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# A Method for Solving Super-Efficiency Infeasibility by Adding virtual DMUs with Mean Values

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## Abstract

Using super-efficiency, with regard to ranking efficient units, is increasing in DEA. However, this model has some problems such as the infeasibility. Thus, this article studies infeasibility of the input-based super-efficiency model (because of the zero inputs and outputs), and presents a solution by adding two virtual DMUs with mean values (one for inputs and one for outputs). Adding virtual DMUs to Production Possibility Set (PPS) changed the basic super-efficiency model, so a new model is proposed for solving this problem. Finally, the newly developed model is illustrated with a real-world data set.

## Keywords

DEA, super-efficiency, infeasibility, mean values.

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## Introduction

Charnes, Cooper, and Rhodes (CCR) (1979) devised the way to change a fractional linear measure of efficiency into a Linear Programming (LP) format and that led to the creation of DEA in 1978, the result of which was the assessment of Decision-Making Units (DMUs) based on multiple inputs and outputs, even if the production function was unknown. A DMU is efficient provided that its performance is not improvable in comparison to other DMUs from the sample.

In the standard DEA method, the efficiency score for inefficient DMUs is less than one from which a ranking can be derived. All efficient DMUs, however, have an efficiency of 1, so no ranking can be given for these units. Andersen and Petersen (1993) suggested the Super-Efficiency (SE) model for ranking efficient DMUs. They suggested modifying the LP formulation in order to remove the corresponding column of the DMU under evaluation from the coefficient matrix. Removing the DMU under evaluation from the Production Possibility Set (PPS) can play major roles in different situations. The SE was used by Zhu (1996) and Charnes et al. (1992) to study the sensitivity of the efficiency classifications (Seiford & Zhu, 1998; Charnes et al., 1996; Seiford & Thrall, 1990; Banker & Thrall, 1992).

As Thrall (1996) indicated, the super-efficiency CCR model may be an infeasible model. Besides, the super-efficiency CCR model, as Zhu (1996) showed, is infeasible if and only if certain zero patterns appear in the data domain. In recent years, some ways and models are proposed to solve this problem (Seiford et al., 1999; Mehrabian et al., 1999). In this paper, a new approach with mean values is proposed that can be used for solving super-efficiency infeasibility.

This paper is organized in the following manner: Next section describes super-efficiency infeasibility problem and presents the proposed model with mean values for solving this problem. The third section presents an illustrative example. Concluding remarks and future research extensions are summarized in the final section.

## Super-Efficiency Infeasibility and Mean Values

Assume that we have *n* DMUs $\{DMU_j: j = 1,2,3,\dots,n\}$ with $m$ inputs $x_{ij}$ $(i = 1,2,3,\dots,m)$ and $s$ outputs, $y_{rj}$ $(r = 1,2,3,\dots,s)$. On the basis of the super-efficiency DEA model provided in Andersen and Petersen (1993), the SE-CCR model (input-based) can be displayed as:

$$\begin{aligned}
&Min\ \theta_o \\
&s.t. \sum_{j=1, j\neq o}^{n} \lambda_j x_{ij} \leq \theta_o x_{io} \quad , \ i = 1,2,3,\dots,m \\
&\qquad \sum_{j=1, j\neq o}^{n} \lambda_j y_{rj} \geq y_{ro} \quad , \ r = 1,2,3,\dots,s \\
&\qquad \lambda_j \geq 0 \ \ for \ \ j = 1,2,3,\dots,n \neq o
\end{aligned} \tag{1}$$

where $(x_o, y_o)$ represents $DMU_o$.

SE-CCR represents the super-efficiency CCR model which assumes Constant Returns To Scale (CRS).

Despite its advantages, Model 1 has some problems. For instance, consider three observations with two inputs and one output as $DMU_1 = (8,0,12)$, $DMU_2 = (8,7,10)$ and $DMU_3 = (4,4,8)$. The SE-CCR efficiency for $DMU_2$ and $DMU_3$ are 0.64 and 1.33, respectively. But calculating the super-efficiency for $DMU_1$ is infeasible. Figure (1a) shows the production possibility set (Scaled to $\mathrm{y} = 10$ ), before and after $DMU_1$ is removed with the dash line. As seen in Figure (1a), after removing $DMU_1$ from the production possibility set, the line passing along the point (0,0,10) and $DMU_3$ does not cross PPS in any place, and this justifies the infeasibility of the super-efficiency for $DMU_3$.

Now, consider another example with one input and two outputs as $DMU_1 = (8,5,5)$, $DMU_2 = (12,7,0)$ and $DMU_3 = (10,9,0)$. The SE-CCR efficiency for $DMU_2$ and $DMU_3$ are 0.65 and 1.44, respectively. But calculating the super-efficiency for $DMU_1$ is infeasible. The PPS of these DMUs is shown in Figure (1b). Removing the $DMU_1$ will reduce one dimension of PPS, and make SE-CCR (for $DMU_1$)to be infeasible.

The following theorem suggests that if a certain pattern of zero inputs/outputs is involved, then the SE-CCR model is infeasible.

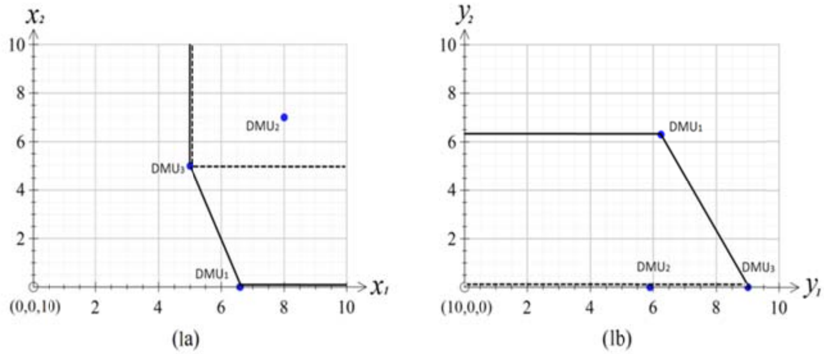


***Figure 1.*** **The PPS before and after removing DMU1 in two examples**

**Theorem 1.** Suppose $DMU_o$, and let $I_o = \{i|x_{io} = 0\}$ and $I_j = \{i|x_{ij} = 0\}$, $j \neq o$; $H_o = \{r|y_{ro} \neq 0\}$ and $H_j = \{r|y_{rj} \neq 0\}$, $j \neq o$. If the input-based super-efficiency Model (1) is infeasible, then there exists no $l$ such that $I_o \cap I_l = I_o$ and $H_o \cap H_l = H_o$.

**Proof.** See Tone (2002).

Based on previous hints and Theorem 1, we try to solve the problems by adding virtual DMUs with mean values. So the mean values for each input and output are calculated separately and shown with $M_i^{in}$ and $M_r^{out}$ that is

$$M_i^{in} = \frac{\sum_{j=1}^{n} x_{ij}}{n} \quad , \quad i = 1,2,3,\dots,m$$

$$M_r^{out} = \frac{\sum_{j=1}^{n} y_{rj}}{n} \quad , \quad r = 1,2,3,\dots,s$$

Now, with the use of the values obtained, the following virtual DMUs are produced and added to the PPS (for under evaluation $DMU_o$).

$$DMU_o^{input} = (x_o, \mathrm{M}^{\text{output}}) \quad , \quad DMU_o^{output} = (\mathrm{M}^{\text{input}}, y_o)$$

in which $\mathrm{M}^{\text{input}}$ and $\mathrm{M}^{\text{output}}$ are

$$\mathrm{M}^{\text{input}} = (M_1^{in}, M_2^{in}, M_3^{in}, \dots, M_m^{in}),$$

$$\mathrm{M}^{\text{output}} = (M_1^{out}, M_2^{out}, M_3^{out}, \dots, M_s^{out})$$

Accomplishing this, the PPS for $DMU_o$ changes as follows:

$$T_o^+ = \left\{ (x,y) \middle| \begin{array}{l} x \ge \sum_{j=1, j\neq o}^{n} \lambda_j x_j + \lambda_{n+1} x_o + \lambda_{n+2} \mathrm{M}^{\text{input}}, \\ y \le \sum_{j=1, j\neq o}^{n} \lambda_j y_j + \lambda_{n+1} \mathrm{M}^{\text{output}} + \lambda_{n+2} y_o, \\ \lambda_j \ge 0 \quad for \quad j = 1,2,3, \dots, n, n+1, n+2 \quad and \quad j \neq o \end{array} \right\}$$

On the basis of the $T_o^+$, the SE-CCR model changes as follows:

$$\begin{aligned} & Min\ \theta_o \\ & s.t. \sum_{j=1, j\neq o}^{n} \lambda_j x_j + \lambda_{n+1} x_o + \lambda_{n+2} \mathrm{M}^{\text{input}} \le \theta_o x_o \\ & \sum_{j=1, j\neq o}^{n} \lambda_j y_j + \lambda_{n+1} \mathrm{M}^{\text{output}} + \lambda_{n+2} y_o \ge Y_o \\ & \lambda_j \ge 0 \quad for \quad j = 1,2,3, \dots, n, n+1, n+2 \quad and \quad j \neq o \end{aligned} \tag{2}$$

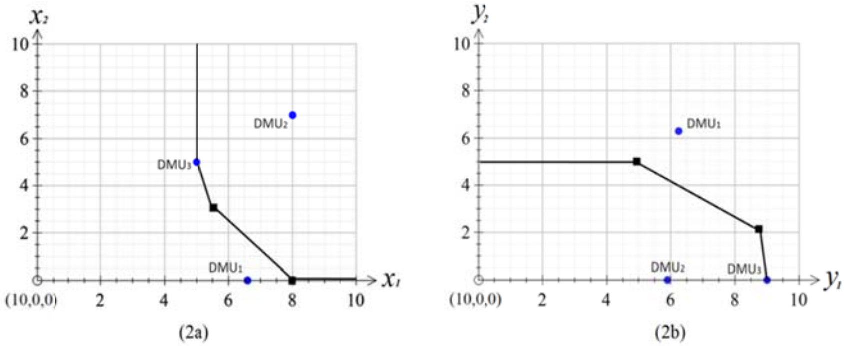


***Figure 2.*** **The PPS after adding the virtual DMUs**

After adding these two virtual DMUs, the efficiency will not increase and Theorem 2 tells this.

**Theorem 2.** Let $\theta_o^*$ and $\theta_o^{*\prime}$ be the optimal values of Models (1), and (2), respectively, then $\theta_o^* \ge \theta_o^{*\prime}$

**Proof.** Let the optimal solution to Model (1) be $(\lambda_{j\neq o}^*, \theta_o^*)$. Clearly, $(\lambda_{j\neq o}^*, \lambda_{n+1} = 0, \lambda_{n+1} = 0, \theta_o^*)$ is a feasible solution to Model (2), indicating $\theta_o^* \ge \theta_o^{*\prime}$. ■

Adding virtual DMUs with mean values leads to more sensitivity in evaluating efficiency. In fact, the DMU under evaluation is evaluated by frontiers, as well as the mean input, and output values. Moreover, this solves the infeasibility in super-efficiency models discussed in the following theorem.

**Theorem 3.** Model (2) is always feasible.

**Proof.** Consider $DMU_o$. Based on Theorem 1, it is sufficient to prove that a $k$ exists such that $I_o \cap I_k = I_o$ and $H_o \cap H_k = H_o$. Accordingly, $\mathrm{M}^{\mathrm{output}}$ is bigger than zero (on the basis that for all $r = 1,2,3,\ldots,s$ exists at least one $DMU_j$ with $y_{rj} > 0$), and $DMU_o^{input} = (x_o, \mathrm{M}^{\mathrm{output}}) \in T_o^+$ we have $I_o \cap I_{n+1} = I_o$ and $H_o \cap H_{n+1} = H_o$ by considering $DMU_o^{input} = (x_{n+1}, y_{n+1})$. Hence, Model (2) is always feasible ■

After adding two virtual DMUs to PPS, the values of $\theta_o^*$ for the efficient DMU (i.e. $DMU_o$) might be less than 1. Encountered by this problem, one could use Adjusted Index Number (AIN) Sueyoshi (1999) to solve it as follows:

$$AIN = 1 + \left[\frac{\theta_o^* - min_{i\epsilon E}\theta_i^*}{max_{i\epsilon E}\theta_i^* - min_{i\epsilon E}\theta_i^*}\right]$$

in which $E$ is a set of efficient DMUs.

Among the benefit of AIN is that, it belongs to a 100% and 200% range. As result, the DEA efficiency score belongs to range of 0 to 100%, and AIN is in another range.

It is not, however, necessary to use AIN. In fact, we can use virtual DMUs to evaluate all DMUs (both efficient and inefficient). This improves efficiency frontiers, and the DMU is also evaluated with the mean values, and this improves the ranking models.

Now, we study the efficiency of the infeasible examples above with new Model (2). Since the mean values $(\mathrm{M}^{\mathrm{input}}, \mathrm{M}^{\mathrm{output}})$ for the first and second examples are $(6.67,3.67,10)$, and $(10,7,1.67)$, respectively, the PPS for evaluating the efficiency of $DMU_1$ will be as that in Figure 2, after adding the virtual DMUs (virtual DMUs are represented by black squares).

As seen in the Figure 2, both PPSs have answers amounting 1.20, and 1.25 , respectively. As the value of $\theta_1^*$ is above 1, using AIN is not necessary.

This section analyzed the details of the method for solving the infeasibility of super-efficiency. The next sections will present comprehensive example to clarify the subject matter.

**Example 1.** Consider 5 DMUs; that each DMU consumes three inputs for producing three outputs, shown in Table 1. The CCR and SE-CCR efficiency are calculated and shown in the first and second columns in Table 2. As seen, calculating the super-efficiency for $DMU_1$, $DMU_4$, $DMU_5$ are not possible.

**Table 2. Data for the numerical example.**

| DMUs | Input1 | Input2 | Output1 | Output2 | Output3 |
|---|---|---|---|---|---|
| DMU1 | 32 | 54 | 6 | 0 | 27 |
| DMU2 | 37 | 45 | 0 | 14 | 22 |
| DMU3 | 24 | 65 | 0 | 0 | 17 |
| DMU4 | 39 | 0 | 0 | 12 | 0 |
| DMU5 | 0 | 71 | 0 | 9 | 30 |

Now, we apply Model (2) on the data in Table 1. The following mean values are used to create two virtual DMUs for each efficient DMU.

$$\mathrm{M}^{\mathrm{input}} = (26.4,47) \quad , \quad \mathrm{M}^{\mathrm{output}} = (1.2,7,19.2)$$

For example, consider $DMU_1$ as an efficient DMU. Two following virtual DMUs are added to the PPS:

$$DMU_1^{input} = (32,54,1.2,7,19.2) \ , \ \ DMU_1^{output} = (26.4,47,6,0,27)$$

The result of solving Model (2), with this virtual DMUs, are shown in the third column of Table 2. After solving Model (2) for all efficient DMUs, we use the AIN to change $\theta^*$ range between 100% and 200%. The AIN results are listed in the fourth column of Table 2.

Based on the AIN results, ranks of DMUs are shown in the last column of Table 2.

**Table 3. Results of the numerical example.**

| DMUs | CCR | Super-Eff. | SE with mean | AIN | Rank |
|---|---|---|---|---|---|
| DMU1 | 1 | infeasible | 0.8704 | 1 | 4 |
| DMU2 | 1 | 1.0949 | 1.0273 | 1.1860 | 3 |
| DMU3 | 0.5555 | 0.5555 | - | - | 5 |
| DMU4 | 1 | infeasible | 1.7143 | 2 | 1 |
| DMU5 | 1 | infeasible | 1.5625 | 1.8201 | 2 |

## Illustration

In this section, we use our approach to the twenty Japanese companies in 1999 used in Chen (2004)(see Table 3).

**Table 3. Japanese companies data.**

| DMU | Company | Asset | Equity | Employee | Revenue |
|---|---|---|---|---|---|
| 1 | MITSUI & CO. | 50905.3 | 5137.9 | 40,000 | 106793.2 |
| 2 | ITOCHU CORP. | 51432.5 | 2333.8 | 5775 | 106184.1 |
| 3 | MITSUBISHI CORP. | 67553.2 | 7253.2 | 36,000 | 104656.3 |
| 4 | TOYOTA CORP. | 112698.1 | 47,177 | 183,879 | 97387.6 |
| 5 | MARUBENI CORP. | 49742.9 | 2704.3 | 5844 | 91361.7 |
| 6 | SUMITOMO CORP. | 41168.4 | 4351.5 | 30,700 | 86,921 |
| 7 | NIPPON TELEGRAPH & TEL. | 133008.8 | 47467.1 | 138,150 | 74323.4 |
| 8 | NISSHO IWAI CORP. | 35581.9 | 1274.4 | 19,461 | 66,144 |
| 9 | HITACHI LTD. | 73,917 | 21914.2 | 328,351 | 60937.9 |
| 10 | MATSUSHITA ELECTRIC INDL. | 60,639 | 26988.4 | 282,153 | 58361.6 |
| 11 | SONY CORP. | 48117.4 | 13930.7 | 177,000 | 51,903 |
| 12 | NISSAN MOTOR | 52842.1 | 9583.6 | 39,467 | 50263.5 |
| 13 | HONDA MOTOR | 38455.8 | 13473.8 | 112,200 | 47597.9 |
| 14 | TOSHIBA CORP. | 46,013 | 8023.3 | 198,000 | 40492.7 |
| 15 | FUJITSU LTD. | 39052.2 | 8901.6 | 188,000 | 40050.3 |
| 16 | TOKYO ELECTRIC POWER | 110055.8 | 12157.7 | 50,558 | 38869.5 |
| 17 | NEC CORP. | 38,015 | 6517.4 | 157,773 | 36356.4 |
| 18 | TOMEN CORP. | 16,696 | 676.1 | 3654 | 30205.3 |
| 19 | JAPAN TOBACCO | 17023.6 | 10816.6 | 31,000 | 29612.2 |
| 20 | MITSUBISHI ELECTRIC CORP. | 31,997 | 4129.6 | 116,479 | 28982.2 |

The inputs are assets (million \$), equity (million \$) and number of employees and the DEA output is revenue (million \$). Adding $\sum_{j=1}^{n+2} \lambda_j = 1$ indicates that five of them are VRS-efficient (Banker et al., 1984)(see the third columns in Table 4). The VRS-Super efficiency of all DMUs are shown in fourth columns of Table 4. As seen, $DMU_1$ is infeasible under input-oriented model. So, by considering that mean values of inputs and outputs are $\mathrm{M}^{\text{input}} = (55745.75, 12740.61, 107222.2), \mathrm{M}^{\text{output}} = (62370.19)$,

two virtual DMUs are added to PPS for each efficient DMU. The result of solving Model (2) with adding $\sum_{j=1}^{n+2} \lambda_j = 1$ are shown in fifth columns of Table 4 for efficient DMUs. As seen, the infeasibility for $DMU_1$ is solved, and now we can rank all DMUs with AIN. The Ranks based on AIN results are shown in the last column of Table 4.

**Table 4. Result of the ranking with Model (2) (VRS)**

| DMU | Company | VRS | VRS of Model (1) | VRS of Model (2) | AIN | Rank |
|---|---|---|---|---|---|---|
| 1 | MITSUI & CO. | 1.000 | Infeasible | 2.680 | 1.354 | 2 |
| 2 | ITOCHU CORP. | 1.000 | 6.693 | 6.692 | 2.000 | 1 |
| 3 | MITSUBISHI CORP. | 0.742 | 0.742 | - | | 8 |
| 4 | TOYOTA CORP. | 0.411 | 0.411 | - | | 17 |
| 5 | MARUBENI CORP. | 0.917 | 0.917 | - | | 7 |
| 6 | SUMITOMO CORP. | 1.000 | 1.021 | 1.007 | 1.084 | 4 |
| 7 | NIPPON TELEGRAPH & TEL. | 0.269 | 0.269 | - | | 19 |
| 8 | NISSHO IWAI CORP. | 1.000 | 1.146 | 1.072 | 1.095 | 3 |
| 9 | HITACHI LTD. | 0.405 | 0.405 | - | | 18 |
| 10 | MATSUSHITA ELECTRIC INDL. | 0.476 | 0.476 | - | | 15 |
| 11 | SONY CORP. | 0.542 | 0.542 | - | | 10 |
| 12 | NISSAN MOTOR | 0.480 | 0.480 | - | | 14 |
| 13 | HONDA MOTOR | 0.629 | 0.629 | - | | 9 |
| 14 | TOSHIBA CORP. | 0.459 | 0.459 | - | | 16 |
| 15 | FUJITSU LTD. | 0.536 | 0.536 | - | | 11 |
| 16 | TOKYO ELECTRIC POWER | 0.186 | 0.186 | - | | 20 |
| 17 | NEC CORP. | 0.509 | 0.509 | - | | 13 |
| 18 | TOMEN CORP. | 1.000 | 2.900 | 0.484 | 1.000 | 5 |
| 19 | JAPAN TOBACCO | 0.981 | 0.981 | - | | 6 |
| 20 | MITSUBISHI ELECTRIC CORP. | 0.522 | 0.522 | - | | 12 |

## Conclusion and Future Extensions

As seen, the SE-CCR model might be infeasible because zero exists in the inputs or outputs. Therefore, the second section presented a method by adding virtual DMUs with mean values in the inputs, and outputs to improve efficiency frontier and solve the problem, it solved the infeasibility of the SE-CCR on the basis of the above theorems. At last, a numerical example is presented with the use of AIN for a complete classification.

This article opens the way to study the use of mean values in super-efficiency BCC Model (2). Another subject suggested by the research, is working on the super-efficiency form of the other DEA models (Esmaeili & Rostamy-Malkhalifeh, 2017; Thrall, 1996).

## References

Andersen, P., & Petersen, N. (1993). A procedure for ranking efficient units in data envelopment analysis. *Management Science, 39*(10) , 1261-1264.

Banker, R. D., Charnes, A., & Cooper, W. W. (1984). Some models for estimating technical and scale inefficiencies in data envelopment analysis. *Management Science*, *30*(9), 1078-1092.

Banker, R. D., & Thrall, R. (1992). Estimation of returns to scale using data envelopment analysis. *European Journal of Operational Research*, *62*(1), 74-84.

Charnes, A., Cooper, W., & Rhodes, E. (1979). Measuring the efficiency of decision-making units. *European Journal of Operational Research*, *3*(4), 339.

Charnes, A., Haag, S., Jaska, P., & Semple, J. (1992). Sensitivity of efficiency classifications in the additive model of data envelopment analysis. *International Journal of Systems Science*, *23*(5), 789-798.

Charnes, A., Rousseau, J. J., & Semple, J. H. (1996). Sensitivity and stability of efficiency classifications in data envelopment analysis. *Journal of Productivity Analysis*, *7*(1), 5-18.

Chen, Y. (2004). Ranking efficient units in DEA. *OMEGA*, *32*, 213-219.

Esmaeili, F. S., & Rostamy-Malkhalifeh, M. (2017). Data envelopment analysis with fixed inputs, undesirable outputs and negative data. *Data Envelopment Analysis and Decision Science*, *2017*(1), 1-6.

Mehrabian S., Alirezaee M. R., & Jahanshahloo G. R. (1999). A complete efficiency ranking of decision making units in data envelopment analysis. *Computer Optimization Apply*, *14*, 261-266.

Rezai, F. R., Balf, H. Z., Jahanshahloo, G., & Lotfi, F. H. (2012). Ranking efficient DMUs using the Tchebycheff norm. *Applied Mathematical Modeling*, *36*(1), 46-56.

Seiford L. M., & Thrall R. M. (1990). Recent developments in DEA:

The mathematical programming approach to frontier analysis. *Journal of Econometrics*, *46*, 7-38.

Seiford, L. M., & Zhu, J. (1998). Stability regions for maintaining efficiency in data envelopment analysis. *European Journal of Operational Research*, *108*(1), 127-139.

Seiford, L. M., & Zhu, J. (1999). Infeasibility of super-efficiency data envelopment analysis models. *INFOR: Information Systems and Operational Research*, *37*(2), 174-187.

Sueyoshi, T. (1999). DEA non-parametric ranking test and index measurement: Slack-adjusted DEA and an application to Japanese agriculture cooperatives. *Omega*, *27*(3), 315-326.

Thrall R. M. (1996). Duality, classification and slack in data envelopment analysis. *The Annals of Operational Research*, *66*, 109-138.

Tone, K. (2002). A slacks-based measure of super-efficiency in data envelopment analysis. *European Journal of Operational Research*, *143*(1), 32-41.

Zhu, J. (1996). Robustness of the efficient DMUs in data envelopment analysis. *European Journal of Operational Research*, *90*(3), 451-460.