

## Size dependent on vibration and flexural sensitivity of atomic force microscope cantilevers

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### ABSTRACT

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In this paper, the free vibration behaviors and flexural sensitivity of atomic force microscope cantilevers with small-scale effects are investigated. To study the small-scale effects on natural frequencies and flexural sensitivity, the consistent couple stress theory is applied. In this theory, the couple stress is assumed skew-symmetric. Unlike the classical beam theory, the new model contains a material-length-scale parameter and can capture the size effect. For this purpose, the Euler–Bernoulli beam theory is used to develop the AFM cantilever. The tip interacts with the sample that is modeled by a spring with constant of. The equation of motion is obtained through a variational formulation based on Hamilton’s principle. In addition, the analytical expressions for the natural frequency and sensitivity are also derived. At the end, some numerical results are selected to study the effects of material-length-scale parameter and dimensionless thickness on the natural frequency and flexural sensitivity.

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### 1. INTRODUCTION

Micro and nano technologies include a wide range of advanced techniques used to fabricate and study artificial systems with dimensions ranging from several micrometers to a few nanometers [1, 2]. At nano and micro scales, size effects often become important parameters [3]. The results of both experimental and Molecular dynamics simulation have shown that the small-scale effects cannot be neglected in analyzing the mechanical properties of nano and microstructures, so the classical continuum theories become unusable in these scales [4-7]. Molecular dynamics simulation is a convenient method for simulating the mechanical behavior of small size structures; however, it is computationally expensive for structures with large number of atoms [8-16]. Thus, researchers were stimulated to develop higher-order continuum theories such as nonlocal theory [17-24], strain gradient theory [25] and etc. with the capability of predicting size effect by considering

material length scale parameters. In 1960s, the couple stress theory introduced by Toupin [26], Mindlin and Tiersten [27], and Koiter [28], became a popular non-classical theory for analyzing micro and nano scale structures due to the higher predicted value of the stiffness of micro and nano scale structures compared to the classical theory. Free vibration of a C-CST Euler-Bernoulli nano-beams made of arbitrary bidirectional functionally graded materials have illustrated by Nejad et al. [11]. Farajpour et al. [29] have investigated the nonlinear buckling of magneto-electro-elastic (MEE) hybrid nanoshells in thermal environment using a size-dependent continuum model. The stability analysis and vibration of rotating nanobeams under the effect of the compressive loading and a non-uniform magnetic field have investigated by Beghani et al. [30]. The influence of thermo-electro-mechanical loads on the free and forced vibration of a piezoelectric nanowire consideration of the framework of Timoshenko beam theory

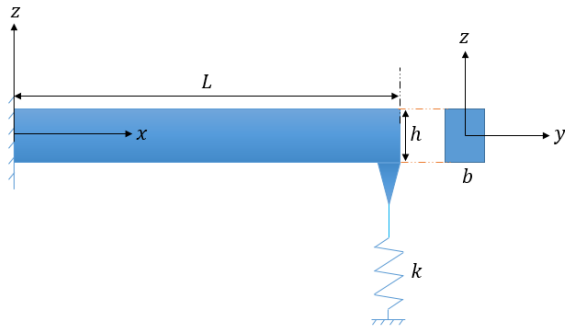
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within the nonlocal elasticity theory is illustrated with a numerical solution procedure by Kordani et al. [31]. In this paper effect of surface piezoelectricity, surface elasticity and residual surface stress is investigated. An analytical approach to study buckling of bundles of MTs considering surface effects provided by Farajpour et al. [32]. Yang, Chong, Lam, & Tong [33] proposed the modified version of the couple stress theory and considered the couple stress tensor to be symmetric. In their theory, two higher-order material-length-scale parameters are introduced in addition to the two Lamé constants. One of the superiority of this theory compared to other theories was that the four additional parameters in the micro polar theory and five additional parameters in the strain gradient theory were reduced to only two additional parameters in this theory. In recent years, this property has attracted researchers to derive formulations for the mechanical analysis of the micro-beams and micro-plates and investigate their mechanical behavior based on this theory. Investigations about the formulations and mechanical behavior have been presented in recent years based on the modified couple stress theory for homogeneous linear micro-beams [34, 35], homogenous nonlinear micro-beams [36-38], functionally graded linear micro-beams [39], functionally graded nonlinear micro-beams [40-42], linear micro-plates [43-45], nonlinear micro-plates [46, 47], composite laminated beams [48, 49]. Recently, by considering true continuum kinematical displacement and rotation, Hajesfandiari & Dargush [50] have demonstrated that the couple tensor is skew-symmetric and present the consistent couple stress theory by adopting the skew-symmetric part of the rotation gradients as the curvature tensor. In this article, free vibration behaviors and flexural sensitivity of atomic force microscope cantilevers with small-scale effects are investigated. To study the small-scale effects on natural frequencies and flexural sensitivity, the consistent couple stress theory is applied.

## 2. ANALYSIS

A schematic diagram of an atomic force microscope probe cantilevered at one end is shown in Fig. 1. The cantilever has length  $L$ , thickness  $h$  and width  $b$ . A mass  $M$  is attached at the free end of the cantilever, which interacts with the sample by a spring constant  $k$ . Cartesian coordinates  $(x,y,z)$  are considered as presented in the figure.



**Fig. 1.** Euler–Bernoulli micro-cantilever of an AFM probe with the end spring.

According to the consistent couple stress developed by Hajesfandiari [50], first variation of the strain energy for an isotropic linear elastic material with volume  $\Omega$  experiencing an infinitesimal displacement is defined as:

$$\delta U = \int_{\Omega} (\sigma_{ij} \delta e_{ij} + m_{ij} \delta \kappa_{ji}) dv \quad (1)$$

Where,  $\sigma_{ij}$ ,  $m_{ij}$ ,  $e_{ij}$  and  $\kappa_{ij}$  represent the stress, couple stress, strain and skew-symmetric curvature tensors, respectively. These tensors are defined by

$$\sigma_{ij} = \lambda e_{kk} \delta_{ij} + 2\mu e_{ij} \quad (2)$$

$$e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad (3)$$

$$m_{ij} = -8\mu l^2 \kappa_{ij} \quad (4)$$

$$\kappa_{ij} = \frac{1}{2} (\omega_{i,j} - \omega_{j,i}) \quad (5)$$

Where,  $\lambda$  and  $\mu$  are Lamé's constants, and  $u_i$  and  $\omega_j$  are the components of the displacement and the rotation vectors, respectively. The size-dependent parameter,  $l$ , is dependent on the material and scale and varies from one material to another or from one scale to another scale. This parameter should be obtained via conducting experiments for various dimensions in different working conditions. Also, it can be approximated by more accurate techniques such as molecular dynamics simulation. Where the  $\omega_j$ , rotation vector is defined as:

$$\omega_i = \frac{1}{2} \varepsilon_{ijk} u_{k,j} \quad (6)$$

In which,  $\varepsilon_{ijk}$ , denotes the permutation or Levi-Civita symbol.

Components of displacement vector ( $u_1$ ,  $u_2$  and  $u_3$ ) for micro beams can be expressed based on Euler–Bernoulli beam theories as follows:

$$u_1 = -z \frac{dw}{dx} \quad (7)$$

$$u_2 = 0 \quad (8)$$

$$u_3 = w(x) \quad (9)$$

Substitution of Eqs (6-9) into the Eq. (5) yields to the expression for the skew-symmetric curvature tensor as

$$\kappa = \frac{1}{2} \frac{\partial^2 w}{\partial x^2} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (10)$$

From Eq. (4), the couple stress tensor is defined as follows

$$m = 4\mu l^2 \frac{\partial^2 w}{\partial x^2} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (11)$$

From the displacement field, the strain components can be calculated by substituting Eqs. (7-9) into equation, Eq. (3).

$$e = -z \frac{\partial^2 w}{\partial x^2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (12)$$

For a slender beam with a large aspect ratio, the Poisson effect is secondary and can be disregarded for simplifying the beam

theory formulation. Hence, the stress component is presented as

$$\sigma = -Ez \frac{\partial^2 w}{\partial x^2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (13)$$

Where,  $E$  is the modulus of elasticity. Considering the first variation of the kinetic energy on time interval  $[0,t]$  the following relation can be obtained:

$$\int_0^t \delta K dt = \int_0^t \int_{\Omega} \rho (\dot{u}_1 \delta \dot{u}_1 + \dot{u}_2 \delta \dot{u}_2 + \dot{u}_3 \delta \dot{u}_3) dV dt$$

Where,  $\rho$  is the volume density. In order to obtain governing equations and corresponding boundary conditions for the AFM cantilever flexural vibration, the Hamilton's principle can be used as follows:

$$\delta \int_0^t (U - K) dt = 0$$

In which  $U$  and  $K$  are strain energy and kinematic energy, respectively. The governing equation of the AFM cantilever flexural vibration is:

$$(EI + 4\mu l^2 A) \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = 0 \quad (16)$$

Where,  $A$  is the cross-sectional area of the AFM cantilever. The corresponding boundary conditions are:

$$\begin{cases} w(0,t) = 0 \\ \frac{\partial w(0,t)}{\partial x} = 0 \\ \frac{\partial^2 w(L,t)}{\partial x^2} = 0 \\ (EI + 4\mu l^2 A) \frac{\partial^3 w(L,t)}{\partial x^3} = kw(L,t) + M \frac{\partial^2 w(L,t)}{\partial t^2} \end{cases} \quad (17)$$

For free vibration, we assume that the  $w(x,t)$  varies harmonically with respect to the time variable  $t$  as follows:

$$w(x,t) = f(x) e^{i\omega t} \quad (18)$$

For convenience and generality, the following non-dimensional variables are introduced:

$$\begin{aligned} \bar{x} &= \frac{x}{L} & \bar{f} &= \frac{f}{L} & \bar{k} &= \frac{kL^3}{EI} \\ \eta &= \frac{4\mu l^2 A}{EI} & \gamma^2 &= \frac{\rho L^4 \omega^2 A}{EI} & \bar{M} &= \frac{M}{\rho AL} \end{aligned}$$

Use of the above non-dimensional parameters in the ordinary differential equation and boundary conditions results in:

$$\frac{d^4 \bar{f}}{d\bar{x}^4} - \frac{\gamma^2}{1+\eta} \bar{f} = 0 \quad (19)$$

$$\begin{cases} \bar{f}(0) = 0 \\ \left. \frac{d\bar{f}}{d\bar{x}} \right|_{\bar{x}=0} = 0 \\ \left. \frac{d^2 \bar{f}}{d\bar{x}^2} \right|_{\bar{x}=1} = 0 \\ (1+\eta) \left. \frac{d^3 \bar{f}}{d\bar{x}^3} \right|_{\bar{x}=1} = (\bar{k} - \bar{M} \gamma^2) \bar{f}(1) \end{cases} \quad (20)$$

From equation (19), the general solution for  $f$  is given below:

$$\bar{f} = C_1 \sin a\bar{x} + C_2 \cos a\bar{x} + C_3 \sinh a\bar{x} + C_4 \cosh a\bar{x} \quad (21)$$

Where

$$a = \frac{\gamma^{\frac{1}{2}}}{(1+\eta)^{\frac{1}{4}}} \quad (22)$$

In which  $C_i$  are some constants. By applying the introduced boundary conditions, the characteristic equation can be obtained as:

$$C = a^3 (1+\eta) (1 + \cos a \cosh a) + (\bar{k} - \bar{M} a^4 (1+\eta)) (\cosh a \sin a - \cos a \sinh a) \quad (23)$$

And the natural frequency can be determined from the characteristic equation. The flexural sensitivity of the cantilever to the surface stiffness variations can be calculated from the following equation:

$$S = \frac{\partial \gamma}{\partial k} = \frac{\partial \gamma}{\partial a} \frac{\partial a}{\partial k} \quad (24)$$

$\frac{\partial \gamma}{\partial a}$  can be calculated from equation (22) as

$$\frac{\partial \gamma}{\partial a} = 2a(1+\eta)^{\frac{1}{2}} \quad (25)$$

And from equation (23) we have:

$$\frac{\partial a}{\partial k} = -(\cosh a \sin a - \cos a \sinh a)$$

$$\begin{aligned} & \left[ 3a^2 (1+\eta) (1 + \cos a \cosh a) \right. \\ & \times \left. + a^3 (1+\eta) (\cos a \sinh a - \cosh a \sin a) \right. \\ & \left. + 2(\bar{k} - \bar{M} \gamma^2) \sin a \sinh a \right]^{-1} \quad (26) \end{aligned}$$

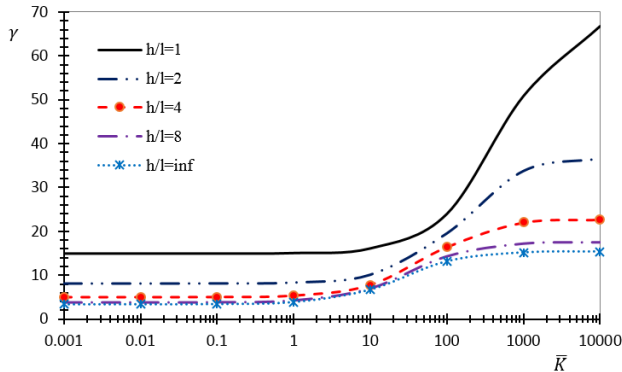
Substituting Eqs. (25) and (26) in Eq.(24), the flexural sensitivity of the AFM cantilever is expressed as

$$\begin{aligned} S &= -2a(1+\eta)^{\frac{1}{2}} (\cosh a \sin a - \cos a \sinh a) \\ & \left[ 3a^2 (1+\eta) (1 + \cos a \cosh a) \right. \\ & \left. + a^3 (1+\eta) (\cos a \sinh a - \cosh a \sin a) \right. \\ & \left. + 2(\bar{k} - \bar{M} \gamma^2) \sin a \sinh a \right]^{-1} \quad (27) \end{aligned}$$

### 3. Result and discussion

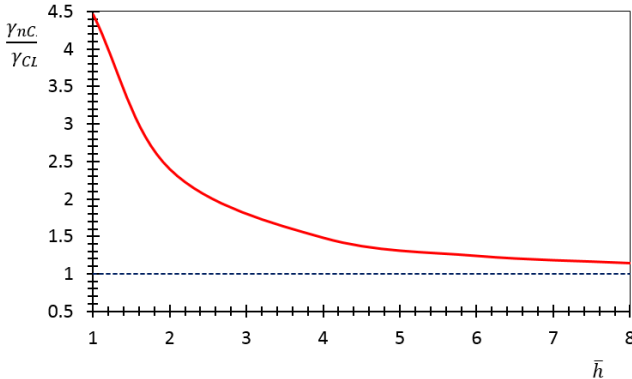
In this section, the free vibration and sensitivity analysis of AFM cantilever are investigated by numerical results based on a consistent couple stress theory.

In AFM micro cantilever, the variation of the natural frequency of the first modes versus to the parameter  $\bar{k}$  for various values of  $h/l$  is shown in Figure 2. As can be seen, the natural frequency increases by increasing the contact stiffness. It should be noted that the  $\frac{h}{l} = \infty$  case presents the classical beam theory results. It is also seen that for a constant value of  $\bar{k}$ , the natural frequency decreases as  $h/l$  increases. Therefore, the natural frequency obtained by the consistent couple stress theory is significantly greater than the frequency obtained by the classical beam theory. Moreover, in very high values of  $\bar{k}$ , changing  $\bar{k}$  has little effect on the natural frequency.

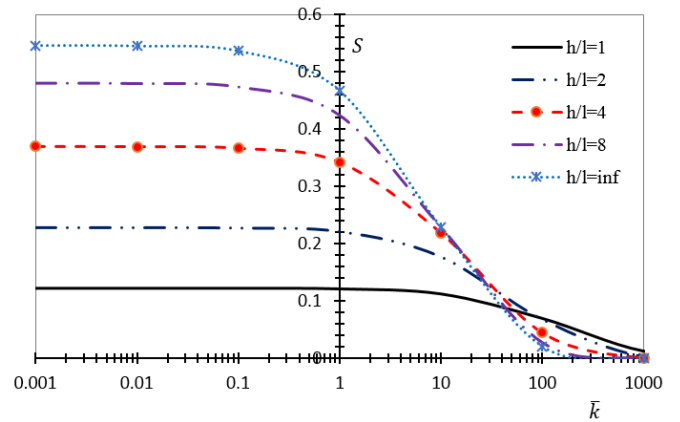


**Fig. 2.** The changes of the natural frequency of the first modes of an AFM micro-cantilever versus to the parameter  $\bar{k}$  for various values  $h/l$

Figure 3 shows the ratio of natural frequency in the case of considering consistent couple stress,  $\gamma_{CL}$ , to the classic case,  $\gamma_{CL}$ , in terms of dimensionless thickness,  $h/l$  for  $\bar{k} = 0.01$ . As it can be seen, with increasing the dimensionless thickness, the natural frequency ratio tends to 1, which indicates the reduction in the couple stress effect by increasing the thickness against size scale parameter. If the dimensionless thickness equals to 1, relative natural frequency becomes 4.46, which shows the difference between the classic and consistent couple stress theory in small sizes.



**Fig. 3.** Natural frequency ratio of the first modes of an AFM micro-cantilever versus to dimensionless thickness for  $\bar{k} = 0.01$ .



**Fig. 4.** The changes of the sensitivity of the first modes of an AFM micro-cantilever versus to the parameter  $\bar{k}$  for various values  $h/l$

The distribution of the first mode sensitivity of an AFM micro-cantilever versus to the parameter  $\bar{k}$  for various values  $\frac{h}{l}$  is shown in Figure 4. As it can be seen, with increasing  $\bar{k}$ , the sensitivity tends to zero. It may be important that for low values of contact stiffness, the decrease in the  $\frac{h}{l}$  leads to the reduction of the sensitivity, while the opposite trend is observed for high values of contact stiffness. As an important result, it is observed that when the thickness of micro-cantilevers is close to the internal material-length-scale parameter, the difference between the sensitivities obtained from the coupled stress theory and the sensitivities predicted by the classical beam theory is relatively high.

### 4. Conclusion

In this paper, using an Euler–Bernoulli beam theory, the sensitivity of the flexural vibration modes for an atomic force microscope cantilever has been analyzed by the consistent couple stress theory. According to the conducted analysis, the sensitivity of the cantilever obtained by the consistent couple stress theory was smaller compared to the one obtained by classical beam theory. The presented results in this work may provide useful guidance for design and development of AFM cantilever-based micro devices.

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