NAKHODA

Journal of Algorithms and Computation

journal homepage: http://jac.ut.ac.ir

Vertex Switching in 3-Product Cordial Graphs

P. Jeyanthi*1, A. Maheswari†2 and M.VijayaLakshmi‡3

[1]Research Centre, Department of Mathematics, Govindammal Aditanar College for Women Tiruchendur-628 215, Tamil Nadu, India.

[2]Department of Mathematics Kamaraj College of Engineering and Technology Virudhunagar-626 001, Tamil Nadu, India.

[3]Department of Mathematics, Dr.G.U. Pope College of Engineering, Sawyerpuram,Thoothukudi District, Tamil Nadu, India.

A mapping $f : V(G) \rightarrow \{0, 1, 2\}$ is called 3-product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i, j \in \{0, 1, 2\}$, where $v_f(i)$ denotes the number of vertices labeled with $i, e_f(i)$ denotes the number of edges xy with $f(x)f(y) \equiv i(mod\ 3)$. A graph with 3-product cordial labeling is called 3-product cordial graph. In this paper we establish that vertex switching of wheel,gear graph and degree splitting of bistar are 3-product cordial graphs.

Article history:
Received 23, January 2018
Received in revised form 11, May 2018
Accepted 28 May 2018
Available online 01, June 2018

Keyword: cordial labeling, product cordial labeling, 3-product cordial labeling, 3-product cordial graph, vertex switching

AMS subject Classification: 05C78.

1 Introduction

All graphs considered here are simple, finite, connected and undirected. For basic notations and terminology, we follow [3]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. There are several types of labeling

*Corresponding author: P. Jeyanthi. Email: jeyajeyanthi@rediffmail.com
†bala_nithin@yahoo.co.in
‡viji_mac@rediffmail.com

and a complete survey of graph labeling is available in [2]. Cordial labeling is a weaker version of graceful labeling and harmonious labeling introduced by Cahit in [1]. Let f be a function from the vertices of G to $\{0,1\}$ and for each edge xy assign the label $|f(x)f(y)|$. f is called a cordial labeling of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1. Sundaram et al. introduced the concept of EP-cordial labeling in [11]. A vertex labeling $f : V(G) \to \{-1,0,1\}$ is said to be an EP-cordial labeling if it induces the edge labeling f^* defined by $f^*(uv) = f(u)f(v)$ for each and if $|e_f(i) - e_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i \neq j$, $i,j \in \{-1,0,1\}$ where $v_f(x)$ and $e_f(x)$ denotes the number of vertices and edges of G having the label $x \in \{-1,0,1\}$. In [10] it is remarked that any EP-cordial labeling is a 3-product cordial labeling. A mapping $f : V(G) \to \{0,1,2\}$ is called 3-product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i,j \in \{0,1,2\}$, where $v_f(i)$ denotes the number of vertices labeled with $i, e_f(i)$ denotes the number of edges xy with $f(x)f(y) \equiv i(mod 3)$. A graph with 3-product cordial labeling is called 3-product cordial graph.
Jeyanthi and Maheswari [4]-[9] proved that the graphs $\langle B_{n,n} : W \rangle$, $C_n \cup P_n$, $C_m \circ \overline{K_n}$ if $m \geq 3$ and $n \geq 1$, $P_m \circ \overline{K_n}$ if $m,n \geq 1$, duplicating arbitrary vertex in cycle C_n, duplicating arbitrary edge in cycle C_n, duplicating arbitrary vertex in wheel W_n, middle graph of P_n, the splitting graph of P_n, the total graph of P_n, $P_n[P_2]$, P_n^2, $K_{2,n}$, vertex switching of C_n, ladder L_n, triangular ladder TL_n and the graph $\left\langle W_n^{(1)} : W_n^{(2)} : \cdots : W_n^{(k)} \right\rangle$, the splitting graphs $S^{'}(K_{1,n})$, $S^{'}(B_{n,n})$, the shadow graph $D_2(B_{n,n})$ and the square graph $B_{n,n}^2$ are 3-product cordial graphs.

Also they proved that a complete graph K_n is a 3-product cordial graph if and only if $n \leq 2$. In addition, they proved that if $G(p,q)$ is a 3-product cordial graph $(i) p \equiv 1(mod\ 3)$ then $q \leq \dfrac{p^2-2p+7}{3}$ $(ii) p \equiv 2(mod\ 3)$ then $q \leq \dfrac{p^2-p+4}{3}$ $(iii) p \equiv 0(mod\ 3)$ then $q \leq \dfrac{p^2-3p+6}{3}$ and if G_1 is a 3-product cordial graph with $3m$ vertices and $3n$ edges and G_2 is any 3-product cordial graph then $G_1 \cup G_2$ is also 3-product cordial graphs. In addition they established that alternate triangular snake, double alternate triangular snake, triangular snake graph, vertex switching of an apex vertex in closed helm, double fan, book graph $K_{1,n} \times K_2$ and permutation graph $P(K_2 + mK_1, I)$ admit 3-product cordial labeling. In this paper we establish that vertex switching of wheel, gear graph and degree splitting of bistar are 3-product cordial graph.

We use the following definitions in the subsequent section.

Definition 1.1. *The vertex switching G_v of a graph G is the graph obtained by taking a vertex v of G, by removing all the edges incident with v and joining the vertex v to every vertex which is not adjacent to v in G.*

Definition 1.2. *A gear graph G_n is obtained from the wheel W_n by adding a vertex* *between every pair of adjacent vertices of C_n.*

Definition 1.3. *Let $G = (V, E)$ be a graph with $V = S_1 \cup S_2 \cup \cdots \cup S_t \cup T$, where each S_i is a set of vertices and having the same degree and $T = V - \cup S_i$. The degree splitting graph of G is denoted by $DS(G)$ and is obtained from G by adding the vertices $w_1, w_2, \cdots, w_t$ and joining w_i to each vertex of S_i, $1 \le i \le t$.*
For any real number n, $\lceil n \rceil$ denotes the smallest integer $\ge n$ and $\lfloor n \rfloor$ denotes the greatest integer $\le n$.

2 Main Results

In this section we establish that the vertex switching of any vertex of gear graph, vertex switching of any rim vertex of wheel W_n, $DS(B_{n,n})$ are 3-product cordial graphs.

Theorem 2.1. *The graph obtained by vertex switching of any vertex of gear graph is a 3-product cordial graph.*

Proof. Let u_0 be the apex vertex and $u_1, u_2, \cdots, u_{2n}$ be the other vertices of gear graph G_n, where $deg(u_i) = 2$ when i is even and $deg(u_i) = 3$ when i is odd. Now the graph obtained by vertex switching of rim vertices u_i and u_j of degree 2 are isomorphic to each other for all i and j. Similarly the graph obtained by vertex switching of rim vertices u_i and u_j of degree 3 are isomorphic to each other for all i and j. Hence it is necessary to discuss two cases: (i) vertex switching of an arbitrary vertex say u_1 of G_n of degree 3.(ii) vertex switching of an arbitrary vertex say u_2 of G_n of degree 2.
Let $G = (G_n)_{u_i}$ denote the vertex switching of G_n with respect to the vertex u_i ,$i = 1, 2$. Then $|V(G)| = 2n + 1$ and
$|E(G)| = \begin{cases} 5n-6 & \text{if vertex switching } u_i \text{ is of degree 3} \\ 5n-4 & \text{if vertex switching } u_i \text{ is of degree 2} \end{cases}$.

Define $f : V(G) \to \{0, 1, 2\}$ by considering the following three cases.

Case(i). $n \equiv 0(mod\ 3)$.
In case of vertex switching of $u_1 (n > 3)$:
$f(u_0) = 2$, $f(u_1) = 1$, $f(u_i) = 0$ if $2 \le i \le \dfrac{2n-3}{3}$,
For $1 \le i \le \dfrac{4n-3}{3}$, $f(u_{\frac{2n}{3}-1+i}) = \begin{cases} 1 & \text{if } i \equiv 1, 2(mod\ 4) \\ 2 & \text{if } i \equiv 0, 3(mod\ 4) \end{cases}$
and $f(u_i) = 0$ if $i = 2n - 1, 2n$.
From the above labeling, we have $v_f(0) = v_f(1) - 1 = v_f(2) = \left\lfloor \dfrac{2n+1}{3} \right\rfloor$,
$e_f(0) = e_f(1) = e_f(2) = \dfrac{5n-6}{3}$.
In case of vertex switching of u_2:

$f(u_0) = 2, f(u_1) = 0,\ f(u_2) = 1,\ f(u_i) = 0$ if $3 \leq i \leq \dfrac{2n+3}{3}$,
For $1 \leq i \leq \dfrac{4n-3}{3}$, $f(u_{\frac{2n}{3}+1+i}) = \begin{cases} 1 & \text{if } i \equiv 1, 2(mod\ 4) \\ 2 & \text{if } i \equiv 0, 3(mod\ 4) \end{cases}$.

From the above labeling, we have
$v_f(0) = v_f(1) - 1 = v_f(2) = \left\lfloor \dfrac{2n+1}{3} \right\rfloor$, $e_f(0) = e_f(1) + 1 = e_f(2) = \left\lceil \dfrac{5n-4}{3} \right\rceil$.

Case(ii). $n \equiv 1(mod\ 3)$.
In case of vertex switching of u_1:
$f(u_0) = 2,\ f(u_1) = 1,\ f(u_i) = 0$ if $2 \leq i \leq \left\lceil \dfrac{n}{3} \right\rceil$,
For $1 \leq i \leq \left\lfloor \dfrac{4n-3}{3} \right\rfloor$, $f(u_{\left\lceil \frac{n}{3} \right\rceil+i}) = \begin{cases} 1 & \text{if } i \equiv 1, 2(mod\ 4) \\ 2 & \text{if } i \equiv 0, 3(mod\ 4) \end{cases}$
and $f(u_{n+1-i}) = 0$ if $1 \leq i \leq \left\lfloor \dfrac{n}{3} \right\rfloor$.
From the above labeling, we have $v_f(0) = v_f(1) = v_f(2) = \dfrac{2n+1}{3}$,
$e_f(0) = e_f(1) + 1 = e_f(2) = \left\lceil \dfrac{5n-6}{3} \right\rceil$.
In case of vertex switching of u_2:
$f(u_0) = 2, f(u_1) = 0,\ f(u_2) = 1,\ f(u_i) = 0$ if $3 \leq i \leq \dfrac{2n+4}{3}$,
For $1 \leq i \leq \dfrac{4n-4}{3}$, $f(u_{\frac{2n+4}{3}+i}) = \begin{cases} 1 & \text{if } i \equiv 1, 2(mod\ 4) \\ 2 & \text{if } i \equiv 0, 3(mod\ 4) \end{cases}$.
From the above labeling, we have
$v_f(0) = v_f(1) = v_f(2) = \dfrac{2n+1}{3}$, $e_f(0) - 1 = e_f(1) = e_f(2) = \left\lfloor \dfrac{5n-4}{3} \right\rfloor$.

Case(iii). $n \equiv 2(mod\ 3)$.
In case of vertex switching of u_1:
$f(u_0) = 2,\ f(u_1) = 1,\ f(u_i) = 0$ if $2 \leq i \leq \dfrac{2n+2}{3}$,
For $1 \leq i \leq \dfrac{4n-5}{3}$, $f(u_{\frac{2n+2}{3}+i}) = \begin{cases} 1 & \text{if } i \equiv 1, 2(mod\ 4) \\ 2 & \text{if } i \equiv 0, 3(mod\ 4) \end{cases}$
and $f(u_n) = 0$.
From the above labeling, we have $v_f(0) = v_f(1) = v_f(2) + 1 = \left\lceil \dfrac{2n+1}{3} \right\rceil$,
$e_f(0) - 1 = e_f(1) = e_f(2) = \left\lfloor \dfrac{5n-6}{3} \right\rfloor$.
In case of vertex switching of u_2:

$f(u_0) = 2, f(u_1) = 2,\ f(u_2) = 1,\ f(u_i) = 0$ if $3 \leq i \leq \dfrac{2n+5}{3}$,
For $1 \leq i \leq \dfrac{4n-5}{3}$, $f(u_{\frac{2n+5}{3}+i}) = \begin{cases} 1 & \text{if } i \equiv 1, 2 (mod\ 4) \\ 2 & \text{if } i \equiv 0, 3 (mod\ 4) \end{cases}$.

From the above labeling, we have
$v_f(0) + 1 = v_f(1) = v_f(2) = \left\lceil \dfrac{2n+1}{3} \right\rceil$, $e_f(0) = e_f(1) = e_f(2) = \dfrac{5n-4}{3}$.

Hence, we have $|v_f(i) - v_f(j)| \leq 1$ and $e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0, 1, 2$.
Thus, f is a 3-product cordial labeling. Therefore, G is a 3-product cordial graph.
An example for the 3-product cordial labeling for the graph obtained by vertex switching of gear graph G_8 with respect to vertex of degree 3 in shown in Figure 1.

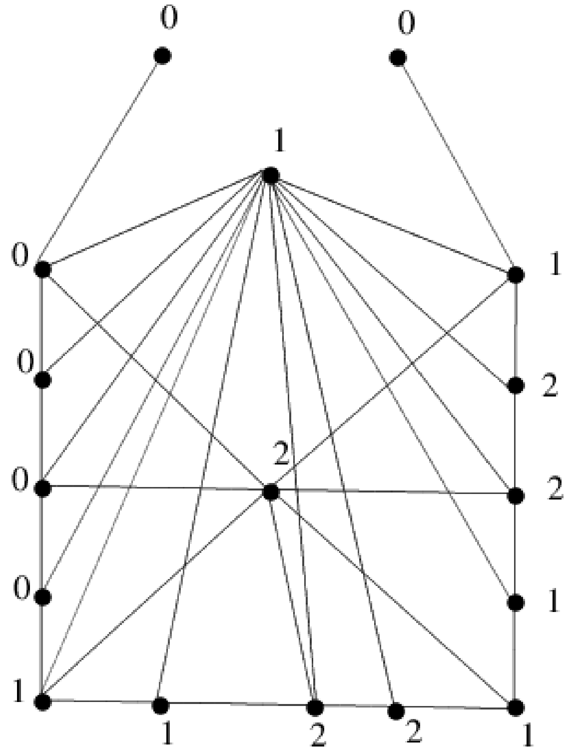

Figure 1

□

Theorem 2.2. *The graph obtained by vertex switching of any rim vertex of wheel W_n*

(except the apex vertex) is a 3-product cordial graph if and only if $n \equiv 0, 1(mod\ 3)$.

Proof. Let u_0 be the apex vertex and $u_1, u_2, \cdots, u_n$ be the other vertices of wheel graph W_n. Now the graph obtained by vertex switching of rim vertices u_i is isomorphic to the graph obtained by vertex switching of rim vertex u_j ,$i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, n$.
Hence it is necessary to discuss the case of an arbitrary rim vertex say u_1 of W_n.
Let $(W_n)_{u_i}$ denote the vertex switching of W_n with respect to the vertex u_1 of W_n.
Let $G = (W_n)_{u_1}$. Then $|V(G)| = n + 1$ and $|E(G)| = 3n - 6$.
Define $f : V(G) \to \{0, 1, 2\}$ by considering the following cases.
Case(i). $n \equiv 0(mod\ 3)$.
$f(u_0) = 2$, $f(u_1) = 1$, $f(u_i) = 0$ if $2 \le i \le \frac{n}{3}$, $f(u_n) = 0$,

For $1 \le i \le \frac{2n-3}{3}$, $f(u_{\frac{n}{3}+i}) = \begin{cases} 2 & \text{if } i \equiv 1, 2(mod\ 4) \\ 1 & \text{if } i \equiv 0, 3(mod\ 4) \end{cases}$.

From the above labeling, we have
$v_f(0) = v_f(1) = v_f(2) - 1 = \left\lfloor \frac{n+1}{3} \right\rfloor$, $e_f(0) = e_f(1) = e_f(2) = n - 2$.
Case(ii). $n \equiv 1(mod\ 3)$.

$$f(u_0) = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases},$$

$$f(u_1) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases},$$

$f(u_i) = 0$ if $2 \le i \le \left\lfloor \frac{n}{3} \right\rfloor + 1$,

For $1 \le i \le \frac{2n-2}{3}$, $f(u_{\left\lfloor \frac{n}{3} \right\rfloor + i + 1}) = \begin{cases} 2 & \text{if } i \equiv 1, 2(mod\ 4) \\ 1 & \text{if } i \equiv 0, 3(mod\ 4) \end{cases}$.

From the above labeling, we have $v_f(0) + 1 = v_f(1) = v_f(2) = \left\lceil \frac{n+1}{3} \right\rceil$,
$e_f(0) = e_f(1) = e_f(2) = n - 2$.
Thus, $|v_f(i) - v_f(j)| \le 1$ and $|e_f(i) - e_f(j)| = 0$ for all $i, j = 0, 1, 2$.
Hence, f is a 3-product cordial labeling. Therefore, G is a 3-product cordial graph for $n \equiv 0, 1(mod\ 3)$.

Conversely, assume that $n \equiv 2(mod\ 3)$ and take $n = 3k + 2$. Then $|V(G)| = 3k + 3$ and $|E(G)| = 9k$.
Let f be a 3-product cordial labeling of $(W_n)_{u_1}$. Hence, we have $v_f(0) = v_f(1) = v_f(2) = k + 1$ and $e_f(0) = e_f(1) = e_f(2) = 3k$. If both $f(u_0)$ and $f(u_1)$ are zero then all the edges receive the label zero and hence $e_f(0) = 6k$. If one of $f(u_i)$ is 0, then $e_f(0) \ge 3k + 1$. Hence, both $f(u_i)$ and $f(u_0)$ cannot be 0.
From the above arguments, we get a contradiction to f is a 3-product cordial labeling.
Hence, $(W_n)_{u_1}$ is not a 3-product cordial graph if $n \equiv 2(mod\ 3)$.

An example for the 3-product cordial labeling for the graph obtained by vertex switching of graph W_6 with respect to rim vertex is shown in Figure 2.

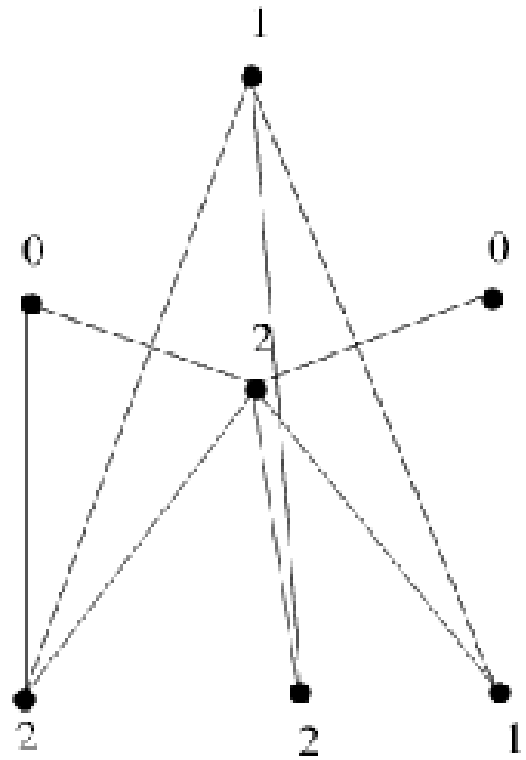

Figure 2

□

Theorem 2.3. *The graph $DS(B_{n,n})$ is a 3-product cordial graph if and only is $n \equiv 2(mod\ 3)$.*

Proof. Consider $B_{n,n}$ with $V(B_{n,n}) = \{u, v, u_i, v_i : 1 \leq i \leq n\}$ where u_i, v_i are pendant vertices. Here$V(B_{n,n}) = V_1 \cup V_2$ where $V_1 = \{u_i, v_i : 1 \leq i \leq n\}$ and $V_2 = \{u, v\}$. Now in order to obtain $DS(B_{n,n})$ from G, we add w_1, w_2 corresponding to V_1, V_2.
Then $|V(DS(B_{n,n}))| = 2n + 4$ and
$E(DS(B_{n,n})) = \{uv, uw_2, vw_2\} \cup \{uu_i, vv_i, w_1u_i, w_1v_i : 1 \leq i \leq n\}$ and so
$|E(DS(B_{n,n}))| = 4n + 3$,
$deg(u_i) = deg(v_i) = 2$ for $1 \leq i \leq n$, $deg(u) = deg(v) = n + 2$, $deg(w_1) = 2n$ and

$deg(w_2) = 2$.
Define $f : V(DS(B_{n,n})) \to \{0, 1, 2\}$ by $f(u) = f(v) = 2$, $f(w_1) = f(w_2) = 1$, $f(u_i) = f(v_i) = 0$ if $1 \leq i \leq \left\lceil \frac{n}{3} \right\rceil$, $f(u_i) = 1$, $f(v_i) = 2$ if $\left\lceil \frac{n}{3} \right\rceil + 1 \leq i \leq n$.
From the above labeling, we have $v_f(0) + 1 = v_f(1) = v_f(2) = \left\lceil \frac{2n+4}{3} \right\rceil$, $e_f(0) = e_f(1) + 1 = e_f(2) = \left\lceil \frac{4n+3}{3} \right\rceil$.
Thus, $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0, 1, 2$.
Hence, f is a 3-product coridal labeling. Therefore, $DS(B_{n,n})$ is a 3-product cordial graph for $n \equiv 2(mod\ 3)$.
Conversely, assume that $n \equiv 1(mod\ 3)$ and take $n = 3k + 1$. Then $|V(G)| = 6k + 6$ and $|E(G)| = 12k + 7$.
Let f be a 3-product cordial labeling of $DS(B_{n,n})$. We have $v_f(0) = v_f(1) = v_f(2) = 2k+2$ and $e_f(0) = 4k + 2$ or $4k + 3$.
If $f(u) = f(v) = f(w_1) = f(u_i) = 0$ or $f(u) = f(v) = f(u_i) = f(w_2) = 0$ or $f(u) = f(v) = f(w_1) = f(v_i) = 0$ or $f(u) = f(v) = f(v_i) = f(w_2) = 0$ for any i.
Hence we get $e_f(0) = 12k + 7$.
If $f(v_i) = 0$ or $f(u_i) = 0$. Hence $e_f(0) = 4k + 4$.
From the above arguments, we get a contradiction to f is a 3-product cordial labeling.
Therefore, $DS(B_{n,n})$ is not a 3-product cordial graph if $n \equiv 1(mod\ 3)$.
Assume that $n \equiv 0(mod\ 3)$ and take $n = 3k$. Then $|V(G)| = 6k+4$ and $|E(G)| = 12k+3$.
Let f be a 3-product cordial labeling of $DS(B_{n,n})$.
We have $v_f(0) + 1 = v_f(1) = v_f(2) + 1 = 2k + 2$ or $v_f(0) - 1 = v_f(1) = v_f(2) = 2k + 1$ and $e_f(0) = 4k + 1$.
If $f(u) = f(v) = f(w_1) = f(u_i) = 0$ or $f(u) = f(v) = f(u_i) = f(w_2) = 0$ or $f(u) = f(v) = f(w_1) = f(v_i) = 0$ or $f(u) = f(v) = f(v_i) = f(w_2) = 0$ for any i.
Hence we get $e_f(0) = 12k + 3$.
If $f(v_i) = 0$ or $f(u_i) = 0$. Hence $e_f(0) = 4k + 2$.
From the above arguments, we get a contradiction to f is a 3-product cordial labeling.
Therefore, $DS(B_{n,n})$ is not a 3-product cordial graph if $n \equiv 0(mod\ 3)$.
An example for the 3-product cordial labeling for the graph obtained by degree splitting of bistar graph $B_{5,5}$ is shown in Figure 3.

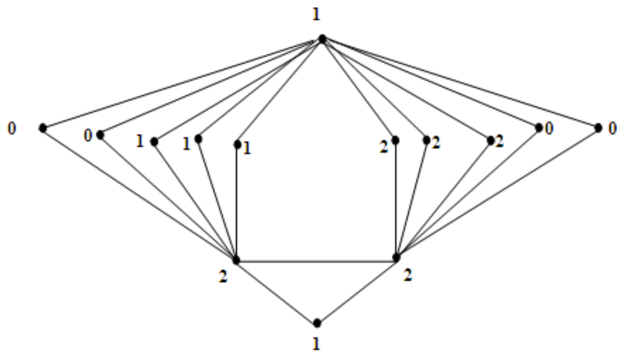

Figure 3

□

References

[1] Cahit I., Cordial Graphs: A weaker version of graceful and harmonious graphs, *Ars Combinatoria*, 23 (1987),201-207.

[2] Gallian J. A., A Dynamic Survey of Graph Labeling, *The Electronics Journal of Combinatorics*, 2017, # DS6.

[3] Harary F., Graph Theory, *Addison-Wesley, Massachusetts*, 1972.

[4] Jeyanthi P., and Maheswari A., 3-Product cordial labeling of some graphs, *International Journal on Mathematical Combinatorics*, 1 (2012), 96-105.

[5] Jeyanthi P., and Maheswari A., 3-Product cordial labeling, *SUT Journal of Mathematics*, 48(2012),231-140.

[6] Jeyanthi P., and Maheswari A., 3-Product cordial labeling of star graphs, *South east Asian Bulletin of Mathematics*, 39(3)(2015),429-437.

[7] Jeyanthi P., and Maheswari A., Some results on 3-Product cordial labeling, *Utitlas Math.*, 99 (2016),215-229.

[8] Jeyanthi P., Maheswari A., and Vijaya Lakshmi M., 3-Product cordial labeling of some snake graphs, *Proyecciones Journal of Mathematics*, to appear.

[9] Jeyanthi P., Maheswari A., and Vijaya Lakshmi M., Further results on 3-Product cordial labeling, *Proyecciones Journal of Mathematics*, to appear.

[10] Ponraj R., Sivakumar M., and Sundaram M., k-Product Cordial Labeling of Graphs, *Int. J. Contemp. Math. Sciences*, 7(15)(2012),733-742.

[11] Sundaram M., Ponraj R., and Somasundaram S., EP-cordial labelings of graphs, *Varahmihir Journal of Mathematical Sciences*, 7 (1) (2007), 183-194.