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Nilpotent Elements in Skew Polynomial Rings

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Abstract

Let R be a ring with an endomorphism α and an α -derivation δ. Antoine studied the structure of the set of nilpotent elements in Armendariz rings and introduced nil-Armendariz rings. In this paper we introduce and investigate the notion of nil-(α,δ)-compatible rings. The class of nil-(α,δ)-compatible rings are extended through various ring extensions and many classes of nil-(α,δ)-compatible rings are constructed. We also prove that, if R is nil-α-compatible and nil-Armendariz ring of power series type with $nil(R)$ nilpotent, then $nil(R[[x;\alpha]]) = nil(R)[[x;\alpha]]$. We show that, if R is a *nil-Armendariz ring of power series type*, with $nil(R)$ nilpotent and nil-(α,δ)-compatible ring, then $nil(R[x;\alpha,\delta]) = nil(R)[x;\alpha,\delta]$. As a consequence, several known results are unified and extended to the more general setting. Also examples are provided to illustrate our results.

Keywords: $(\alpha,\delta)-$compatible ring; Skew polynomial ring; Skew power series ring.

Introduction

Throughout this article, all rings are associative whit identity. Let R be a ring, α be an endomorphism and δ an α -derivation of R, that δ is an additive map such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$, for all $a,b \in R$. We denote $R[x;\alpha,\delta]$ the Ore extension whose elements are the polynomials over R, the addition is defined as usual and the multiplication subject to the relation $xa = \alpha(a)x + \delta(a)$ for any $a \in R$. We also denote the skew power series ring $R[[x;\alpha]]$, whose elements are the power series over R, the addition is defined as usual and the multiplication subject to the relation $xa = \alpha(a)x$ for any $a \in R$.

Recall that a ring R is *reduced* if R has no nonzero nilpotent elements. Another generalization of a reduced ring is an Armendariz ring. A ring R is said to be *Armendariz* if the product of two polynomials in $R[x]$ is zero it implies that the products of their coefficients are zero. This definition was coined by Rege and Chhawchharia in [26] in recognition of Armendariz's proof in [4, Lemma 1] that reduced rings satisfy this condition.

According to Antoine [3], a ring R is called *nil-Armendariz*, if $f(x)g(x) \in nil(R)[x]$ implies $a_ib_j \in nil(R)$, for all

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$f(x)=\sum_{i=0}^{n} a_i x^i, g(x)=\sum_{j=0}^{m} b_j x^j \in R[x]$.

When R is a 2-primal ring, then the polynomial ring $R[x]$ and the Laurent polynomial ring $R[x,x^{-1}]$ are 2-primal and nil-Armendariz, and $nil(R[x]) = nil(R)[x]$. This condition is strongly connected to the question of whether or not a polynomial ring $R[x]$ over a nil ring R is nil, which is related to a question of Amitsur [1]. This is true for any 2-primal ring R (i.e. the lower nil radical $Nil_*(R)$ coincides with $nil(R)$).

In [13], M. Habibi and A. Moussavi, say, a ring R with an endomorphism α is *nil-Armendariz of skew power series type*, if $f(x).g(x) \in nil(R)[[x;\alpha]]$ implies that $a_i\alpha^i(b_j) \in nil(R)$, for all i,j and for all $f(x)=\sum_{i=0}^{\infty} a_i x^i, g(x)=\sum_{j=0}^{\infty} b_j x^j \in R[[x;\alpha]]$.

In this paper, we are concerned with nil-Armendariz rings of skew power series type, which is a generalization of nil-Armendariz rings.

According to Krempa [15], an endomorphism α of a ring R is called *rigid* if $a\alpha(a)=0$ implies $a=0$ for each $a \in R$. A ring R is called α-rigid if there exists a rigid endomorphism α of R.

In [9], E. Hashemi and A. Moussavi, say a ring R is α*-compatible* if for each $a, b \in R$, $ab=0$ if and only if $a\alpha(b) = 0$. Moreover, R is said to be δ*-compatible* if for each $a, b \in R$, $ab = 0$ implies $a\delta(b) = 0$. If R is both $\alpha-$compatible and δ-compatible, R is said to be (α,δ)-compatible. By [22], R is called weak $\alpha-$compatible, if $ab \in nil(R)$ if and only if $a\alpha(b) \in nil(R)$ for each $a, b \in R$, and R is said to be weak $\delta-$compatible if for each $a, b \in R$, $ab \in nil(R)$ implies $a\delta(b) \in nil(R)$.

Unifying and extending the above notions, we say R is a *nil-*α*-compatible* ring if for each $a, b \in R$, $aRb \subseteq nil(R)$ if and only if $aR\alpha(b) \subseteq nil(R)$. Moreover, we say R is *nil-*δ*-compatible* if for each $a,b \in R$, $aRb \subseteq nil(R)$ implies $aR\delta(b) \subseteq nil(R)$.

If R is both nil-α-compatible and nil-δ-compatible, we say that R is *nil-*(α,δ)*-compatible*.

We extend the class of nil-(α,δ)-compatible rings through various ring extensions. We show that R is a nil-(α,δ)-compatible ring if and only if the ring of triangular matrix $T_n(R)$ is nil-$(\bar{\alpha},\bar{\delta})$-compatible, where $\bar{\delta}$ is an $\bar{\alpha}$-derivation of $T_n(R)$. If R is a nil-Armendariz ring of power series type and nil-(α,δ)-compatible then $R[x;\alpha]$ is a nil-$(\bar{\alpha},\bar{\delta})$-compatible ring, where $\bar{\delta}$ is an $\bar{\alpha}$-derivation of $R[x;\alpha]$.

As a consequence, several properties of (α,δ)-compatible rings are generalized to a more general setting.

We show that if R is a nil-α-compatible and nil-Armendariz ring of power series type with $nil(R)$ nilpotent, then $nil(R[[x;\alpha]]) = nil(R)[[x;\alpha]]$. We also show that, if R is nil-Armendariz ring of power series type and nil-(α,δ)-compatible, with $nil(R)$ nilpotent, then $nil(R[x;\alpha,\delta]) = nil(R)[x;\alpha,\delta]$. Moreover we show that, when R is nil-(α,δ)-compatible, 2-primal, and either R is a right Goldie ring or R has the ascending chain condition (a.c.c.) on ideals or R has the a.c.c. on right and left annihilators or R is a ring with right Krull dimension, then $nil(R[x;\alpha,\delta]) = nil(R)[x;\alpha,\delta]$.

Results and Discussion

We first introduce the concept of a nil-(α,δ)-compatible ring and study its properties.

Definition 1.1. For an endomorphism α and an α-derivation δ, we say that R is *nil-*α*-compatible* if for each $a, b \in R$, $aRb \subseteq nil(R)$ if and only if $aR\alpha(b) \subseteq nil(R)$. Moreover, R is said to be *nil-*δ*-compatible* if for each $a,b \in R$, $aRb \subseteq nil(R)$ implies $aR\delta(b) \subseteq nil(R)$. If R is both nil-α-compatible and nil-δ-compatible, we say that R is *nil-*(α,δ)*-compatible*.

By [9], α-rigid rings are (α,δ)-compatible. Clearly every (α,δ)-compatible ring and hence every α-rigid ring is also nil-(α,δ)-compatible. Although the set of (α,δ)-compatible rings is narrow, we show that nil-(α,δ)-compatible rings are ubiquitous.

By [11], a ring R is *nil-Armendariz of power series type* if $f(x).g(x)\in nil(R)[[x]]$ implies $a_ib_j \in nil(R)$, for all i,j and

$$f(x)=\sum_{i=0}^{n}a_ix^i,\ g(x)=\sum_{j=0}^{m}b_jx^j\in R[[x]].$$

Lemma 1.2. Let R be a nil-(α,δ)-compatible ring. Then

(1) $aRb\subseteq nil(R)$ if and only if $aR\alpha^n(b)\subseteq nil(R)$, for each positive integer number n.

(2) $aRb\subseteq nil(R)$ implies $aR\delta^m(b)\subseteq nil(R)$, for each positive integer number m.

(3) If R is a nil-Armendariz of power series type and $aRb\subseteq nil(R)$ then $\alpha^n(a)R\delta^m(b)\subseteq nil(R)$, $\delta^p(a)R\alpha^q(b)\subseteq nil(R)$ when m,n,p,q are positive integer numbers.

Proof. (1) Since R is nil-(α,δ)-compatible, we have the following implications:
$aRb\subseteq nil(R)\Rightarrow aR\alpha(b)\subseteq nil(R)\Rightarrow aR\alpha^2(b)\Rightarrow\cdots aR\alpha^n(b)\subseteq nil(R)$. Conversely we have $aR\alpha^n(b)\subseteq nil(R)\Rightarrow aR\alpha(\alpha^{n-1}(b))\subseteq nil(R)\Rightarrow aR\alpha^{n-1}(b)\subseteq nil(R)\Rightarrow\cdots\Rightarrow aRb\subseteq nil(R)$.

(2) This is similar to (1).

(3) $aRb\subseteq nil(R)$ implies $bRa\subseteq nil(R)$ because for $u\in bRa$ it implies that $u=bra$, for each $r\in R$, $u^2=(bra)(bra)=br(ab)ra$. But $aRb\subseteq nil(R)$ then $ab\in nil(R)$ since R is nil-Armendariz of power series type, thus $u^2\in nil(R)$ so $u\in nil(R)$ and $bRa\subseteq nil(R)$. We have $aRb\subseteq nil(R)$, so $aR\delta^m(b)\subseteq nil(R)$ by (2). Then we have $\delta^m(b)Ra\subseteq nil(R)$ so $\delta^m(b)R\alpha^n(a)\subseteq nil(R)$ and that $\alpha^n(a)R\delta^m(b)\subseteq nil(R)$. Then we conclude that $\alpha^n(a)R\delta^m(b)$ is contained in $nil(R)$. We do the same for $\delta^p(b)R\alpha^q(a)\subseteq nil(R)$ with positive integers p,q.

Lemma 1.3. Each weak $(\alpha,\delta)-$compatible ring is nil-(α,δ)-compatible.

Proof. Suppose that $aRb\subseteq nil(R)$. So $arb\in nil(R)$ for each $r\in R$. Then $(ar)b\in nil(R)$ and so we have $(ar)\alpha(b)\in nil(R)$, by weak $(\alpha,\delta)-$compatibility. So $aR\alpha(b)\subseteq nil(R)$. Similarly, $(ar)b\in nil(R)$ for all $r\in R$ and by weak $(\alpha,\delta)-$compatibility we have $(ar)\delta(b)\in nil(R)$ and so $aR\delta(b)\subseteq nil(R)$. Next assume, $aR\alpha(b)\subseteq nil(R)$. Then $(ar)\alpha(b)\in nil(R)$ and by weak (α,δ)-compatibility, we have $(ar)b\in nil(R)$ for all $r\in R$, so $aRb\subseteq nil(R)$.

In the following, we will see that the converse is not true. Indeed, there exists a ring R , which is nil-(α,δ)-compatible but it is not weak (α,δ)-compatible. Thus a nil-(α,δ)-compatible ring is a true generalization of a weak (α,δ)-compatible ring (and hence (α,δ)-compatible ring). We then can find various classes of nil-(α,δ)-compatible rings which are not weak (α,δ)-compatible and hence are not (α,δ)-compatible.

Example 1.4. Let K be a field, and $S=K\langle x,y,z\rangle$. Let $R=\dfrac{S=K\langle x,y,z\rangle}{\langle yx\rangle}$.

Also assume that α is an endomorphism of S and $\overline{\alpha}$ be an endomorphism of R, given by:

$$\alpha(k)=k,\alpha(x)=xz,\alpha(z)=x.$$

$$\overline{\alpha}(\overline{k})=\overline{k},\overline{\alpha}(\overline{x})=\overline{x}\,\overline{z},\overline{\alpha}(\overline{z})=\overline{x}.$$

We first show that $\overline{\alpha}$ is well defined. To see this, let $\overline{f}=\overline{g}$ for some $\overline{f},\overline{g}\in R$, so there exists $h,f_i,f_i'\in S$ such that $f=g+\sum_i f_iyxf_i'$. Thus $\alpha(f)=\alpha(g)+\sum_i\alpha(f_i)\alpha(y)\alpha(x)\alpha(f_i')$ $=\alpha(g)+\sum_i\alpha(f_i)yxz\,\alpha(f_i')$. So $\overline{\alpha}(\overline{f})=\overline{\alpha}(\overline{g})$, and α is well defined. Now, we determine the set of nilpotent elements of R . First, we find zero devisor monomials. Let $0\neq f\in R$ be a zero devisor monomial.

We have $f=u_1f_1u_2, g=v_1g_1v_2$, with $u_1,u_2,v_1,v_2\in\{\overline{x},\overline{y},\overline{z}\}$, $f_1,g_1\in R$ such that $fg=0$, so $u_1f_1u_2v_1g_1v_2=0$. If $u_2v_1\neq\overline{yx}$, then there exists $\overline{yx}$ in one of u_1f_1 or f_1 or f_1u_2 or v_1g_1 or g_1 or g_1v_2. But if one of these cases occurs, $f=0$ or $g=0$. So $u_2v_1=\overline{yx}$, and $u_2=\overline{y}$, $v_1=\overline{x}$. Hence $0\neq f\in R$ is left (right) zero divisor if and only if $f=\overline{x}f_1$, $g=g_1\overline{y}$. If f is a nilpotent monomial of R, then $f=\overline{x}f_1\overline{y}$. Moreover $(\overline{x}f_1\overline{y})^2=\overline{0}$. So f is a nilpotent monomial, if and only if $f=\overline{x}f_1\overline{y}$, for some monomial $f_1\in R$.

Now we claim that, if is f a nilpotent polynomial and $f=\sum f_i$ where f_i is monomial for some i, then f_i is nilpotent in R (i.e. $f=\overline{x}f'\overline{y}$ for some polynomial $f'\in R$). Before proving the claim we have the following property:

The $\deg_z(\mathrm{f})$, where f is a monomial in the from $\mathrm{x}^{i_1}y^{j_1}z^{k_1}\mathrm{x}^{i_2}y^{j_2}z^{k_2}\cdots\mathrm{x}^{i_r}y^{j_r}z^{k_r}$, is $\sum_{t=1}^{r}k_t$. Similarly, $\deg_x(\mathrm{f}),\deg_y(\mathrm{f})$ are defined by $\sum_{t=1}^{r}j_t$ and $\sum_{t=1}^{r}i_t$ respectively. Also $\deg(f)=\sum_{t=1}^{r}(i_t+j_t+k_t)$.

Proof of the claim:

Let $A=\{f_i\mid\deg(\mathrm{f}_i)\text{ is maximal and } f_i \text{ is not nilpotent}\}$. Assume that f', f'', $f'''\in A$ are the monomials such that $\deg_x,\deg_y,\deg_z$ they have the largest. One can see that at least one of $\deg_x,\deg_y,\deg_z$ is nonzero. Without loss of generality, let $\deg_x\neq 0$. Since f' is not nilpotent f' is not zero devisor, hence $(f')^n$ is not zero.

Also, it is worth to say that the monomial with largest $\deg_x$ in f^n is $(f')^n$. So it can not be simplified and this means that f is not nilpotent. This contradiction shows that $\deg_x(f)=\deg_y(f)=\deg_z(f)=0$. So f is nilpotent and constant which means that $f=0$. Hence f is either zero or in the form $\overline{x}f_1\overline{y}$ for some $f_1\in R$.

Now, let $fRg\subseteq nil(R)$. Suppose that $f=uf_1$, $g=g_1v$ and $u,v\in\{\overline{x},\overline{y},\overline{z}\}$. If $u\neq\overline{x}, v\neq\overline{y}$, then $fzg\notin nil(R)$. So we have $f=\overline{x}f_1, g=g_1\overline{y}$, hence $fR\alpha(g)=\overline{x}f_1R\alpha(g_1)\overline{y}\subseteq nil(R)$. It is obvious that $fR\alpha(g)\subseteq nil(R)$. Conversely let $fR\alpha(g)\subseteq nil(R)$, with $f=uf_1$, $g=g_1v$ and $u,v\in\{\overline{x},\overline{y},\overline{z}\}$. Since $fR\alpha(g)\subseteq nil(R)$, $u=\overline{x}, \alpha(v)=v'\overline{y}$, so $v=\overline{y}$. Hence $fRg=\overline{x}f_1Rg_1\overline{y}$, which is obviously a subset of $nil(R)$, which shows that R is a nil-α-compatible ring.

But it is easy to see that $\overline{yz}\notin nil(R)$, while $y\alpha(z)=\overline{yz}=0\in nil(R)$. Thus R is not weak α-compatible. Note that $nil(R)$ is not an ideal of R. This is because $\overline{yzx}\in nil(R)$, $\overline{z}\in R$, but $\overline{zyzx}\notin nil(R)$, $\overline{yzxz}\notin nil(R)$.

Let δ be an α-derivation of R. The endomorphism α of R is extended to the endomorphism $\overline{\alpha}:T_n(R)\to T_n(R)$ defined by $\overline{\alpha}((a_{ij}))=(\alpha(a_{ij}))$, also the α-derivation $\overline{\delta}$ is extended to the $\overline{\alpha}$-derivation $\overline{\delta}:T_n(R)\to T_n(R)$ defined by $\overline{\delta}((a_{ij}))=(\delta(a_{ij}))$, for each $(a_{ij})\in T_n(R)$. Then we have the following.

Theorem 1.5. A ring R is nil-(α,δ)-compatible if and only if the triangular ring $T_n(R)$ is nil-$(\overline{\alpha},\overline{\delta})$-compatible.

Proof. Suppose that R is a nil-(α,δ)-compatible ring and $A=(a_{ij}),B=(b_{ij})\in T_n(R)$.

We show that $AT_n(R)B\subseteq nil(T_n(R))$ if and only if $AT_n(R)\overline{\alpha}(B)\subseteq nil(T_n(R))$. We observe that

$$nil(T_n(\mathrm{R}))=\begin{pmatrix} nil(R) & R & \cdots & R \\ 0 & nil(R) & \ddots & \vdots \\ \vdots & 0 & \ddots & R \\ 0 & \cdots & 0 & nil(R) \end{pmatrix}.$$

Then for $C=(\mathrm{r}_{ij})\in T_n(R)$, we have $\mathrm{AC}B\in nil(T_n(R))\Leftrightarrow$

$$\begin{pmatrix} a_{11}r_{11}b_{11} & \cdots & * & \cdots & * \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & a_{ii}r_{ii}b_{ii} & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & a_{nn}r_{nn}b_{nn} \end{pmatrix} \in nil(T_n(R)) \Leftrightarrow$$

$a_{ii}r_{ii}b_{ii} \in nil(R)$ for $1 \le i \le n$
$\Leftrightarrow a_{ii}r_{ii}\alpha(b_{ii}) \in nil(R)$, for $1 \le i \le n$, by nil-(α,δ)-compatibility $\Leftrightarrow$

$$\begin{pmatrix} a_{11}r_{11}\alpha(b_{11}) & \cdots & * & \cdots & * \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & a_{ii}r_{ii}\alpha(b_{ii}) & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & a_{nn}r_{nn}\alpha(b_{nn}) \end{pmatrix} \in nil(T_n(R)) \Leftrightarrow$$

$\mathrm{AC}\overline{\alpha}(B) \in nil(T_n(R)) \Leftrightarrow A\,T_n(R)\overline{\alpha}(B) \subseteq nil(T_n(R))$.

The case nil-$\overline{\delta}$ -compatibility is similar. Next suppose that $T_n(\mathrm{R})$ is a nil-(α,δ)-compatible ring and that $a,b,r \in R$, $A=(a)_{ij}, B=(b)_{ij}$, $\mathrm{C}=(\mathrm{c})_{ij}$ are diagonal matrices in $T_n(\mathrm{R})$. Then we have $aRb \subseteq nil(R) \Leftrightarrow$

$$\begin{pmatrix} arb & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & arb & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & arb \end{pmatrix} \in nil(T_n(R)) \Leftrightarrow$$

$\mathrm{AC}\,B \in nil(T_n(R)) \Leftrightarrow \mathrm{AC}\overline{\alpha}(B) \in nil(T_n(R))$, for all $r \in R$, by nil-$(\overline{\alpha},\overline{\delta})$-compatibility $\Leftrightarrow$

$$\begin{pmatrix} ar\alpha(b) & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & ar\alpha(b) & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & ar\alpha(b) \end{pmatrix} \in nil(T_n(R)) \Leftrightarrow$$

$ar\alpha(b) \in nil(R)$ for all $\mathrm{r} \in \mathrm{R} \Leftrightarrow aR\alpha(b) \subseteq nil(R)$. The case nil-$\alpha$ -compatible is similar.

Let R be a ring and let

$$S_n(R) = \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}$$ and

$$T(R,n) = \left\{ \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & a_1 & \cdots & a_{n-1} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & a_1 \end{pmatrix} \mid a_i \in R \right\}$$ with

$n \ge 2$, and let $T(\mathrm{R},\mathrm{R})$ be the trivial extension of R by R. Any endomorphism α of R can be extended to an endomorphism $\overline{\alpha}$ of $S_n(R)$ or $T(\mathrm{R},\mathrm{n})$ or $T(\mathrm{R},\mathrm{R})$ defined by $\overline{\alpha}((a_{ij})) = (\alpha(a_{ij}))$, and any α-derivation δ can be extended to an $\overline{\alpha}$ -derivation $\overline{\delta}$ of $S_n(R)$ (or $T(R,n)$ or $T(R,R)$) defined by $\overline{\delta}((a_{ij})) = (\delta(a_{ij}))$.

Theorem 1.6. Let α be an endomorphism and δ an α-derivation of R. Then the following conditions are equivalent:

(1) R is nil-(α,δ)-compatible.

(2) $S_n(R)$ is nil-$(\overline{\alpha},\overline{\delta})$-compatible.

(3) $T(R,\mathrm{n})$ is nil-$(\overline{\alpha},\overline{\delta})$-compatible.

(4) $T(R,R)$ is nil-$(\overline{\alpha},\overline{\delta})$-compatible.

Proof. Using the same method as in the proof of Theorem 1.5, the result follows.

By [22, Lemma 1.9], it is proved that, if R is 2-primal then $nil(R)[x] = nil(R[x])$. An endomorphism α and α-derivation δ of a ring R are extended to $R[x]$, given by $\overline{\alpha} : R[x] \to R[x]$ defined by $\overline{\alpha}(\sum_{i=0}^{m} a_i x^i) = \sum_{i=0}^{m} \alpha(a_i) x^i$, and $\overline{\delta} : R[x] \to R[x]$ defined by $\overline{\delta}(\sum_{i=0}^{m} a_i x^i) = \sum_{i=0}^{m} \delta(a_i) x^i$. We can easily see that $\overline{\delta}$ is an $\overline{\alpha}$-derivation of the polynomial ring $R[x]$.

Lemma 1.7 Let R be a nil-Armendariz ring of power series type, nil-α-compatible and $\Delta_k = \sum_{i+j=k} a_i \alpha^i(b_j) \in nil(R)$, $a_i, b_j \in R$, such that $k = 0,1,2,\cdots,n$. Then $a_i \alpha^i (b_j) \in nil(R)$ for all $i + j = k$.

Proof. We have the following system of equations:

$$\Delta_0 = a_0 b_0 \in nil(R);$$
$$\Delta_1 = a_0 b_1 + a_1 \alpha(b_0) \in nil(R);$$
$$\vdots$$
$$\Delta_s = a_0 b_s + a_1 \alpha(b_{s-1}) + a_2 \alpha^2 (b_{s-2}) + \cdots + a_s \alpha^s (b_0) \in nil(R). \quad (*)$$

We will show that $a_i \alpha^i (b_j) \in nil(R)$ by induction on $i + j$. If $i + j = 0$, then $a_0 b_0 \in nil(R)$, $b_0 a_0 \in nil(R)$. Now, suppose that s is a positive integer such that $a_i \alpha^i (b_j) \in nil(R)$, when $i + j < s$. We will show that $a_i \alpha^i (b_j) \in nil(R)$, when $i + j = s$. Multiplying equation (*) by b_0 from left, we have $b_0 a_s \alpha^s (b_0) = b_0 \Delta_s - b_0 a_0 b_s - b_0 a_1 \alpha(b_{s-1}) - b_0 a_2 \alpha^2 (b_{s-2}) - \cdots - b_0 a_{s-1} \alpha^{s-1} (b_0)$

By the induction hypothesis $a_i \alpha^i (b_0) \in nil(R)$, for each $i, 0 \le i < s$. So $a_i R \alpha^i (b_0) \subseteq nil(R)$ by [11, Lemma 3], hence $a_i R b_0 \subseteq nil(R)$, by nil-compatibility. Then $a_i b_0 \in nil(R)$, $b_0 a_i \subseteq nil(R)$, for each $i, 0 \le i < s$. Thus $b_0 a_s \alpha^s (b_0) \in nil(R)$ and so $b_0 a_s R \alpha^s (b_0) \subseteq nil(R)$, so $b_0 a_s R b_0 \subseteq nil(R)$, and hence $a_s \alpha^s (b_0) \in nil(R)$. Multiplying equation (*) by $b_1, b_2, \cdots, b_{s-1}$ from the left side respectively, yields $a_{s-1} \alpha^{s-1} (b_1) \in nil(R), a_{s-2} \alpha^{s-2} (b_2) \in nil(R), \cdots, a_0 b_s \in nil(R)$, in turn. This mean that $a_i \alpha^i (b_j) \in nil(R)$, when $i + j = s$.

Theorem 1.8. If R is a 2-primal and nil-(α,δ)-compatible ring, then the polynomial ring $R[x]$ is nil-$(\overline{\alpha},\overline{\delta})$-compatible.

Proof. Let $f(x) R[x] g(x) \subseteq nil(R[x])$, with $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x]$. Then for all $r(x) = \sum_{t=0}^{p} r_t x^t \in R[x]$, we have

$f(x) r(x) g(x) = \sum_{k=0}^{m+n+p} (\sum_{i+j+t=k} a_i r_t b_j) x^k \in nil(R[x]) = nil(R)[x]$. Hence $\sum_{i+j+t=k} a_i r_t b_j \in nil(R)$ for $k = 0,1,2,\ldots,m+n+p$. But R is 2-primal, so $a_i r_t b_j \in nil(R)$, by method of Lemma 1.7, and by nil-(α,δ)-compatibility we have $a_i r_t \alpha(b_j) \in nil(R)$ for all i, j, t. Thus $\sum_{i+j+t=k} a_i r_t \alpha(b_j) \in nil(R)$. So we can conclude that $f(x) r(x) \overline{\alpha}(g(x)) = \sum_{k=0}^{m+n+p} (\sum_{i+j+t=k} a_i r_t \alpha(b_j) x^k \in nil(R[x])$. Hence we get $f(x) R[x] \overline{\alpha}(g(x)) \subseteq nil(R[x])$. Similarly, we can show that $f(x) R[x] \overline{\delta}(g(x)) \subseteq nil(R[x])$. The converse is similar. Thus $R[x]$ is a nil-$(\overline{\alpha},\overline{\delta})$-compatible ring.

Let δ be an α-derivation of R, and for integers i, j with $0 \le i \le j$, $f_i^j \in End(R,+)$, will denote the map which is the sum of all possible words in α, δ built with i letters α and $j - i$ letters δ. For instance $f_0^0 = 1$, $f_j^j = \alpha^j$, $f_0^j = \delta^j$ and $f_{j-1}^j = \alpha^{j-1}\delta + \alpha^{j-2}\delta\alpha + \cdots + \delta\alpha^{j-1}$. The next lemma appears in [16].

Lemma 1.9. For any positive integer n and $r \in R$ we have $x^n r = \sum_{i=0}^{n} f_i^n (r) x^i$ in the ring $R[x;\alpha,\delta]$.

By [7], a ring R is *nil-semicommutative* if $ab \in nil(R)$ implies $aRb \subseteq nil(R)$, for all $a, b \in R$.

Lemma 1.10. Let R be a nil-(α,δ)-compatible ring and nil-Armendariz of power series type. If $aRb \subseteq nil(R)$ then $aRf_i^j(b) \subseteq nil(R)$ for all $0 \le i \le j$.

Proof. Using Lemma 1.2, the proof is trivial.

By [11, Lemma 3], if R is a nil-Armendariz ring of power series type then it is a nil-semicommutative ring.

Now we have:

Proposition 1.11. Let R be a nil-(α,δ)-compatible ring and nil-Armendariz of power series type. Then we have $nil(R[x;\alpha,\delta]) \subseteq nil(R)[x;\alpha,\delta]$.

Proof. Let $f(x)=\sum_{i=0}^{n} a_i x^i \in nil(R[x;\alpha,\delta])$. There exists a positive integer m such that $f^m(x)=0$. Then we have $a_n\alpha^n(a_n)\alpha^{2n}(a_n)\ldots\alpha^{(m-1)n}(a_n)x^{mn}+lower\ terms$ $=0$. So $a_n\alpha^n(a_n)\alpha^{2n}(a_n)\ldots\alpha^{(m-1)n}(a_n)=0$ $\in nil(R)$. Thus we have $a_n\alpha^n(a_n)\alpha^{2n}(a_n)..\alpha^{(m-2)n}(a_n)R\alpha^{(m-1)n}(a_n)$ $\subseteq nil(R)$, by [11, Lemma 3]. So we have $a_n\alpha^n(a_n)\alpha^{2n}(a_n)\ldots\alpha^{(m-2)n}(a_n)R\ \alpha^{(m-2)n}(a_n)$ $\subseteq nil(R)$, by nil-(α,δ)-compatibility. It implies that $a_n\alpha^n(a_n)\alpha^{2n}(a_n)\ldots\alpha^{(m-2)n}(a_n).1.\alpha^{(m-2)n}(a_n)\in nil(R)$, then $a_n\alpha^n(a_n)\alpha^{2n}(a_n)\ldots\alpha^{(m-3)n}(a_n)$ $R\alpha^{(m-2)n}(a_na_n)\subseteq nil(R)$, by [11, Lemma 3], so $a_n\alpha^n(a_n)\alpha^{2n}(a_n)\ldots\alpha^{(m-3)n}(a_n).1.a_na_n\in nil(R)$. By following this method, we have $a_n\in nil(R)$. Also $a_n=1.a_n\in nil(R)$, then $1Ra_n\subseteq nil(R)$. We have $1Rf_i^j(a_n)\subseteq nil(R)$, by Lemma 1.10 . So $f_i^j(a_n)\in nil(R)$ for $0\le i\le j$.

Now we fix $A=a_0+a_1x+\ldots+a_{n-1}x^{n-1}$. Then $0=f^m(x)=(A+a_nx^n)^m=A^m+\Delta$ with $\Delta\in R[x;\alpha,\delta]$. Note that the coefficients of Δ can be written as sums of monomials in a_i and $f_s^t(a_j)$, where $a_i, a_j\in\{a_0, a_1, \ldots, a_n\}$ and $t\ge s\ge 0$, are nonnegative integers and each monomial has a_n or $f_s^t(a_i)$. Because R is nil-Armendariz of power series type, we get that $\Delta\in nil(R)[x;\alpha,\delta]$. Now consider the term $A=a_0+a_1x+\ldots+a_{n-1}x^{n-1}$, so we have $A^m\in nil(R)[x;\alpha,\delta]$, then $A^m=a_{n-1}\alpha^{n-1}(a_{n-1})\ldots\alpha^{(m-1)(n-1)}(a_{n-1})x^{m(n-1)}$ $+lower\ terms$ $\in nil(R)[x;\alpha,\delta]$. Hence $a_{n-1}\alpha^{n-1}(a_{n-1})\ldots\alpha^{(m-1)(n-1)}(a_{n-1})\in nil(R)$. As the argument above, then we have $a_{n-1}\in nil(R)$. By following this method we have $a_i\in nil(R)$ for $0\le i\le n$. Hence $f(x)\in nil(R)[x;\alpha,\delta]$.

Corollary 1.12. If $\delta=0$, then $nil(R[x;\alpha])\subseteq nil(R)[x;\alpha]$.

Lemma 1.13. Let R be a nil-(α,δ)-compatible and nil-Armendariz ring of power series type ring. Then $\delta(nil(R))\subseteq nil(R)$.

Proof. Let $u\in\delta(nil(R))$ so $u=\delta(a)$ with $a\in nil(R)$. Hence $\delta(a)Ra\subseteq nil(R)$, and by δ-compatibility we have $\delta(a)R\delta(a)\subseteq nil(R)$. So $\delta^2(a)\in nil(R)$, then $\delta(a)\in nil(R)$.

Theorem 1.14. Let R be a nil-(α,δ)-compatible, nil-Armendariz ring of power series type and $nil(R)$ is nilpotent. Then $nil(R)[x;\alpha,\delta]\subseteq nil(R[x;\alpha,\delta])$.

Proof. Let $f(x)=\sum_{i=0}^{\infty} a_i x^i\in nil(R)[x;\alpha,\delta]$. For a_i any arbitrary coefficient of f, $\delta(a_i)\in nil(R)$. As α is an endomorphism, $\alpha(a_i)\in nil(R)$. Then there exists natural k such that $(nil(R))^k$ is zero. Now we consider $f^k(x)=$ $\sum_{s=0}^{nk}\left(\sum_{i_1+\ldots+i_k=s} a_{i_1}f_{u_2}^{v_2}(a_{i_2})\ldots f_{u_k}^{v_k}(a_{i_k})\right)x^s$. All coefficients of $f^k(x)$ are in the form of $a_{i_1}f_{u_2}^{v_2}(a_{i_2})..f_{u_k}^{v_k}(a_{i_k})$, which is the product of k members of the $nil(R)$ by Lemma 1.10, so it should be equal to zero. Thus $f^k(x)=0$ and hence $f\in nil(R[x;\alpha,\delta])$.

Corollary 1.15. Let R be a nil-(α,δ)-compatible, nil-Armendariz ring of power series type and $nil(R)$ is nilpotent. Then we have $nil(R[x;\alpha,\delta]) = nil(R)[x;\alpha,\delta]$.

Corollary 1.16. Let R be a nil-(α,δ)-compatible and 2-primal ring. Assume that either R is a right Goldie ring or R has the ascending chain condition (a.c.c.) on ideals or R has the a.c.c. on right and left annihilators or R is a ring with right Krull dimension. Then $nil(R[x;\alpha,\delta]) = nil(R)[x;\alpha,\delta]$.

Proof. If R has any of these chain conditions, the upper nilradical $Nil_*(R)$ of R is nilpotent. If R has the a.c.c. on ideals, $Nil_*(R)$ can be characterized as the maximal nilpotent ideal of R. If R has the a.c.c. on both left and right annihilators, $Nil_*(R)$ is nilpotent by a result of Herstein and Small [10, Theorem 1.34], while if R is right Goldie, $Nil_*(R)$ is nilpotent by a result of Lanski [18, Theorem 1]. Also, if R is a ring with right Krull dimension, then by [17], $Nil_*(R)$ is nilpotent. □

Corollary 1.17. If R is a nil-Armendariz ring of power series type and $nil(R)$ is nilpotent and nil-α-compatible ring, then $nil(R[x;\alpha]) = nil(R)[x;\alpha]$.

Proof. By Corollary 1.15, if $\delta = 0$, then $nil(R[x;\alpha]) = nil(R)[x;\alpha]$.

Theorem 1.18. If R is a nil-Armendariz ring of power series type and nil-(α,δ)-compatible ring $nil(R)$ is nilpotent, then $R[x;\alpha]$ is a nil-$(\overline{\alpha},\overline{\delta})$-compatible ring.

Proof. Assume that $f(x)=\sum_{i=0}^{n}a_ix^i, g(x)=\sum_{j=0}^{m}b_jx^j \in R[x;\alpha]$, and let $f(x)R[x;\alpha]\overline{\alpha}(g(x)) \subseteq nil(R[x;\alpha])$. For all $r(x)=\sum_{t=0}^{p}a_tx^t \in R[x;\alpha]$, we have $f(x)r(x)\overline{\alpha}(g(x)) \in nil(R[x;\alpha])$. So by Corollary 1.17, we have $f(x)r(x)\overline{\alpha}(g(x))$
$= \sum_{k=0}^{m+n+p}(\sum_{i+j+t=k} a_i\alpha^i(r_t)\alpha^{i+t+1}(b_j))x^k \in nil(R[x;\alpha])$
$\subseteq nil(R)[x;\alpha]$.

Then $\sum_{i+j+t=k} a_i\alpha^i(r_t)\alpha^{i+t+1}(b_j) \in nil(R)$, for $k=0,1,2,\ldots,m+n+p$. But R is nil-Armendariz ring of power series type, so $a_i\alpha^i(r_t)\alpha^{i+t+1}(b_j) \in nil(R)$, by Lemma 1.7, with $0 \le i \le n, 0 \le j \le m, \quad 0 \le t \le p$. As it is nil-semicommutative, so we get $a_i\alpha^i(r_t)R\alpha^{i+t+1}(b_j)$ $\subseteq nil(R)$, then $a_i\alpha^i(r_t)R\alpha^{i+t}(b_j) \subseteq nil(R)$, by Lemma 1.2. Then $a_i\alpha^i(r_t)\alpha^{i+t}(b_j) \in nil(R)$. Hence $\sum_{i+j+t=k} a_i\alpha^i(r_t)\alpha^{i+t}(b_j) \in nil(R)$. So we have $\sum_{k=0}^{m+n+p}(\sum_{i+j+t=k} a_i\alpha^i(r_t)\alpha^{i+t}(b_j))x^k$ $\in nil(R)[x;\alpha] = nil(R[x;\alpha])$. Therefore we get $f(x)r(x)g(x) \in nil(R[x;\alpha])$.

Conversely assume $f(x)R[x;\alpha]g(x)$ $\subseteq nil(R[x;\alpha])$. So we have

$$f(x)r(x)g(x) = \sum_{k=0}^{m+n+p}(\sum_{i+j+t=k} a_i\alpha^i(r_t)\alpha^{i+t}(b_j))x^k \in nil(R)[x;\alpha],$$

for all $r(x) \in R[x;\alpha]$. Thus we have $\sum_{i+j+t=k} a_i\alpha^i(r_t)\alpha^{i+t}(b_j) \in nil(R)$. Since R is nil-Armendariz ring of power series type, $a_i\alpha^i(r_t)\alpha^{i+t}(b_j)$ $\in nil(R)$, so $a_i\alpha^i(r_t)R\alpha^{i+t}(b_j) \subseteq nil(R)$. Hence $a_i\alpha^i(r_t)R\alpha^{i+t+1}(b_j) \subseteq nil(R)$, so $a_i\alpha^i(r_t)\alpha^{i+t+1}(b_j) \in nil(R)$. Then for all i, j, k, we have $\sum_{i+j+t=k} a_i\alpha^i(r_t)\alpha^{i+t+1}(b_j)$ $\in nil(R)$. So $\sum_{k=0}^{m+n+p}(\sum_{i+j+t=k} a_i\alpha^i(r_t)\alpha^{i+t+1}(b_j))x^k \in nil(R)[x;\alpha]$, and hence $f(x)r(x)\overline{\alpha}(g(x)) \in nil(R[x;\alpha])$, for all $r(x)\in R[x;\alpha]$. Finally we have $f(x)R[x;\alpha]\overline{\alpha}(g(x))\subseteq nil(R[x;\alpha])$. For the case of nil-$\overline{\delta}$-compatibility, let $f(x)R[x;\alpha]g(x)\subseteq nil(R[x;\alpha])$. Then we have $f(x)r(x)g(x)$ $=\sum_{k=0}^{m+n+p}(\sum_{i+j+t=k}a_i\alpha^i(r_t)\alpha^{i+t}(b_j))x^k\in nil(R)[x;\alpha]$, for all $r(x)\in R[x;\alpha]$. Hence $\sum_{i+j+t=k}a_i\alpha^i(r_t)\alpha^{i+t}(b_j)\in nil(R)$. Then $a_i\alpha^i(r_t)\alpha^{i+t}(b_j)\in nil(R)$ for $0\le i\le n, 0\le j\le m, 0\le t\le p$. Based on the assumption we have $a_i\alpha^i(r_t)R\alpha^{i+t}(b_j)$ $\subseteq nil(R)$, so $a_i\alpha^i(r_t)Rb_j\subseteq nil(R)$ and that $a_i\alpha^i(r_t)R\delta(b_j)\subseteq nil(R)$. Hence $\sum_{i+j+t=k}a_i\alpha^i(r_t)\alpha^{i+t}(\delta(b_j))\in nil(R)$, and that $\sum_{k=0}^{m+n+p}(\sum_{i+j+t=k}a_i\alpha^i(r_t)\alpha^{i+t}(\delta(b_j)))x^k\in nil(R)[x;\alpha]$. This implies $f(x)r(x)\overline{\delta}(g(x))\in nil(R[x;\alpha])$, for all $r(x)\in R[x;\alpha]$. Therefore we conclude that $f(x)R[x;\alpha]\overline{\delta}(g(x))\subseteq nil(R[x;\alpha])$.

Now we consider the nilpotent elements in skew polynomial rings when R is a nil-Armendariz ring of power series type.

Theorem 1.19. Let R be a nil-Armendariz ring of power series type and nil-(α,δ)-compatible ring. Let

$$f(x)=\sum_{i=0}^{n}a_ix^i,\ g(x)=\sum_{j=0}^{m}b_jx^j\in R[x;\alpha,\delta].$$

If $f(x)R[x;\alpha,\delta]g(x)\subseteq nil(R[x;\alpha,\delta])$, then $a_iRb_j\subseteq nil(R)$ for $0\le i\le n,\ 0\le j\le m$.

Proof. Let $a_irb_j\in a_iRb_j$ for all $r\in R$, $0\le i\le n,\ 0\le j\le m$. We have $r\in R[x;\alpha,\delta]$ so $f(x)rg(x)\in f(x)R[x;\alpha,\delta]g(x)$, hence we have $(\sum_{i=0}^{n}a_ix^i)\mathrm{r}(\sum_{j=0}^{m}b_jx^j)$ $=(\sum_{i=0}^{n}a_ix^i)(\sum_{j=0}^{m}(rb_j)x^j)\in nil(R[x;\alpha,\delta])$, so $(\sum_{i=0}^{n}a_ix^i)(\sum_{j=0}^{m}(rb_j)x^j)\in nil(R)[x;\alpha,\delta]$ by Proposition 1.11. Therefore $\sum_{k=0}^{n+m}\left(\sum_{i+j=k}\left(a_i\sum f_s^t(rb_j)\right)\right)x^k$ $\in nil(R)[x;\alpha,\delta]$, with $t\le s$. Put $\Delta_k=\sum_{i+j=k}\left(a_i\sum f_s^t(rb_j)\right),\ k=0,1,2,\ldots,m+n$, hence $\Delta_k\in nil(R)$. We have the following equation:

$$\Delta_{m+n}=a_m\alpha^m(rb_n)\in nil(R);$$
$$\Delta_{m+n-1}=a_m\alpha^m(rb_{n-1})+a_{m-1}\alpha^{m-1}(rb_n)+\mathrm{a}_m f_{m-1}^m(rb_n)\in nil(R);$$
$$\Delta_{m+n-2}=a_m\alpha^m(rb_{n-2})+\sum_{i=m-1}^{m}a_if_{m-1}^i(rb_{n-1})+\sum_{i=m-2}^{m}a_if_{m-2}^i(rb_n)\in nil(R);$$
$$\vdots$$
$$\Delta_k=\sum_{s+t=k}(\sum_{i=s}^{m}a_if_s^i(rb_t))\in nil(R).$$

Then since R is nil-semicommutative by [11, Lemma 3], applying the method in the proof of [22, Theorem 2.14], we obtain $a_i(rb_j)\in nil(R)$, then $a_iRb_j\subseteq nil(R)$.

Theorem 1.20. Let R be a nil-(α,δ)-compatible and I be a nil ideal of R such that $\alpha(I)=I,\ \delta(I)=I$. Then $\overline{R}=\frac{R}{I}$ is a nil-(α,δ)-compatible ring.

Proof. We have to prove $\overline{aRb}\subseteq nil(\overline{R})$ if and only if $\overline{aR}\alpha(\overline{b})\subseteq nil(\overline{R})$, for any $\overline{a},\overline{b}\in\overline{R}$, such that $\overline{a}=\mathrm{a}+I,\ \overline{b}=b+I$. First assume $\overline{aRb}\subseteq nil(\overline{R})$ and $\overline{r}\in\overline{R}$. Then $\overline{a}\,\overline{r}\,\alpha(\overline{b})\in\overline{a}\,\overline{R}\,\alpha(\overline{b})$, so $(a+I)(r+I)(\alpha(b)+I)\in\overline{a}\,\overline{R}\,\alpha(\overline{b})$. Then

$ar\alpha(b)+I \in \overline{a}\overline{R}\alpha(\overline{b})$. But $nil(\frac{R}{I})=\frac{nil(R)}{I}$, so $(arb+I)\in\frac{nil(R)}{I}$, hence $arb\in nil(R)$. As R is nil-(α,δ)-compatible, we have $ar\alpha(b)\in nil(R)$ so $ar\alpha(b)+I\in\frac{nil(R)}{I}=nil(\frac{R}{I})$.

Then $\overline{a}\overline{R}\alpha(\overline{b})\subseteq nil(\overline{R})$. The case nil-$\delta$-compatible is similar. Conversely assume $\overline{a}\overline{R}\alpha(\overline{b})\subseteq nil(\overline{R})$ and $\overline{arb}\in\overline{aRb}$. Then $(arb+I)\in\overline{a}\overline{R}\overline{b}$. Under the assumption $\overline{a}\overline{r}\alpha(\overline{b})\in nil(\overline{R})$, hence $ar\alpha(b)\in nil(R)$. As R is nil-(α,δ)-compatible, we have $arb\in nil(R)$ for all $r\in R$, so we concluded that $\overline{aRb}\subseteq\frac{nil(R)}{I}=nil(\frac{R}{I})$.

Definition 1.21. [13] A ring R is said to be (α,δ)-*skew nil-Armendariz* if whenever $f(x)=\sum_{i=0}^{n}a_ix^i, g(x)=\sum_{j=0}^{m}b_jx^j \in R[x;\alpha,\delta]$ satisfy $f(x).g(x)\in nil(R)[x,\alpha,\delta]$, then $a_ix^ib_jx^j\in nil(R)[x,\alpha,\delta]$, for any i, j.

Lemma 1.22. If R is a nil-(α,δ)-compatible and nil-Armendariz ring of power series type, then R is an (α,δ)-skew nil-Armendariz ring.

Proof. Let $f(x)=\sum_{i=0}^{n}a_ix^i, g(x)=\sum_{j=0}^{m}b_jx^j \in R[x;\alpha,\delta]$ and $f(x).g(x)\in nil(R)[x,\alpha,\delta]$. Therefore $\sum_{k=0}^{n+m}\left(\sum_{i+j=k}\left(a_i\sum f_s^t(b_j)\right)\right)x^k\in nil(R)[x;\alpha,\delta]$, with $t\le s$. So $\sum_{s+t=k}(\sum_{i=s}^{m}a_if_s^i(b_t))\in nil(R)$, since R is nil-semicommutative by [11, Lemma 3], applying the method in the proof of [22, Theorem 2.14], we obtain $a_if_s^i(b_t)\in nil(R)$ with $t\le s$. Then R is a (α,δ)-skew nil-Armendariz ring.

Proposition 1.23. Let R be a nil-(α,δ)-compatible and nil-Armendariz ring of power series type, then for each idempotent element $e\in R$, $\delta(e)\in nil(R)$ and $\alpha(e)=e+u$ such that $u\in nil(R)$.

Proof. We have $\delta(e)=\delta(e^2)=\alpha(e)\delta(e)+\delta(e)e$. By taking polynomials $f(x)=\delta(e)+\alpha(e)x$, $g(x)=(e-1)+(e-1)x$, we see that $f(x).g(x)=0$, which implies that $f(x).g(x)\in nil(R[x;\alpha,\delta])\subseteq nil(R)[x;\alpha,\delta]$ by Proposition 1.11. $\delta(e)(e-1)=\delta(e)e-\delta(e)\in nil(R)$. Now take $h(x)=\delta(e)-(1-\alpha(e))x$, $k(x)=e+ex$. Then we have $h(x).k(x)=0$, so we get $\delta(e)e\in nil(R)$ and so $\delta(e)\in nil(R)$. Now take $p(x)=(1-e)+(1-e)\alpha(e)x$ and $q(x)=e+(e-1)\alpha(e)x\in R[x;\alpha,\delta]$. Then $p(x).q(x)=(1-e)\alpha(e)\delta(e)+(1-e)\alpha(e)\delta(e)\alpha(e)x\in nil(R)[x;\alpha,\delta]$, since $\delta(e)\in nil(R)$ and R is nil-Armendariz ring of power series type. But R is (α,δ)-skew nil-Armendariz by Lemma 1.22, so $(1-e).(e-1)\alpha(e)=e\alpha(e)-\alpha(e)\in nil(R)$ (1). Now take $t(x)=e+e(1-\alpha(e))x$, $s(x)=(1-e)-e(1-\alpha(e))x\in R[x;\alpha,\delta]$. Then we have $t(x)s(x)=-e(1-\alpha(e))\delta(e)-e(1-\alpha(e))\delta(e)x+e(1-\alpha(e))\delta(e)\alpha(e)x$. As $\delta(e)\in nil(R)$, $t(x)s(x)\in nil(R)[x;\alpha,\delta]$.

And so R is a (α,δ)-skew nil-Armendariz ring, thus $e.e(1-\alpha(e)) = e-e\alpha(e) \in nil(R)$ (2). Now by (1) and (2) we obtain $u = e-\alpha(e) \in nil(R)$. Hence $\alpha(e) = e+u$ with $u \in nil(R)$.□

Theorem 1.24. Let R be a nil-(α,δ)-compatible and nil-Armendariz ring of power series type. Then for each idempotent element $e \in R$ and $a \in R$, $ea = ae+u$ with $u \in nil(R)$.

Proof. According to the Proposition 1.23, $\alpha(e) = e+u$ with $u \in nil(R)$, $\delta(e) \in nil(R)$. Now take the polynomials $f(x) = e-ea(1-e)x$, $g(x) = 1-e+ea(1-e)x$ in $R[x;\alpha,\delta]$. Hence $f(x).g(x) = ea(1-e)xe - ea(1-e)x.ea(1-e)x$. On the other hand, $u \in nil(R)$, $\delta(e) \in nil(R)$ and R is nil-Armendariz ring of power series type. So we have

$$ea(1-e)xe = ea(1-e)\alpha(e)x + ea(1-e)\delta(e)$$
$$= ea(1-e)ux + eu(1-e)\delta(e) \in nil(R)[x;\alpha,\delta].$$

Similarly $ea(1-e)xea(1-e)x \in nil(R)[x;\alpha,\delta]$. Then $f(x)g(x) \in nil(R)[x;\alpha,\delta]$, hence we get $e.ea(1-e) \in nil(R)$, and that $ea - eae \in nil(R)$ (1). Let $h(x) = 1-e-(1-e)aex$, $k(x) = e+(1-e)aex$, according to an earlier state we have $(1-e)(1-e)ae \in nil(R)$. Hence $ae - eae \in nil(R)$ (2). Using (1), (2) we have $ea - ae \in nil(R)$, so $ea = ae+u$ with $u \in nil(R)$.

Definition 1.25. For an endomorphism α and an α-derivation δ, an ideal I is said to be *nil-*(α,δ)*-compatible* provided that:

1) $aRb \subseteq nil(R) \Leftrightarrow aR\alpha(b) \subseteq nil(R)$. For all $a,b \in I$.

2) $aRb \subseteq nil(R) \Rightarrow aR\delta(b) \subseteq nil(R)$. For all $a,b \in I$.

Theorem 1.26. Let R be an abelian nil-Armendariz ring of power series type. Then the following statements are equivalent :

1) R is a nil-(α,δ)-compatible ring.

2) For each idempotent $e \in R$ with $\alpha(e) = e+u$, $u \in nil(R)$ and $\delta(e) \in nil(R)$, eR and $(1-e)R$ are nil-(α,δ)-compatible ideals.

Proof. 1⇒2 is trivial. Let eR be a nil-(α,δ)-compatible ideal and $aRb \subseteq nil(R)$ for each $a,b \in R$ so $arb \in nil(R)$, hence $earb \in nil(R)$. Thus $(ea)r(eb) \in nil(R)$. But eR is a nil-(α,δ)-compatible ideal, hence we have that $(ea)r\alpha(e)\alpha(b) = (ea)r(e+u)\alpha(b)$ $= (ea)re\alpha(b) + (ea)ru\alpha(b) \in nil(R)$. Since $u \in nil(R)$, we have $(ea)ru\alpha(b) \in nil(R)$, so $(ea)re\alpha(b) = (ea)r\alpha(b) \in nil(R)$ (1). Now, according to the above argument for $(1-e)R$, we have $(1-e)ar\alpha(b) \in nil(R)$ (2). With (1) and (2) we obtain $ar\alpha(b) \in nil(R)$, for all $r \in R$ or $aR\alpha(b) \subseteq nil(R)$. For the case of nil-$\delta$-compatible, we do in a similar way. Conversely suppose that $aR\alpha(b) \subseteq nil(R)$, then we get $ar\alpha(b) \subseteq nil(R)$, for each $r \in R$. But (ea)rα (eb)= (ea) rα (e)α (b)= (ea) r (e+u) α (b)= (ea) r e α (b)+ (ea) ruα (b) $= ear\alpha(b) + (ea)ru\alpha(b)$ and that

$ear\alpha(b), (ea)ru\alpha(b) \in nil(R)$.

Then $(ea)r\alpha(eb) \in nil(R)$, since eR is nil-(α,δ)-compatible ideal, thus we have $(ea)r(eb) = earb \in nil(R)$ (3). Similarly, we have $(1-e)arb \in nil(R)$ (4). Therefore (3),(4) implies $arb \in nil(R)$, for all $r \in R$. Hence $aRb \subseteq nil(R)$.

We continue to extend nil-α -compatible condition on $R[x,x^{-1}]$ and $R[x,x^{-1};\alpha]$. If $f(x)=\sum_{i=k}^{n} a_i x^i \in R[x,x^{-1}]$, we define $\overline{\alpha}$ $(\sum_{i=k}^{n} a_i x^i) = \sum_{i=k}^{n} \alpha(a_i) x^i$, for each integer numbers k,n.

Theorem 1.27. If R is a 2-primal and nil-α-compatible ring, then $R[x,x^{-1}]$ is a nil-$\overline{\alpha}$-compatible ring.

Proof. Let $\Delta = \{1, x, x^2, \ldots\} \subseteq R[x]$. Then we have $R[x,x^{-1}] = \Delta^{-1}R[x]$. Hence for $f(x)=\sum_{i=t}^{n} a_i x^i \in R[x,x^{-1}]$ with the integer number t, we have $x^t f(x) = \sum_{j=0}^{m} a_j x^j \in R[x]x^t$, hence $x^t f(x) = g(x) \in R[x]$, so $\overline{\alpha}(f(x)) = x^{-t}\overline{\alpha}(g(x))$. Now by Theorem 1.8, $R[x,x^{-1}]$ is an nil-$\overline{\alpha}$ -compatible ring.□

Recall that a ring R is called of *bounded index of nilpotency,* if there exists a positive number n such that $x^n = 0$, for each $x \in nil(R)$.

Lemma 1.28. [11, Lemma 2] If R is a nil-Armendariz ring of power series type, then $nil(R[[x]]) \subseteq nil(R)[[x]]$.

Theorem 1.29. Let R be a nil-Armendariz ring of power series type and of bounded index. Then $nil(R[[x]]) = nil(R)[[x]]$.

Proof. By Lemma 1.28 it is sufficient to prove that $nil(R)[[x]] \subseteq nil(R[[x]])$. Since R is nil-Armendariz of power series type, $nil(R)$ is nil and of bounded index, as a ring, by [12, Theorem 2.5]. Then $R[[x]]$ is a nil ring of bounded index. Hence we get $nil(R)[[x]] \subseteq nil(R[[x]])$.

Lemma 1.30. [11, Lemma1] Let R be a nil-Armendariz ring of power series type. Let $f_1, f_2, \ldots, f_n \in R[[x]]$ and $f_1 f_2 \ldots f_n \in nil(R)[[x]]$. Then $a_1 a_2 \ldots a_n \in nil(R)$, for all coefficients a_i of f_i.

Theorem 1.31. Let R be a nil-(α,δ)-compatible and nil-Armendariz of power series type ring with bounded index. Then $R[[x]]$ is a nil-$(\overline{\alpha},\overline{\delta})$-compatible ring.

Proof. Let $f(x)=\sum_{i=0}^{\infty} a_i x^i, g(x)=\sum_{j=0}^{\infty} b_j x^j \in R[[x]]$ and assume that $f(x)R[[x]]g(x) \subseteq nil(R[[x]])$. For each $r(x)=\sum_{t=0}^{\infty} c_t x^t \in R[[x]]$ we have $f(x)r(x)g(x) \in nil(R[[x]])$. If u is an arbitrary element of $f(x)R[[x]]\overline{\alpha}(g(x))$, then $u = f(x)r(x)\overline{\alpha}(g(x))$, for all $r(x) \in R[[x]]$. Under the assumption we have $f(x)r(x)g(x) \in nil(R[[x]]) = nil(R)[[x]]$, since R is nil-Armendariz of power series type so $a_i c_t b_j \in nil(R)$ and since R is nil-(α,δ)-compatible $a_i c_t \alpha(b_j) \in nil(R)$ and

$\sum_{i+t+j=k} a_i c_t \alpha(b_j) \in nil(R)$ for $k=0,1,2,\ldots$. Hence $\sum_{k=0}^{\infty}(\sum_{i+t+j=k} a_i c_t \alpha(b_j))\ x^k \in nil(R)[[x]]$, so we have $f(x)r(x)\overline{\alpha}(g(x)) \in nil(R)[[x]] = nil(R[[x]])$. And this means $f(x)R[[x]]\overline{\alpha}(g(x)) \subseteq nil(R[[x]])$. Conversely, we prove that $f(x)R[[x]]g(x) \subseteq nil(R[[x]])$. If $f(x)r(x)g(x) \in f(x)R[[x]]g(x)$, then by the assumption $f(x)r(x)\overline{\alpha}(g(x)) \in nil(R[[x]]) = nil(R)[[x]]$, and since R is nil-Armendariz of power series type, we have $a_i c_t \alpha(b_j) \in nil(R)$. Hence $a_i c_t b_j \in nil(R)$ and that $\sum_{i+t+j=k} a_i c_t b_j \in nil(R)$ for $k=0,1,2,\ldots$. So $\sum_{k=0}^{\infty}(\sum_{i+t+j=k} a_i c_t b_j)\ x^k \in nil(R)[[x]] = nil(R[[x]])$. And this means that $f(x)R[[x]]g(x) \subseteq nil(R[[x]])$. For the case of nil-$\overline{\delta}$ -compatible, we do in a similar method. Then $R[[x]]$ is a nil-$(\overline{\alpha},\overline{\delta})$-compatible ring.

Definition 1.32. A ring R with an α endomorphism is *skew nil-Armendariz of power series type,* if whenever for all $f(x)=\sum_{i=0}^{\infty} a_i x^i, g(x)=\sum_{j=0}^{\infty} b_j x^j \in R[[x;\alpha]]$, $f(x).g(x) \in nil(R)[[x;\alpha]]$, then $a_i \alpha^i(b_j) \in nil(R)$ for all i, j.

Proposition 1.33. Let R be a nil-α-compatible and nil-Armendariz ring of power series type. Then R is skew nil-Armendariz ring of power series type.

Proof. Let $f(x)=\sum_{i=0}^{\infty} a_i x^i, g(x)=\sum_{j=0}^{\infty} b_j x^j \in R[[x;\alpha]]$, $f(x).g(x) \in nil(R)[[x;\alpha]]$, thus $\sum_{k=0}^{\infty}(\sum_{i+j=k} a_i \alpha^i(b_j))x^k \in nil(R)[[x;\alpha]]$, so $\sum_{i+j=k} a_i \alpha^i(b_j) \in nil(R)$, thus $a_i \alpha^i(b_j) \in nil(R)$, for all i, j by Lemma 1.7. Then R is skew nil-Armendariz ring of power series type.□

Lemma 1.34. Let R be a nil-(α,δ)-compatible and skew nil-Armendariz ring of power series type. Then R is nil-semicommutative.

Proof. Let $r \in R$ and $ab \in nil(R)$. Then $a(1-rx)(1+rx+(rx)^2+\cdots)b \in nil(R)[[x;\alpha]]$. So $ar\alpha(b) \in nil(R)$ and hence $arb \in nil(R)$.

Lemma 1.35. Let R be a skew nil-Armendariz ring of power series type and assume that $f_1 f_2 \cdots f_n \in nil(R)[[x;\alpha]]$. Then $(a_{i_1}x^{i_1})(a_{i_2}x^{i_2})\cdots(a_{i_n}x^{i_n}) \in nil(R)[[x;\alpha]]$, for all coefficients a_{i_j} of f_{i_j}.

Proof. We will show that $a_{i_1}\alpha^{i_1}(a_{i_2})\alpha^{i_1+i_2}(a_{i_3})\cdots\alpha^{i_1+i_2\cdots+i_{n-1}}(a_{i_n}) \in nil(R)$ by induction on n. Suppose that $a_{i_1}\alpha^{i_1}(a_{i_2})\alpha^{i_1+i_2}(a_{i_3})\cdots\alpha^{i_1+i_2\cdots+i_{k-1}}(a_{i_k}) \in nil(R)$ for $k<n$. Since $\alpha^{i_1+i_2\cdots+i_k}(a_{i_{k+1}}) \in$R, we have $a_{i_1}\alpha^{i_1}(a_{i_2})\alpha^{i_1+i_2}(a_{i_3})\cdots\alpha^{i_1+i_2\cdots+i_{k-1}}(a_{i_k})\alpha^{i_1+i_2\cdots+i_k}(a_{i_{k+1}}) \in nil(R)$. This is because, if $a \in nil(R)$, $b \in R$, we have $\text{a}(1-bx)(1+(bx)+(bx)^2+\cdots) = \text{a} \in nil(R)[[x;\alpha]]$. So $ab \in nil(R)$ and hence $(a_{i_1}x^{i_1})(a_{i_2}x^{i_2})\cdots(a_{i_{k+1}}x^{i_{k+1}}) \in nil(R)[[x;\alpha]]$. □

Theorem 1.36. Let R be a nil-(α,δ)-compatible nil-Armendariz ring of power series type. Then $nil(R[[x;\alpha]]) \subseteq nil(R)[[x;\alpha]]$.

Proof. We show that $nil(R[[x;\alpha]]) \subseteq nil(R)[[x;\alpha]]$. Let $f(x)=\sum_{i=0}^{\infty} a_i x^i \in nil(R[[x;\alpha]])$. Then $f^k(x)=0$ for some positive integer k. So we have $0=f^k(x)=\sum_{s=0}^{\infty}(\sum_{i_1+i_2+\cdots+i_k=s} a_{i_1}\alpha^{i_1}(a_{i_2})\alpha^{i_1+i_2}(a_{i_3})\cdots\alpha^{i_1+i_2\cdots+i_{k-1}}(a_{i_k}))x^s$.

If a is an arbitrary member of coefficients of f, then $(ax^t)(ax^t)\ldots(ax^t)\in nil(R)[[x;\alpha]]$ (k times). Hence we have $a\alpha^t(a)\alpha^{2t}(a)\ldots\alpha^{(k-1)t}(a)\in nil(R)$. Then by Lemma 1.2, and Lemma 1.34, we have $a^k\in nil(R)$. Thus $a\in nil(R)$, and hence $f(x)\in nil(R)[[x;\alpha]]$.

Theorem 1.37. Let R be a nil-(α,δ)-compatible and nil-Armendariz ring of power series type and $nil(R)$ be nilpotent.

Then $nil(R)[[x;\alpha]]\subseteq nil(R[[x;\alpha]])$.

Proof. Let $f(x)=\sum_{i=0}^{\infty}a_ix^i\in nil(R)[[x;\alpha]]$. Then $a_i\in nil(R)$ and $\alpha^m(a_i)\in nil(R)$ for all i. Since $nil(R)$ is nilpotent, there exists a positive integer k such that $(nil(R))^k=0$ and any product of k elements from $nil(R)$ is zero.

Now consider $f^k(x)=\sum_{s=0}^{\infty}(\sum_{i_1+i_2+\cdots+i_k=s} a_{i_1}\alpha^{i_1}(a_{i_2})\alpha^{i_1+i_2}(a_{i_3})\cdots\alpha^{i_1+i_2\cdots+i_{k-1}}(a_{i_k}))x^s \in nil(R)[[x;\alpha]]$, so $a_{i_1}\alpha^{i_1}(a_{i_2})\ldots\alpha^{i_1+\ldots+i_{k-1}}(a_{i_k})\in nil(R)$, then $a_{i_1}\alpha^{i_1}(a_{i_2})\ldots\alpha^{i_1+\ldots+i_{k-1}}(a_{i_k})=0$. Hence $f^k(x)=0$ and that $f(x)\in nil(R[[x;\alpha]])$.

Corollary 1.38. Let R be a nil-(α,δ)-compatible nil-Armendariz ring of power series type and $nil(R)$ be nilpotent. Then $nil(R)[[x;\alpha]]=nil(R[[x;\alpha]])$.

Theorem 1.39. Let R be a nil-(α,δ)-compatible, nil-Armendariz ring of power series type and $nil(R)$ be nilpotent. Then $R[[x;\alpha]]$ is a nil-$(\overline{\alpha},\overline{\delta})$-compatible ring.

Proof. Let $f(x)=\sum_{i=0}^{\infty}a_ix^i, g(x)=\sum_{j=0}^{\infty}b_jx^j\in R[[x;\alpha]]$ and $f(x).R[[x;\alpha]].g(x)\subseteq nil(R[[x;\alpha]])$. Then for all $r(x)=\sum_{t=0}^{\infty}r_tx^t\in R[[x;\alpha]]$ we have $f(x).r(x).g(x)\in nil(R[[x;\alpha]])$.

If $u\in f(x)R[[x;\alpha]]\,\overline{\alpha}(g(x))$ is an arbitrary element, then $u=f(x)r(x)\overline{\alpha}(g(x))$, for all $r(x)\in R[[x;\alpha]]$. Under the assumption we have $f(x).r(x).g(x)\in nil(R[[x;\alpha]])=nil(R)[[x;\alpha]]$. Since R is skew nil-Armendariz of power series type, $a_i\alpha^i(r_t)\alpha^t(b_j)\in nil(R)$. Since R is nil-(α,δ)-compatible, $a_i\alpha^i(r_t)\alpha^{i+t+1}(b_j)\in nil(R)$ and $\sum_{i+t+j=k}a_i\alpha^i(r_t)\alpha^{i+t+1}(b_j)\in nil(R)$ for $k\geq 0$. Hence

$$\sum_{k=0}^{\infty}(\sum_{i+t+j=k}a_i\alpha^i(r_t)\alpha^{i+t+1}(b_j))x^k\in nil(R)[[x;\alpha]].$$

Then we get $f(x).r(x).\overline{\alpha}(g(x))\in nil(R)[[x;\alpha]]=nil(R[[x;\alpha]])$. And this means that $f(x)R[[x;\alpha]]\overline{\alpha}(g(x))\subseteq nil(R[[x;\alpha]])$. Conversely, we must prove that $f(x).R[[x;\alpha]].g(x)\subseteq nil(R[[x;\alpha]])$. If $u=f(x).r(x).g(x)\in f(x).[[x;\alpha]].g(x)$, then under the assumption $f(x)R[[x;\alpha]]\overline{\alpha}(g(x))\subseteq nil(R[[x;\alpha]])=nil(R)[[x;\alpha]]$, and since R is skew nil-Armendariz of power series type we have $a_i\alpha^i(r_t)\alpha^{i+t+1}(b_j)\in nil(R)$. Hence $a_i\alpha^i(r_t)\alpha^{i+t}(b_j)\in nil(R)$ and that $\sum_{i+t+j=k}a_i\alpha^i(r_t)\alpha^{i+t}(b_j)\in nil(R)$ for $k\geq 0$.

Hence $\sum_{k=0}^{\infty}(\sum_{i+t+j=k} a_i\alpha^i(r_t)\alpha^{i+t}(b_j))\,x^k$ $\in nil(R)[[x;\alpha]]$, for $k\geq 0$. And this means that $f(x).R[[x;\alpha]].g(x)\subseteq nil(R[[x;\alpha]])$. For the case of nil-$\overline{\delta}$ -compatibility, we can do in a similar way. Thus $R[[x;\alpha]]$ is a nil-$(\overline{\alpha},\overline{\delta})$-compatible ring. □

Theorem 1.40. Let R be a nil-(α,δ)-compatible and nil-Armendariz ring of power series type. If $f(x).g(x)\in nil(R)[[x;\alpha]]$, then $f(x).R[[x;\alpha]].g(x)\subseteq nil(R)[[x;\alpha]]$ for all $f,g\in R[[x;\alpha]]$.

Proof. Let $f(x).g(x)\in nil(R)[[x;\alpha]]$, and for $r(x)=\sum_{t=0}^{\infty}c_t x^t\in R[[x;\alpha]],$ assume that $u=f(x)r(x)g(x)\in f(x)\mathrm{R}[[x;\alpha]]g(x)$. Then $u=\sum_{k=0}^{\infty}(\sum_{i+t+j=k} a_i\alpha^i(\mathrm{c}_t)\alpha^{i+t}(b_j))x^k$. But $f(x).g(x)=\sum_{k=0}^{\infty}(\sum_{i+j=k} a_i\alpha^i(b_j))x^k\in nil(R)[[x;\alpha]],$ and R is skew nil-Armendariz of power series type, so $a_i\alpha^i(b_j)\in nil(R)$ for all i,j. By Lemma 1.34, R is nil-semicommutative, which yields $a_iR\alpha^i(b_j)\subseteq nil(R)$. By Lemma 1.2, $a_iR\alpha^{i+t}(b_j)\subseteq nil(R)$. Thus $a_i\alpha^i(\mathrm{c}_t)\alpha^{i+t}(b_j)$ $\in nil(R)$ and hence $\sum_{i+t+j=k} a_i\alpha^i(\mathrm{c}_t)\alpha^{i+t}(b_j)$ $\in nil(R)$, for all i,j,t,k, which yields $\sum_{k=0}^{\infty}(\sum_{i+t+j=k} a_i\alpha^i(r_t)\alpha^{i+t}(b_j))x^{\mathbf{k}}$ $\in nil(R)[[x;\alpha]].$ Therefore we have $f(x).R[[x;\alpha]].g(x)\subseteq nil(R)[[x;\alpha]]$.

Corollary 1.41. Let R be a skew nil-Armendariz ring of power serieswise type, and nil-(α,δ)-compatible. Then $R[[x;\alpha]]$ is a nil-semicommutative ring.

Proof. We prove that, if $f(x).g(x)\in nil(R[[x;\alpha]]),$ then for all $f,g\in R[[x;\alpha]]$ we get $f(x).R[[x;\alpha]].g(x)\subseteq nil(R[[x;\alpha]])$. We have $nil(R[[x;\alpha]])\subseteq nil(R)[[x;\alpha]]$. Then $R[[x;\alpha]]$ is a nil-semicommutative ring by Lemma 1.34.

We remark that, the above results enable us to produce large classes of rings which satisfy the condition $nil(R[x;\alpha,\delta])=nil(R)[x;\alpha,\delta]$.

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