) = -R_{11}^{-1}B_1^TK_1x \tag{21}$$

$$\tilde{m}(t) = -R_{22}^{-1}B_2^TK_2x \tag{22}$$

Where K_1 and K_2 satisfy the set of coupled algebraic Riccati Equations,

$$\begin{aligned} -(A - S_2K_2)^TK_1 - K_1(A - S_2K_2) + K_1S_1K_1 - Q_1 - K_2S_{21}K_2 \\ - K_1M_1K_1 = 0 \end{aligned} \tag{23}$$

$$\begin{aligned} -(A - S_1K_1)^TK_2 - K_2(A - S_1K_1) + K_2S_2K_2 - Q_2 - K_1S_{12}K_1 \\ - K_2M_2K_2 = 0 \end{aligned} \tag{24}$$

And such matrices $A - S_1K_1 - S_2K_2 + M_1K_1$, $A - S_1K_1 - S_2K_2 + M_2K_2$ as well as $A - S_1K_1 - S_2K_2$ are stable.

In addition, x solves the differential equation:

$$\dot{x}(t) = (A - S_1K_1 - S_2K_2)x(t) + E\tilde{w}(t); \quad x(0) = \begin{bmatrix} d_0 - \bar{d} \\ 1 \end{bmatrix}. \tag{25}$$

The worst-case signal $w_i, i = f, m$ for both two players are respectively:

$$\tilde{w}_i(t) = V_i^{-1}E^TK_i e^{(A - S_1K_1 - S_2K_2 + M_iK_i)t}x_0 \tag{26}$$

Appendix B: Cooperative Case

Let $\tilde{f}(t) := (f(t) - \bar{f}) e^{-\frac{1}{2}\rho t}$, $\tilde{m}(t) := (m(t) - \bar{m}) e^{-\frac{1}{2}\rho t}$, $\tilde{d}(t) := (d(t) - \bar{d}) e^{-\frac{1}{2}\rho t}$, $\tilde{w}(t) := w(t) e^{-\frac{1}{2}\rho t}$, where $w(\cdot)$ is as in the non-cooperative case, and

$$x(t) := \begin{bmatrix} \tilde{d}(t) \\ e^{-\frac{1}{2}\rho t} \end{bmatrix} \rightarrow \dot{x}(t) := \begin{bmatrix} \dot{\tilde{d}}(t) \\ -\frac{1}{2} e^{-\frac{1}{2}\rho t} \end{bmatrix};$$

The problem (9) can be rewritten as:

$$\min_{f,m} \max_w Y = \int_0^\infty \left\{ x^T(t)Qx(t) + [\tilde{f}(t) \quad \tilde{m}(t)]R \begin{bmatrix} \tilde{f}(t) \\ \tilde{m}(t) \end{bmatrix} - v\tilde{w}^2(t) \right\} dt, \tag{27}$$

Subject to

$$\dot{x}(t) = Ax(t) + B \begin{bmatrix} \tilde{f}(t) \\ \tilde{m}(t) \end{bmatrix} + E\tilde{w}(t) \tag{28}$$

Where

$$A = \begin{bmatrix} (r - g - \frac{1}{2}\rho) & ((r - g)\bar{d} + \bar{f} - \bar{m}) \\ 0 & -\frac{1}{2}\rho \end{bmatrix}; B = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}; E = \begin{bmatrix} 1 \\ 0 \end{bmatrix};$$

$$Q = \begin{bmatrix} \theta\omega + (1 - \omega)\tau & 0 \\ 0 & 0 \end{bmatrix}; R = \begin{bmatrix} \omega + (1 - \omega)\eta & 0 \\ 0 & \varphi\omega + (1 - \omega) \end{bmatrix}; v = v$$

Next, introduce

$$S = BR^{-1}B^T = \begin{bmatrix} \frac{1}{\omega + (1 - \omega)\eta} + \frac{1}{\varphi\omega + (1 - \omega)} & 0 \\ 0 & 0 \end{bmatrix}; M = EV_1^{-1}E^T$$

$$= \begin{bmatrix} \frac{1}{v} & 0 \\ 0 & 0 \end{bmatrix};$$

Additionally defining

$$K = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{13} \end{bmatrix},$$

The solution of optimization problem 9 is given by (Engwerda, 2005):

$$\begin{bmatrix} \tilde{f}(t) \\ \tilde{m}(t) \end{bmatrix} = - \begin{bmatrix} \frac{1}{\omega + (1 - \omega)\eta} & 0 \\ 0 & \frac{1}{\varphi\omega + (1 - \omega)} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{13} \end{bmatrix} \begin{bmatrix} \tilde{d}(t) \\ e^{-\frac{1}{2}\rho t} \end{bmatrix} \quad (29)$$

Where K satisfies the algebraic Riccati Equations,

$$Q + A^T K + KA - KSK + KMK = 0 \quad (30)$$

And is such that matrices $A - SK + MK$ and $A - SK$ are stable.

Furthermore, x solves the differential equation:

$$\dot{x}(t) = (A - SK)x(t) + E\tilde{w}(t); \quad x(0) = \begin{bmatrix} d_0 - \bar{d} \\ 1 \end{bmatrix} \quad (31)$$

The Minimax worst-case control is given by:

$$\tilde{w}(t) = V^{-1}E^T K e^{(A - SK + MK)t} x_0 \quad (32)$$

Appendix C: Stackelberg Case

Let $\tilde{f}(t) := (f(t) - \bar{f}) e^{-\frac{1}{2}\rho t}$, $\tilde{m}(t) := (m(t) - \bar{m}) e^{-\frac{1}{2}\rho t}$, $\tilde{d}(t) := (d(t) - \bar{d}) e^{-\frac{1}{2}\rho t}$, $\tilde{w}(t) := w(t) e^{-\frac{1}{2}\rho t}$, where $w(\cdot)$ is again as in the non-cooperative case. Under assumption that the government announces its policy,

$$\tilde{f}(t) = \alpha_f \tilde{d}(t) + \beta_f e^{-\frac{1}{2}\rho t} \quad (33)$$

The follower solves first his Minimax control problem and the game can be written as follows:

$$\min_m \max_w \frac{1}{2} \int_0^\infty \{x^T(t)Qx(t) + \tilde{m}^2(t) - v_m \tilde{w}^2(t)\} dt \quad (34)$$

Subject to:

$$\dot{x}(t) = Ax(t) + B\tilde{m}(t) + E\tilde{w}(t) \quad (35)$$

Here,

$$A = \begin{bmatrix} (r - g - \frac{1}{2}\rho + \alpha_f) & ((r - g)\bar{d} + \bar{f} - \bar{m} + \beta_f) \\ 0 & -\frac{1}{2}\rho \end{bmatrix}; B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}; E = \begin{bmatrix} 1 \\ 0 \end{bmatrix};$$

$$Q = \begin{bmatrix} \eta\alpha_f^2 + \tau & \alpha_f\beta_f\eta \\ \alpha_f\beta_f\eta & \eta\beta_f^2 \end{bmatrix}; S = BR^{-1}B^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; M = EV_1^{-1}E^T = \begin{bmatrix} 1 & 0 \\ v_m & 0 \\ 0 & 0 \end{bmatrix}; R$$

$$= 1; v = v_m;$$

Next define:

$$K = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{13} \end{bmatrix};$$

Where K satisfies the algebraic Riccati Equations,

$$Q_{\alpha,\beta} + A^TK + KA - KSK + KMK = 0 \quad (36)$$

And is such that matrices $A - SK + MK$ and $A - SK$ are stable.

Then, the solution of optimization problem for follower is given by (Engwerda, 2005):

$$\tilde{m}(t) = k_{11}\tilde{d}(t) + k_{12}e^{-\frac{1}{2}\rho t} \quad (37)$$

Next we assume that the government (leader) solves the model with:

$$\tilde{f}(t) = \alpha_f \tilde{d}(t) + \beta_f e^{-\frac{1}{2}\rho t} \quad (38)$$

The game can be rewritten then as follows:

$$\min_{\alpha,\beta} \max_w \frac{1}{2} \int_0^\infty \{x^T(t)Qx(t) - v_f \tilde{w}^2(t)\} dt \quad (39)$$

Subject to:

$$\dot{x}(t) = Ax(t) + B\tilde{w}(t) \quad (40)$$

Here

$$A = \begin{bmatrix} (r - g - \frac{1}{2}\rho + \alpha_f - k_{11}) & ((r - g)\bar{d} + \bar{f} - \bar{m} + \beta_f - k_{12}) \\ 0 & -\frac{1}{2}\rho \end{bmatrix}; B = \begin{bmatrix} 1 \\ 0 \end{bmatrix};$$

$$Q = \begin{bmatrix} \alpha_f^2 + \phi k_{11}^2 + \theta & \alpha_f \beta_f + \phi k_{11} k_{12} \\ \alpha_f \beta_f + \phi k_{11} k_{12} & \beta_f^2 + \phi k_{12}^2 \end{bmatrix}; S = BR^{-1}B^T = \begin{bmatrix} -\frac{1}{v_f} & 0 \\ 0 & 0 \end{bmatrix}; R = -v_f;$$

Next introduce

$$\bar{K} = \begin{bmatrix} \bar{k}_{11} & \bar{k}_{12} \\ \bar{k}_{12} & \bar{k}_{13} \end{bmatrix};$$

Where \bar{K} is obtained as the solution of algebraic Riccati equation,

$$Q_{\alpha_f, \beta_f} + A^T \bar{K} + \bar{K} A - \bar{K} S \bar{K} = 0 \quad (41)$$

For which matrix $A - S \bar{K}$ is stable.

The worst case signal for the government is then:

$$\tilde{w}(t) = -R^{-1} B^T \bar{K} x \quad (42)$$

Government, next tries to minimize this worst case cost by choice α_f and β_f such that:

$$\min_{\alpha_f, \beta_f} x_0^T \bar{K}_{\alpha_f, \beta_f} x_0, \quad (43)$$

Where x is the solution of the differential equation,

$$\dot{x}(t) = (A - S \bar{K})x(t) \quad (44)$$

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