Vertex Switching in 3-Product Cordial Graphs

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A mapping $f : V(G) \to \{0, 1, 2\}$ is called 3-product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$
for any $i, j \in \{0, 1, 2\}$, where $v_f(i)$ denotes the number of vertices labeled with $i$, $e_f(i)$ denotes the number of edges
$xy$ with $f(x)f(y) \equiv i$ (mod 3). A graph with 3-product cordial labeling is called 3-product cordial graph. In this paper we establish that vertex switching of wheel, gear
graph and degree splitting of bistar are 3-product cordial graphs.

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1 Introduction

All graphs considered here are simple, finite, connected and undirected. For basic notations and terminology, we follow [3]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. There are several types of labeling

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and a complete survey of graph labeling is available in \cite{2}. Cordial labeling is a weaker version of graceful labeling and harmonious labeling introduced by Cahit in \cite{1}. Let $f$ be a function from the vertices of $G$ to \{0, 1\} and for each edge $xy$ assign the label $|f(x)f(y)|$. $f$ is called a cordial labeling of $G$ if the number of vertices labeled 0 and the number of vertices labeled 1 differ at most 1, and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1. Sundaram et al. introduced the concept of EP-cordial labeling in \cite{11}. A vertex labeling $f : V(G) \rightarrow \{0, 1\}$ is said to be an EP-cordial labeling if it induces the edge labeling $f^*$ defined by $f^*(uv) = f(u)f(v)$ for each and if $|ef(i) - ef(j)| \leq 1$ and $|ef(i) - ef(j)| \leq 1$ for any $i \neq j$, $i, j \in \{-1, 0, 1\}$ where $v_f(x)$ and $e_f(x)$ denotes the number of vertices and edges of $G$ having the label $x \in \{-1, 0, 1\}$.

In \cite{10} it is remarked that any EP-cordial labeling is a 3-product cordial labeling. A mapping $f : V(G) \rightarrow \{0, 1, 2\}$ is called 3-product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i, j \in \{0, 1, 2\}$, where $v_f(i)$ denotes the number of vertices labeled with $i$, $e_f(i)$ denotes the number of edges $xy$ with $f(x)f(y) \equiv i (\text{mod} 3)$. A graph with 3-product cordial labeling is called 3-product cordial graph.

Jeyanthi and Maheswari \cite{4}-\cite{9} proved that the graphs $\langle B_{n,n} : W \rangle$, $C_n \cup P_n$, $C_m \circ K_n$ if $m \geq 3$ and $n \geq 1$, $P_m \circ K_n$ if $m, n \geq 1$, duplicating arbitrary vertex in cycle $C_n$, duplicating arbitrary edge in cycle $C_n$, duplicating arbitrary vertex in wheel $W_n$, middle graph of $P_n$, the splitting graph of $P_n$, the total graph of $P_n$, $P_n \cup P_2$, $P_n^2$, $K_2,n$, vertex switching of $C_n$, ladder $L_n$, triangular ladder $TL_n$ and the graph $\langle W_n^{(1)} : W_n^{(2)} : \cdots : W_n^{(k)} \rangle$, the splitting graphs $S'(K_{1,n})$, $S'(B_{n,n})$, the shadow graph $D_2(B_{n,n})$ and the square graph $B_{n,n}^2$ are 3-product cordial graphs.

Also they proved that a complete graph $K_n$ is a 3-product cordial graph if and only if $n \leq 2$. In addition, they proved that if $G(p, q)$ is a 3-product cordial graph $(i)p \equiv 1(\text{mod} 3)$ then $q \leq \frac{p^2 - 2p + 7}{3}$ \((ii)p \equiv 2(\text{mod} 3)\) then $q \leq \frac{p^2 - p + 4}{3}$ \((iii)p \equiv 0(\text{mod} 3)\) then $q \leq \frac{p^2 - 3p + 6}{3}$ and if $G_1$ is a 3-product cordial graph with $3m$ vertices and $3n$ edges and $G_2$ is any 3-product cordial graph then $G_1 \cup G_2$ is also 3-product cordial graphs. In addition they established that alternate triangular snake, double alternate triangular snake, triangular snake graph, vertex switching of an apex vertex in closed helm, double fan, book graph $K_{1,n} \times K_2$ and permutation graph $P(K_2 + mK_1, l)$ admit 3-product cordial labeling. In this paper we establish that vertex switching of wheel, gear graph and degree splitting of bistar are 3-product cordial graph.

We use the following definitions in the subsequent section.

**Definition 1.1.** The vertex switching $G_v$ of a graph $G$ is the graph obtained by taking a vertex $v$ of $G$, by removing all the edges incident with $v$ and joining the vertex $v$ to every vertex which is not adjacent to $v$ in $G$.

**Definition 1.2.** A gear graph $G_n$ is obtained from the wheel $W_n$ by adding a vertex
between every pair of adjacent vertices of $C_n$.

**Definition 1.3.** Let $G = (V, E)$ be a graph with $V = S_1 \cup S_2 \cup \cdots \cup S_t \cup T$, where each $S_i$ is a set of vertices and having the same degree and $T = V - \cup S_i$. The degree splitting graph of $G$ is denoted by $DS(G)$ and is obtained from $G$ by adding the vertices $w_1, w_2, \cdots, w_t$ and joining $w_i$ to each vertex of $S_i$, $1 \leq i \leq t$.

For any real number $n$, $\lceil n \rceil$ denotes the smallest integer $\geq n$ and $\lfloor n \rfloor$ denotes the greatest integer $\leq n$.

## 2 Main Results

In this section we establish that the vertex switching of any vertex of gear graph, vertex switching of any rim vertex of wheel $W_n$, $DS(B_{n, n})$ are 3-product cordial graphs.

**Theorem 2.1.** The graph obtained by vertex switching of any vertex of gear graph is a 3-product cordial graph.

**Proof.** Let $u_0$ be the apex vertex and $u_1, u_2, \cdots, u_{2n}$ be the other vertices of gear graph $G_n$, where $deg(u_i) = 2$ when $i$ is even and $deg(u_i) = 3$ when $i$ is odd. Now the graph obtained by vertex switching of rim vertices $u_i$ and $u_j$ of degree 2 are isomorphic to each other for all $i$ and $j$. Similarly the graph obtained by vertex switching of rim vertices $u_i$ and $u_j$ of degree 3 are isomorphic to each other for all $i$ and $j$. Hence it is necessary to discuss two cases: (i) vertex switching of an arbitrary vertex say $u_1$ of $G_n$ of degree 3. (ii) vertex switching of an arbitrary vertex say $u_2$ of $G_n$ of degree 2.

Let $G = (G_n)_{u_i}$ denote the vertex switching of $G_n$ with respect to the vertex $u_i$, $i = 1, 2$. Then $|V(G)| = 2n + 1$ and

$|E(G)| = \begin{cases} 
5n - 6 & \text{if vertex switching } u_i \text{ is of degree 3} \\
5n - 4 & \text{if vertex switching } u_i \text{ is of degree 2}
\end{cases}$

Define $f : V(G) \to \{0, 1, 2\}$ by considering the following three cases.

**Case(i).** $n \equiv 0 (mod \ 3)$.

In case of vertex switching of $u_1(n > 3)$:

$f(u_0) = 2$, $f(u_i) = 1$, $f(u_i) = 0$ if $2 \leq i \leq \frac{2n - 3}{3}$,

For $1 \leq i \leq \frac{4n - 3}{3}$, $f(u_{2n} \frac{2n}{3} - 1 + i) = \begin{cases} 
1 & \text{if } i \equiv 1, 2 (mod \ 4) \\
2 & \text{if } i \equiv 0, 3 (mod \ 4)
\end{cases}$

and $f(u_i) = 0$ if $i = 2n - 1, 2n$.

From the above labeling, we have $v_f(0) = v_f(1) - 1 = v_f(2) = \left\lfloor \frac{2n + 1}{3} \right\rfloor$.

$e_f(0) = e_f(1) = e_f(2) = \frac{5n - 6}{3}$.

In case of vertex switching of $u_2$:
f(u_0) = 2, f(u_1) = 0, f(u_2) = 1, f(u_i) = 0 if 3 \leq i \leq \frac{2n + 3}{3},

For 1 \leq i \leq \frac{4n - 3}{3}, f(u_{2n + 1 + i}) = \begin{cases} 
1 & \text{if } i \equiv 1, 2(\text{mod } 4) \\
2 & \text{if } i \equiv 0, 3(\text{mod } 4) 
\end{cases}.

From the above labeling, we have
\[ v_f(0) = v_f(1) - 1 = v_f(2) = \left\lfloor \frac{2n + 1}{3} \right\rfloor, \]
\[ e_f(0) = e_f(1) + 1 = e_f(2) = \left\lceil \frac{5n - 4}{3} \right\rceil. \]

Case (ii). \( n \equiv 1(\text{mod } 3). \)

In case of vertex switching of \( u_1: \)
\[ f(u_0) = 2, f(u_1) = 1, f(u_i) = 0 \text{ if } 2 \leq i \leq \left\lfloor \frac{n}{3} \right\rfloor, \]
For 1 \leq i \leq \left\lfloor \frac{4n - 3}{3} \right\rfloor, f(u_{n + 1 + i}) = \begin{cases} 
1 & \text{if } i \equiv 1, 2(\text{mod } 4) \\
2 & \text{if } i \equiv 0, 3(\text{mod } 4) 
\end{cases}.

From the above labeling, we have
\[ v_f(0) = v_f(1) = v_f(2) = \frac{2n + 1}{3}, \]
\[ e_f(0) = e_f(1) + 1 = e_f(2) = \frac{5n - 6}{3}. \]

In case of vertex switching of \( u_2: \)
\[ f(u_0) = 2, f(u_1) = 0, f(u_2) = 1, f(u_i) = 0 \text{ if } 3 \leq i \leq \frac{2n + 4}{3}, \]
For 1 \leq i \leq \frac{4n - 4}{3}, f(u_{2n + 4 + i}) = \begin{cases} 
1 & \text{if } i \equiv 1, 2(\text{mod } 4) \\
2 & \text{if } i \equiv 0, 3(\text{mod } 4) 
\end{cases}.

From the above labeling, we have
\[ v_f(0) = v_f(1) = v_f(2) = \frac{2n + 1}{3}, \]
\[ e_f(0) - 1 = e_f(1) = e_f(2) = \left\lfloor \frac{5n - 6}{3} \right\rfloor. \]

Case (iii). \( n \equiv 2(\text{mod } 3). \)

In case of vertex switching of \( u_1: \)
\[ f(u_0) = 2, f(u_1) = 1, f(u_i) = 0 \text{ if } 2 \leq i \leq \frac{2n + 2}{3}, \]
For 1 \leq i \leq \frac{4n - 5}{3}, f(u_{2n + 2 + i}) = \begin{cases} 
1 & \text{if } i \equiv 1, 2(\text{mod } 4) \\
2 & \text{if } i \equiv 0, 3(\text{mod } 4) 
\end{cases}.

and \( f(u_n) = 0. \)

From the above labeling, we have
\[ v_f(0) = v_f(1) = v_f(2) + 1 = \left\lceil \frac{2n + 1}{3} \right\rceil, \]
\[ e_f(0) - 1 = e_f(1) = e_f(2) = \left\lfloor \frac{5n - 6}{3} \right\rfloor. \]

In case of vertex switching of \( u_2: \)
\[ f(u_0) = 2, f(u_1) = 2, f(u_2) = 1, f(u_i) = 0 \text{ if } 3 \leq i \leq \frac{2n+5}{3}, \]

For \(1 \leq i \leq \frac{4n-5}{3}\),
\[ f(u_{2n+5}+i) = \begin{cases} 1 & \text{if } i \equiv 1, 2 \text{(mod 4)} \\ 2 & \text{if } i \equiv 0, 3 \text{(mod 4)}. \end{cases} \]

From the above labeling, we have
\[ v_f(0) + 1 = v_f(1) = v_f(2) = \lceil \frac{2n+1}{3} \rceil, \]
\[ e_f(0) = e_f(1) = e_f(2) = \frac{5n-4}{3}. \]

Hence, we have \(|v_f(i) - v_f(j)| \leq 1\) and \(e_f(i) - e_f(j)| \leq 1\) for all \(i, j = 0, 1, 2\).

Thus, \(f\) is a 3-product cordial labeling. Therefore, \(G\) is a 3-product cordial graph.

An example for the 3-product cordial labeling for the graph obtained by vertex switching of gear graph \(G_8\) with respect to vertex of degree 3 is shown in Figure 1.

**Figure 1**

**Theorem 2.2.** The graph obtained by vertex switching of any rim vertex of wheel \(W_n\)
From the above labeling, we have

**Proof.** Let \( u_0 \) be the apex vertex and \( u_1, u_2, \ldots, u_n \) be the other vertices of wheel graph \( W_n \). Now the graph obtained by vertex switching of rim vertices \( u_i \) is isomorphic to the graph obtained by vertex switching of rim vertex \( u_j \), \( i = 1, 2, \ldots, n \), \( j = 1, 2, \ldots, n \). Hence it is necessary to discuss the case of an arbitrary rim vertex say \( u_1 \) of \( W_n \).

Let \( (W_n)_{u_1} \) denote the vertex switching of \( W_n \) with respect to the vertex \( u_1 \) of \( W_n \).

Let \( G = (W_n)_{u_1} \). Then \( |V(G)| = n + 1 \) and \( |E(G)| = 3n - 6 \).

Define \( f : V(G) \to \{0, 1, 2\} \) by considering the following cases.

**Case (i).** \( n \equiv 0(\text{mod } 3) \).

\[ f(u_0) = 2, \ f(u_1) = 1, \ f(u_i) = 0 \text{ if } 2 \leq i \leq \frac{n}{3} \text{,} \ f(u_n) = 0, \]

For \( 1 \leq i \leq \frac{2n - 3}{3} \), \( f(u_{\frac{n}{3} + i}) = \begin{cases} 2 & \text{if } i \equiv 1, 2(\text{mod } 4) \\ 1 & \text{if } i \equiv 0, 3(\text{mod } 4) \end{cases} \)

From the above labeling, we have

\[ v_f(0) = v_f(1) = v_f(2) - 1 = \left\lfloor \frac{n + 1}{3} \right\rfloor, \ e_f(0) = e_f(1) = e_f(2) = n - 2. \]

**Case (ii).** \( n \equiv 1(\text{mod } 3) \).

\[ f(u_0) = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}, \]

\[ f(u_1) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}, \]

\[ f(u_i) = 0 \text{ if } 2 \leq i \leq \left\lceil \frac{n}{3} \right\rceil + 1, \]

For \( 1 \leq i \leq \frac{2n - 2}{3} \), \( f(u_{\frac{n}{3} + i + 1}) = \begin{cases} 2 & \text{if } i \equiv 1, 2(\text{mod } 4) \\ 1 & \text{if } i \equiv 0, 3(\text{mod } 4) \end{cases} \)

From the above labeling, we have \( v_f(0) + 1 = v_f(1) = v_f(2) = \left\lceil \frac{n + 1}{3} \right\rceil, \ e_f(0) = e_f(1) = e_f(2) = n - 2. \)

Thus, \( |v_f(i) - v_f(j)| \leq 1 \) and \( |e_f(i) - e_f(j)| = 0 \) for all \( i, j = 0, 1, 2 \).

Hence, \( f \) is a 3-product cordial labeling. Therefore, \( G \) is a 3-product cordial graph for \( n \equiv 1(\text{mod } 3) \).

Conversely, assume that \( n \equiv 2(\text{mod } 3) \) and take \( n = 3k + 2 \). Then \( |V(G)| = 3k + 3 \) and \( |E(G)| = 9k \).

Let \( f \) be a 3-product cordial labeling of \( (W_n)_{u_1} \). Hence, we have \( v_f(0) = v_f(1) = v_f(2) = k + 1 \) and \( e_f(0) = e_f(1) = e_f(2) = 3k \). If both \( f(u_0) \) and \( f(u_1) \) are zero then all the edges receive the label zero and hence \( e_f(0) = 6k \). If one of \( f(u_i) \) is 0, then \( e_f(0) \geq 3k + 1 \). Hence, both \( f(u_i) \) and \( f(u_0) \) cannot be 0.

From the above arguments, we get a contradiction to \( f \) is a 3-product cordial labeling. Hence, \( (W_n)_{u_1} \) is not a 3-product cordial graph if \( n \equiv 2(\text{mod } 3) \).
An example for the 3-product cordial labeling for the graph obtained by vertex switching of graph $W_6$ with respect to rim vertex is shown in Figure 2.

\begin{center}
\begin{figure}
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Figure 2}
\end{figure}
\end{center}

**Theorem 2.3.** The graph $DS(B_{n,n})$ is a 3-product cordial graph if and only is $n \equiv 2 (\text{mod } 3)$.

*Proof.* Consider $B_{n,n}$ with $V(B_{n,n}) = \{u, v, u_i, v_i : 1 \leq i \leq n\}$ where $u_i, v_i$ are pendant vertices. Here $V(B_{n,n}) = V_1 \cup V_2$ where $V_1 = \{u_i, v_i : 1 \leq i \leq n\}$ and $V_2 = \{u, v\}$. Now in order to obtain $DS(B_{n,n})$ from $G$, we add $w_1, w_2$ corresponding to $V_1, V_2$. Then $|V(DS(B_{n,n}))| = 2n + 4$ and

$E(DS(B_{n,n})) = \{uv, uwx, vwz\} \cup \{uu_i, vv_i, w_uu, w_vv : 1 \leq i \leq n\}$ and so

$|E(DS(B_{n,n}))| = 4n + 3$,

$\deg(u_i) = \deg(v_i) = 2$ for $1 \leq i \leq n$, $\deg(u) = \deg(v) = n + 2$, $\deg(w_1) = 2n$ and
Conversely, assume that $f : V(DS(B_{n,n})) \to \{0, 1, 2\}$ by $f(u) = f(v) = 2$, $f(w_1) = f(w_2) = 1$,

\[ f(u_i) = f(v_i) = 0 \text{ if } 1 \leq i \leq \left \lceil \frac{n}{3} \right \rceil, \quad f(u_i) = 1, \quad f(v_i) = 2 \text{ if } \left \lceil \frac{n}{3} \right \rceil + 1 \leq i \leq n. \]

From the above labeling, we have $v_f(0) + 1 = v_f(1) = v_f(2) = \left \lceil \frac{2n + 4}{3} \right \rceil$.

\[ e_f(0) = e_f(1) + 1 = e_f(2) = \left \lceil \frac{4n + 3}{3} \right \rceil. \]

Thus, $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0, 1, 2$.

Hence, $f$ is a 3-product cordial labeling. Therefore, $DS(B_{n,n})$ is a 3-product cordial graph

for $n \equiv 2(mod\ 3)$.

Conversely, assume that $n \equiv 1(mod\ 3)$ and take $n = 3k + 1$. Then $|V(G)| = 6k + 6$ and $|E(G)| = 12k + 7$.

Let $f$ be a 3-product cordial labeling of $DS(B_{n,n})$. We have $v_f(0) = v_f(1) = v_f(2) = 2k + 2$ and $e_f(0) = 4k + 2$ or $4k + 3$.

If $f(u) = f(v) = f(w_1) = f(u_1) = 0$ or $f(u) = f(v) = f(u_i) = f(w_2) = 0$ or $f(u) = f(v) = f(w_1) = f(v_i) = 0$ or $f(u) = f(v) = f(v_i) = f(w_2) = 0$ for any $i$.

Hence we get $e_f(0) = 12k + 7$.

If $f(v_i) = 0$ or $f(u_i) = 0$. Hence $e_f(0) = 4k + 4$.

From the above arguments, we get a contradiction to $f$ is a 3-product cordial labeling.

Therefore, $DS(B_{n,n})$ is not a 3-product cordial graph if $n \equiv 1(mod\ 3)$.

Assume that $n \equiv 0(mod\ 3)$ and take $n = 3k$. Then $|V(G)| = 6k + 4$ and $|E(G)| = 12k + 3$.

Let $f$ be a 3-product cordial labeling of $DS(B_{n,n})$.

We have $v_f(0) + 1 = v_f(1) = v_f(2) = 2k + 2$ or $v_f(0) - 1 = v_f(1) = v_f(2) = 2k + 1$ and $e_f(0) = 4k + 1$.

If $f(u) = f(v) = f(w_1) = f(u_1) = 0$ or $f(u) = f(v) = f(u_i) = f(w_2) = 0$ or $f(u) = f(v) = f(w_1) = f(v_i) = 0$ or $f(u) = f(v) = f(v_i) = f(w_2) = 0$ for any $i$.

Hence we get $e_f(0) = 12k + 3$.

If $f(v_i) = 0$ or $f(u_i) = 0$. Hence $e_f(0) = 4k + 2$.

From the above arguments, we get a contradiction to $f$ is a 3-product cordial labeling.

Therefore, $DS(B_{n,n})$ is not a 3-product cordial graph if $n \equiv 0(mod\ 3)$.

An example for the 3-product cordial labeling for the graph obtained by degree splitting of bistar graph $B_{5,5}$ is shown in Figure 3.
References


