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Z_k-Magic Labeling of Some Families of Graphs

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ABSTRACT

For any non-trivial abelian group A under addition a graph G is said to be *A-magic* if there exists a labeling $f : E(G) \to A - \{0\}$ such that, the vertex labeling f^+ defined as $f^+(v) = \sum f(uv)$ taken over all edges uv incident at v is a constant. An *A-magic* graph G is said to be Z_k-magic graph if the group A is Z_k the group of integers modulo k. These Z_k-magic graphs are referred to as *k-magic* graphs. In this paper we prove that the total graph, flower graph, generalized prism graph, closed helm graph, lotus inside a circle graph, $G \odot \overline{K_m}$, m-splitting graph of a path and m-shadow graph of a path are Z_k-magic graphs.

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1 Introduction

Graph labeling is currently an emerging area in the research of graph theory. A graph labeling is an assignment of integers to vertices or edges or both subject to certain con-

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ditions. A detailed survey was done by Gallian in [5]. If the labels of edges are distinct positive integers and for each vertex v the sum of the labels of all edges incident with v is the same for every vertex v in the given graph then the labeling is called a magic labeling. Sedláček [8] introduced the concept of A-magic graphs. A graph with real-valued edge labeling such that distinct edges have distinct non-negative labels and the sum of the labels of the edges incident to a particular vertex is same for all vertices. Low and Lee [7] examined the A-magic property of the resulting graph obtained from the product of two A-magic graphs. Shiu, Lam and Sun [9] proved that the product and composition of A-magic graphs were also A-magic.
For any non-trivial Abelian group A under addition a graph G is said to be *A-magic* if there exists a labeling $f : E(G) \to A - \{0\}$ such that, the vertex labeling f^+ defined as $f^+(v) = \sum f(uv)$ taken over all edges uv incident at v is a constant. An *A-magic* graph G is said to be Z_k-magic graph if the group A is Z_k, the group of integers modulo k. These Z_k-magic graphs are referred to as *k-magic graphs*. Shiu and Low [10] determined all positive integers k for which fans and wheels have a Z_k-magic labeling with a magic constant 0. Kavitha and Thirusangu [6] obtained a Z_k-magic labeling of two cycles with a common vertex. Motivated by the concept of A-magic graph in [8] and the results in [7], [9] and [10] Jeyanthi and Jeya Daisy [1]-[4] proved that the open star of graphs, subdivision graphs, square graph, middle graph, $m\Delta_n$-snake graph, shell graph, generalised jahangir graph, $(P_n + P_1) \times P_2$ graph, double wheel graph, mongolian tent graph, flower snark, slanting ladder, double step grid graph, double arrow graph and semi jahangir graph admit Z_k-magic labeling. We use the following definitions in the subsequent section.

Definition 1.1. *Total graph $T(G)$* is a graph with the vertex set $V(G) \cup E(G)$ in which two vertices are adjacent whenever they are either adjacent or incident in G.
Definition 1.2. A *helm graph H_n*, $n \geq 3$, is obtained from a wheel W_n by adjoining a pendant edge at each vertex of the wheel except the center.
Definition 1.3. A *flower graph Fl_n*, $n \geq 3$, is obtained from a helm H_n by joining each pendent vertex to the central vertex of the helm.
Definition 1.4. A Cartesian product of a cycle C_n, $n \geq 3$, and a path on m vertices is called a *generalized prism graph $C_n \times P_m$*.
Definition 1.5. A *closed helm graph CH_n*, $n \geq 3$, is obtained from a helm H_n by joining each pendent vertex to form a cycle.
Definition 1.6. A *lotus inside a circle LC_n*, $n \geq 3$, is obtained from a wheel W_n by subdividing every edge forming the outer cycle and joining these new vertices to form a cycle.
Definition 1.7. If G has order n, *the corona of G with H, $G \odot H$* is the graph obtained by taking one copy of G and n copies of H and joining the i^{th} vertex of G with an edge to every vertex in the i^{th} copy of H.
Definition 1.8. A *m-shadow graph $D_m(G)$* is constructed by taking m-copies of G, say $G_1, G_2, G_3, \ldots, G_m$, then join each vertex u in G_i to the neighbors of the corresponding vertex v in G_j, $1 \leq i, j \leq m$.
Definition 1.9. A *m-splitting graph $Spl_m(G)$* is obtained by adding to each vertex v of G new m vertices, say $v^1, v^2, v^3, \ldots, v^m$, such that v^i , $1 \leq i \leq m$ is adjacent to every vertex that is adjacent to v in G.

2 Z_k-Magic Labeling of Some Families of Graphs

In this section we prove that the total graph of a path, flower graph, generalized prism graph, closed helm graph, lotus inside a circle graph, $G \odot \overline{K_m}$, m-splitting graph of a path and m-shadow graph of a path are Z_k-magic graphs.

Theorem 2.1. The total graph of the path $T(P_n)$ is Z_k-magic for all $n > 2$.

Proof. Let the vertex set and the edge set of $T(P_n)$ be $V(T(P_n)) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_{n-1}\}$ and $E(T(P_n)) = \{u_iu_{i+1} : 1 \le i \le n-1\} \cup \{v_iv_{i+1} : 1 \le i \le n-2\} \cup \{u_{i+1}v_i : 1 \le i \le n-1\} \cup \{u_iv_i : 1 \le i \le n-1\}$ respectively.
Let a be an integer such that $a \in \{1, 2, \ldots, \frac{k}{2} - 1\}$ if k is even and $a \in \{1, 2, \ldots, \frac{k-1}{2}\}$ if k is odd.
Define the edge labeling $f : E(T(P_n)) \to Z_k - \{0\}$ as follows:
$f(u_1v_1) = a,\ f(u_2v_1) = a,$
$$f(u_iu_{i+1}) = \begin{cases} k-a, & \text{if } i = 1, n-1, \\ k-2a, & \text{if } 2 \le i \le n-2, \end{cases}$$
$f(v_iv_{i+1}) = k - 2a$ for $1 \le i \le n-2$,
$f(u_iv_i) = 2a$ for $2 \le i \le n-2$,
$f(u_{i+1}v_i) = 2a$ for $2 \le i \le n-2$,
$f(u_nv_{n-1}) = a;\quad f(u_{n-1}v_{n-1}) = a.$
Then the induced vertex labeling $f^+ : V(T(P_n)) \to Z_k - \{0\}$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(T(P_n))$. Thus f^+ is constant and it is equal to $0(mod\ k)$. Hence the total graph of the path $T(P_n)$ is Z_k-magic. □

An example of Z_5-magic labeling of $T(P_5)$ is shown in Figure 1. An example of Z_5-magic labeling of $T(P_5)$ is shown in Figure 1.

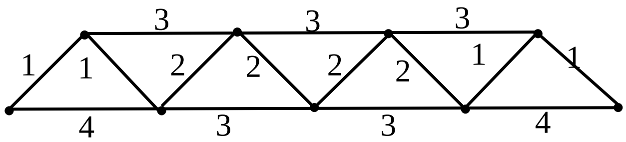


Figure 1: Z_5-magic labeling of $T(P_5)$

Theorem 2.2. The flower graph Fl_n is Z_k-magic for all $n > 2$.

Proof. Let the vertex set and the edge set of Fl_n be $V(Fl_n) = \{v, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ and $E(Fl_n) = \{vv_i : 1 \le i \le n\} \cup \{v_iv_{i+1} : 1 \le i \le n-1\} \cup \{v_nv_1\} \cup \{u_iv_i : 1 \le i \le n\} \cup \{vu_i : 1 \le i \le n\}$ respectively.
Let $a \in Z_k - \{0\}$.
Define the edge labeling $f : E(Fl_n) \to Z_k - \{0\}$ as follows:
$f(vv_i) = k - a$ for $1 \le i \le n$, $f(v_iv_{i+1}) = a$ for $1 \le i \le n$,
$f(v_nv_1) = a$; $f(u_iv_i) = k - a$ for $1 \le i \le n$,
$f(vu_i) = a$ for $1 \le i \le n$.
Then the induced vertex labeling $f^+ : V(Fl_n) \to Z_k - \{0\}$ is $f^+(v) \equiv 0(mod\ k)$ for all

$v \in V(Fl_n)$. Thus f^+ is constant and it is equal to $0(mod\ k)$. Hence the flower graph Fl_n is Z_k-magic. □

An example of Z_3-magic labeling of Fl_5 is shown in Figure 2.

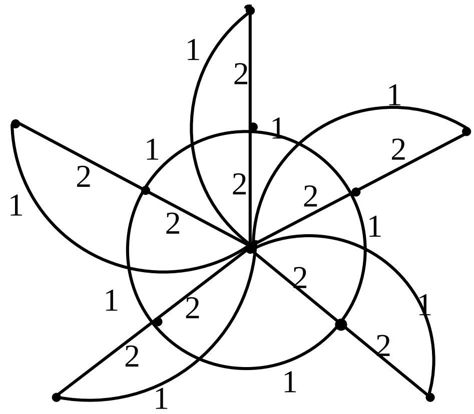


Figure 2: Z_3-magic labeling of Fl_5

Theorem 2.3. The generalized prism graph $C_m \times P_n$ is Z_k-magic for all $n \geq 2$ and $m \geq 3$.

Proof. Let the vertex set and the edge set of $C_m \times P_n$ be $V(C_m \times P_n) = \{v_i^j : 1 \leq i \leq m$ and $1 \leq j \leq n\}$ and $E(C_m \times P_n) = \{v_i^j v_{i+1}^j : 1 \leq i \leq m$ and $1 \leq j \leq n\} \cup \{v_i^j v_i^{j+1} : 1 \leq i \leq m$ and $1 \leq j \leq n-1\}$ respectively.
Let a be an integer such that $a \in \{1, 2, \ldots, \frac{k}{2} - 1\}$ if k is even and $a \in \{1, 2, \ldots, \frac{k-1}{2}\}$ if k is odd.
Define the edge labeling $f : E(C_m \times P_n) \to Z_k - \{0\}$ as follows:
$f(v_i^j v_i^{j+1}) = k - 2a$ for $1 \leq i \leq m$ and $1 \leq j \leq n-1$,

$$f(v_i^j v_{i+1}^j) = \begin{cases} a, & \text{if } 1 \leq i \leq m;\ \ j = 1, n, \\ 2a, & \text{if } 1 \leq i \leq m;\ \ 2 \leq j \leq n-1. \end{cases}$$

Then the induced vertex labeling $f^+ : V(C_m \times P_n) \to Z_k - \{0\}$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(C_m \times P_n)$. Thus f^+ is constant and it is equal to $0(mod\ k)$. Hence the generalized prism graph $C_m \times P_n$ is Z_k-magic. □

An example of Z_4-magic labeling of $C_4 \times P_3$ is shown in Figure 3.

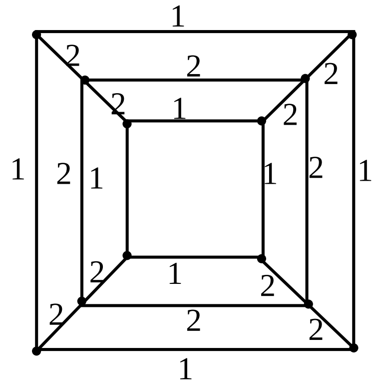


Figure 3: Z_4-magic labeling of $C_4 \times P_3$

Theorem 2.4. The closed helm graph CH_n is Z_k-magic when k is even, $k > 4$ and $n > 2$.

Proof. Let the vertex set and the edge set of CH_n be $V(CH_n) = \{v, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ and $E(CH_n) = \{vv_i : \ 1 \leq i \leq n\} \cup \{v_iv_{i+1} : \ 1 \leq i \leq n-1\} \cup \{v_nv_1\} \cup \{u_iv_i : 1 \leq i \leq n\} \cup \{u_iu_{i+1} : \ 1 \leq i \leq n-1\} \cup \{u_nu_1\}$ respectively.
We consider the following two cases.
Case(i): n is odd.
Subcase(i): $k \equiv \ 0(mod\ 4)$.
Define the edge labeling $f : E(CH_n) \to Z_k - \{0\}$ as follows:
$f(vv_i) \ = \frac{k}{2}$ for $1 \leq i \leq n$,
$f(v_iv_{i+1}) = \frac{k}{4}$ for $1 \leq i \leq n-1$,
$f(u_iv_i) \ = \frac{k}{2}$ for $1 \leq i \leq n$,
$f(u_iu_{i+1}) = \frac{k}{2}$ for $1 \leq i \leq n-1$,
$f(u_nu_1) \ = \frac{k}{2}, \ f(v_nv_1) = \frac{k}{4}$.
Then the induced vertex labeling $f^+ : V(CH_n) \to Z_k - \{0\}$ is $f^+(v) \equiv \frac{k}{2}(mod\ k)$ for all $v \in V(CH_n)$. Hence f^+ is constant and it is equal to $\frac{k}{2}(mod\ k)$.
Subcase(ii): $k \equiv \ 2(mod\ 4)$.
Define the edge labeling $f : E(CH_n) \to Z_k - \{0\}$ as follows:
$f(vv_i) \ = \frac{k}{2}$ for $1 \leq i \leq n-1$, $f(vv_n) = k-2$.
For $1 \leq i \leq n-1$.

$$f(v_iv_{i+1}) = \begin{cases} \frac{k}{2} - 1, & \text{if } i \ \ is\ odd, \\ 1, & \text{if } i \ \ is\ even, \end{cases}$$

$f(v_nv_1) \ = 1, \ f(u_iv_i) = k-2$ for $1 \leq i \leq n$,
$f(u_iu_{i+1}) = \frac{k}{2}$ for $1 \leq i \leq n-1, \ f(u_nu_1) = \frac{k}{2}$.
Then the induced vertex labeling $f^+ : V(CH_n) \to Z_k - \{0\}$ is $f^+(v) \equiv (k-2)(mod\ k)$ for all $v \in V(CH_n)$. Hence $f^+$is constant and it is equal to $(k-2)(mod\ k)$.
Case(ii): n is odd.
Subcase(i): $k \equiv \ 0(mod\ 4)$.
Define the edge labeling $f : E(CH_n) \to Z_k - \{0\}$ as follows:
$f(vv_i) \ = 1$ for $i = 1, 2, \ f(vv_i) = \frac{k}{2}$ for $3 \leq i \leq n$.
For $1 \leq i \leq n-1$.

$$f(v_iv_{i+1}) = \begin{cases} \frac{3k}{4} - 1, & \text{if } i \text{ is even,} \\ \frac{3k}{4} + 1, & \text{if } i \text{ is odd, } i \neq 1. \end{cases}$$

$f(v_1v_2) \;\; = \frac{k}{4}, \;\; f(v_nv_1) = \frac{3k}{4} - 1,$
$f(u_iv_i) \;\; = 2 \quad \text{for } 1 \leq i \leq n,$
$f(u_iu_{i+1}) \;\; = \frac{k}{2} \text{ for } 1 \leq i \leq n-1,$
$f(u_nu_1) \;\; = \frac{k}{2}.$

Then the induced vertex labeling $f^+ : V(CH_n) \rightarrow Z_k - \{0\}$ is $f^+(v) \equiv 2(mod\ k)$ for all $v \in V(CH_n)$. Hence $f^+$is constant and it is equal to $2(mod\ k)$.

Subcase(ii): $k \equiv 2(mod\ 4)$.

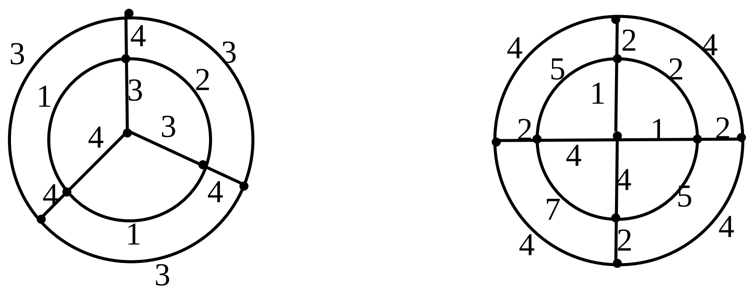
Define the edge labeling $f : E(CH_n) \rightarrow Z_k - \{0\}$ as follows:
$f(vv_i) \;\; = \frac{k}{2} + 1$ for $i = 1, 2$, $\;f(vv_i) = \frac{k}{2}$ for $3 \leq i \leq n$.
For $1 \leq i \leq n - 1$.

$$f(v_iv_{i+1}) = \begin{cases} \frac{k-2}{4}, & \text{if } i \ \ is\ even, \\ \frac{k+2}{4}, & \text{if } i \ \ is\ odd, i \neq 1. \end{cases}$$

$f(v_1v_2) \;\; = \frac{k-2}{4}, \;\; f(v_nv_1) = \frac{k-2}{4},$
$f(u_iv_i) \;\; = 2 \quad \text{for } 1 \leq i \leq n,$
$f(u_iu_{i+1}) = \frac{k}{2} \quad \text{for } 1 \leq i \leq n-1,$
$f(u_nu_1) \;\; = \frac{k}{2}.$

Then the induced vertex labeling $f^+ : V(CH_n) \rightarrow Z_k - \{0\}$ is $f^+(v) \equiv 2(mod\ k)$ for all $v \in V(CH_n)$. Thus f^+ is constant and it is equal to $2(mod\ k)$. Hence the closed helm graph CH_n is Z_k-magic. □

The examples of Z_6-magic labeling of CH_3 and Z_8-magic labeling of CH_4 are shown in Figure 4.



Z_6-magic labeling of CH_3 $\quad$ Z_8-magic labeling of CH_4

Figure 4

Theorem 2.5. The lotus inside a circle graph LC_n is Z_{4k}-magic when $n > 2$.

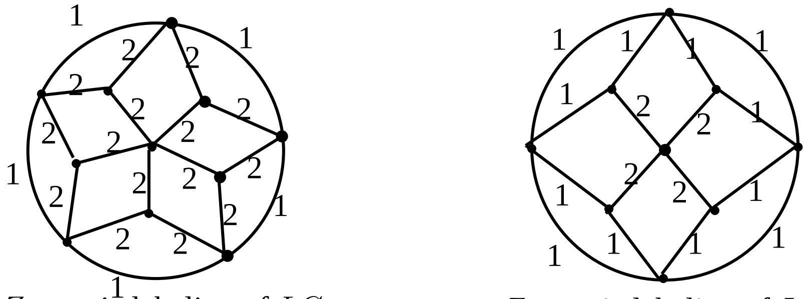
Proof. Let the vertex set and the edge set of LC_n be $V(G) = \{v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ and $E(G) = \{v_0v_i : \ 1 \leq i \leq n\} \cup \{u_iv_i : \ 1 \leq i \leq n\} \cup \{v_iu_{i+1} : \ 1 \leq i \leq n-1\} \cup \{v_nu_1\} \cup \{u_iu_{i+1} : \ 1 \leq i \leq n-1\} \cup \{u_nu_1\}$ respectively.

We consider the following two cases.

Case(i): n is odd.

Define the edge labeling $f : E(LC_n) \to Z_k - \{0\}$ as follows:
$f(v_0v_i) = \frac{k}{2}$ for $1 \le i \le n$, $f(u_iv_i) = \frac{k}{2}$ for $1 \le i \le n$,
$f(v_iu_{i+1}) = \frac{k}{2}$ for $1 \le i \le n-1$, $f(v_nu_1) = \frac{k}{2}$,
$f(u_iu_{i+1}) = \frac{k}{4}$ for $1 \le i \le n-1$, $f(u_nu_1) = \frac{k}{4}$.
Then the induced vertex labeling $f^+ : V(LC_n) \to Z_k - \{0\}$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(LC_n)$.
Case(ii): n is even.
Define the edge labeling $f : E(LC_n) \to Z_k - \{0\}$ as follows:
$f(v_0v_i) = \frac{k}{2}$ for $1 \le i \le n$, $f(u_iv_i) = \frac{k}{4}$ for $1 \le i \le n$,
$f(v_iu_{i+1}) = \frac{k}{4}$ for $1 \le i \le n-1$, $f(v_nu_1) = \frac{k}{4}$,
$f(u_iu_{i+1}) = \frac{k}{4}$ for $1 \le i \le n-1$, $f(u_nu_1) = \frac{k}{4}$.
Then the induced vertex labeling $f^+ : V(LC_n) \to Z_k - \{0\}$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(LC_n)$. Thus f^+ is constant and it is equal to $0(mod\ k)$. Hence the lotus inside a circle graph LC_n is Z_{4k}-magic. □

The examples of Z_4-magic labeling of LC_5 and LC_4 are shown in Figure 5.



Z_4-magic labeling of LC_5 Z_4-magic labeling of LC_4

Figure 5

Theorem 2.6. The two odd cycles connected by a path P_n denoted by $P_n(C_1, C_2)$ is Z_k-magic when k is even and $k \neq 2$.

Proof. Let $P_n(C_1, C_2)$ be a graph obtained by joining two odd cycles by a path P_n. Let the vertex set and the edge set of $P_n(C_1, C_2)$ be $V(P_n(C_1, C_2)) = \{v_i : 1 \le i \le n\} \cup \{v'_i : 1 \le i \le n\} \cup \{v_n = u_1, u_2, \ldots, u_n = v'_n\}$ and $E(G) = \{v_iv_{i+1} : 1 \le i \le n-1\} \cup \{v_nv_1\} \cup \{u_iu_{i+1} : 1 \le i \le n-1\} \cup \{v'_iv'_{i+1} : 1 \le i \le n-1\} \cup \{v'_nv'_1\}$ respectively.
We consider the following two cases.
Case(i): $k \equiv 0(mod\ 4)$.
Define the edge labeling $f : E(P_n(C_1, C_2)) \to Z_k - \{0\}$ as follows:

$$f(v_iv_{i+1}) = f(v'_iv'_{i+1}) = \begin{cases} \frac{k}{4}, & \text{if } i \text{ is odd,} \\ \frac{3k}{4}, & \text{if } i \text{ is even.} \end{cases}$$

$f(u_iu_{i+1}) = \frac{k}{2}$ for $1 \le i \le n$.
Then the induced vertex labeling $f^+ : V(P_n(C_1, C_2)) \to Z_k - \{0\}$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(P_n(C_1, C_2))$.

Case(ii): $k \equiv 2(mod\ 4)$.
Define the edge labeling $f : E(P_n(C_1, C_2)) \to Z_k - \{0\}$ as follows:

$$f(v_iv_{i+1}) = \begin{cases} \frac{k+2}{4}, & \text{if } i \text{ is odd,} \\ \frac{3k-2}{4}, & \text{if } i \text{ is even.} \end{cases}$$

$$f(u_iu_{i+1}) = \begin{cases} \frac{k}{2} - 1, & \text{if } i \text{ is odd,} \\ \frac{k}{2} + 1, & \text{if } i \text{ is even.} \end{cases}$$

If n is odd

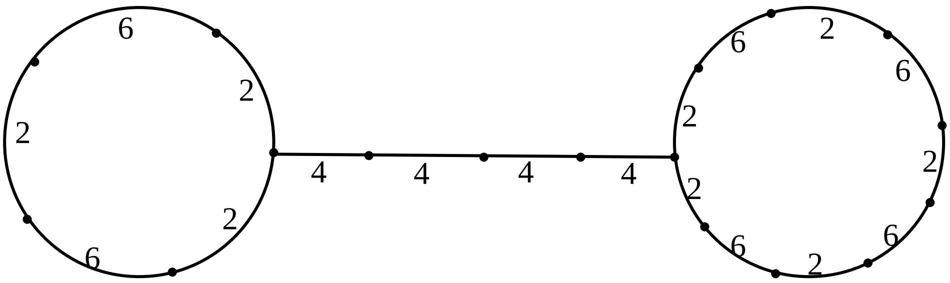
$$f(v'_iv'_{i+1}) = \begin{cases} \frac{k+2}{4}, & \text{if } i \text{ is odd,} \\ \frac{3k-2}{4}, & \text{if } i \text{ is even.} \end{cases}$$

If n is even

$$f(v'_iv'_{i+1}) = \begin{cases} \frac{k+2}{4}, & \text{if } i \text{ is even,} \\ \frac{3k-2}{4}, & \text{if } i \text{ is odd.} \end{cases}$$

Then the induced vertex labeling $f^+ : V(P_n(C_1, C_2)) \to Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(P_n(C_1, C_2))$. Thus f^+ is constant and it is equal to $0(mod\ k)$. Hence $P_n(C_1, C_2)$ is Z_k-magic. □

An example of Z_8-magic labeling of $P_5(C_5, C_9)$ is shown in Figure 6.



Fiure 6 : Z_8-magic labeling of $P_5(C_5, C_9)$

Theorem 2.7. If G is Z_m-magic with the magic constant a then $G \odot \overline{K_m}$ is Z_m-magic for all $m \geq 2$.

Proof. Let G be any graph with the vertex set $\{v_1, v_2, \ldots, v_n\}$. Let G is Z_m-magic with magic constant a where $a \in Z_m - \{0\}$. Therefore $f^+(v) \equiv a(mod\ m)$ for $1 \leq i \leq n$. Let $G \odot \overline{K_m}$ be the corona graph. Let the vertex set and the edge set of $G \odot \overline{K_m}$ is $V(G \odot \overline{K_m}) = V(G) \cup \{v_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(G \odot \overline{K_m}) = E(G) \cup \{v_iv_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$ respectively.
Let $a \in Z_k - \{0\}$.
Define the edge labeling $g : E(G \odot \overline{K_m}) \to Z_m - \{0\}$ as follows:
$g(e) \quad = f(e)$ for $e \in E(G)$,
$g(v_iv_i^j) = a \quad$ for $1 \leq i \leq n, 1 \leq j \leq m$.
Then the induced vertex labeling $g^+ : V(G \odot \overline{K_m}) \to Z_m$ is
$g^+(v_i) = f^+(v_i) + ma$ for $1 \leq i \leq n$,
$\qquad = a + ma \equiv a(mod\ m)$.

$g^+(v_i^j) = a$ for $1 \le i \le n, 1 \le j \le m$.
Thus g^+ is constant and it is equal to $a(mod\ m)$. Hence $G \odot \overline{K_m}$ is Z_m-magic. □

An example of Z_3-magic labeling of $C_4 \odot \overline{K_3}$ is shown in Figure 7.

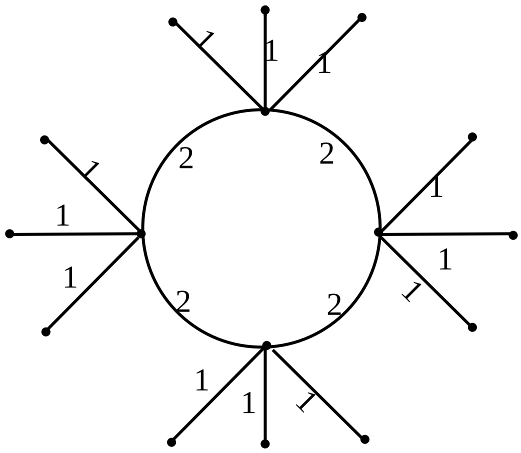


Figure 7: Z_3-magic labeling of $C_4 \odot \overline{K_3}$

Theorem 2.8. The m-splitting graph of a path $Spl_m(P_n)$ is Z_k-magic when n is even and $k > (2m-3)a$ for any integers $m \in N - \{1\}$ and $a \in Z_k - \{0\}$.

Proof. Let P_n be the path $v_1, v_2, \ldots, v_n$. Let $Spl_m(P_n)$ be the m-splitting graph of a path P_n. Let n be even such that $k > (2m-3)a$ for any integers $m \in N - \{1\}$ and $a \in Z_k - \{0\}$. Let the vertex set and the edge set of $Spl_m(P_n)$ be $V(Spl_m(P_n)) = \{v_i : 1 \le i \le n\} \cup \{v_i^j : 1 \le i \le n \text{ and } 1 \le j \le m-1\}$ and $E(Spl_m(P_n)) = \{v_i v_{i+1} : 1 \le i \le n-1\} \cup \{v_i v_{i+1}^j : 1 \le i \le n-1 \text{ and } 1 \le j \le m-1\} \cup \{v_i^j v_{i+1} : 1 \le i \le n-1 \text{ and } 1 \le j \le m-1\}$ respectively.
Let $a \in Z_k - \{0\}$ be an integer and $k > (2m-3)a$.
Define the edge labeling $f : E(Spl_m(P_n)) \to Z_k - \{0\}$ as follows:
$f(v_1 v_2) \ = 2a,$
$f(v_{n-1} v_n) = k - (2m-3)a,$
$$f(v_i v_{i+1}) = \begin{cases} a, & \text{if } i \text{ is odd,} \\ k-(m-1)a, & \text{if } i \text{ is even.} \end{cases}$$
$f(v_i v_{i+1}^j) = k - a$ for $1 \le i \le n-2,\ 1 \le j \le m-1$,
$f(v_i^j v_{i+1}) = 2a$ for $2 \le i \le n-1,\ 1 \le j \le m-1$,
$f(v_n^j v_{n-1}) = a$ for $1 \le j \le m-1$,
$f(v_1^j v_2) \ = a$ for $1 \le j \le m-1$.
Then the induced vertex labeling $f^+ : V(Spl_m(P_n)) \to Z_k$ is $f^+(v) \equiv a(mod\ k)$ for all $v \in V(Spl_m(P_n))$. Thus f^+ is constant and it is equal to $a(mod\ k)$. Hence m-splitting graph of a path $Spl_m(P_n)$ is Z_k-magic. □

An example of Z_k-magic labeling of $Spl_4(P_8)$ is shown in Figure 8.

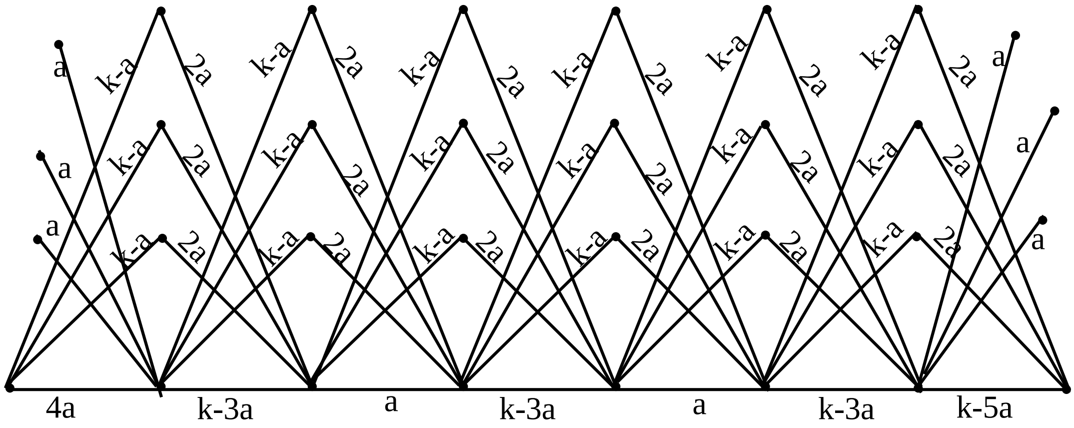


Figure 8: Z_k-magic labeling of $Spl_4(P_8)$

Theorem 2.9. The m-shadow graph of a path $D_m(P_n)$ is Z_k-magic for all $m, n > 2$.

Proof. Consider m copies of P_n. Let $u_1^j, u_2^j, \ldots, u_n^j$ be the vertices of the j^{th}-copy of P_n, $1 \leq j \leq m$. Let $D_m(P_n)$ be a m-shadow graph of a path P_n, then $|V(D_m(P_n))| = mn$ and $|E(D_m(P_n))| = m^2(n-1)$.
Let $a \in Z_k - \{0\}$ be an integer and $k > (m-1)a$.
Define the edge labeling $f : E(D_m(P_n)) \to Z_k - \{0\}$ as follows:
$f(u_i^j u_{i+1}^j) = k - (m-1)a$,
$f(e) \quad = a$ for all other edges of $E(G)$.
Then the induced vertex labeling $f^+ : V(D_m(P_n)) \to Z_k$ is $f^+(v) \equiv 0(mod\ k)$ for all $v \in V(D_m(P_n))$. Thus f^+ is constant and it is equal to $0(mod\ k)$. Hence m-shadow graph of a path $D_m(P_n)$ is Z_k-magic. □

An example of Z_k-magic labeling of $D_3(P_4)$ is shown in Figure 9.

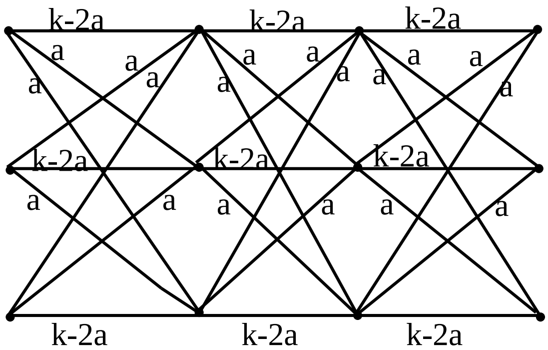


Figure 9: Z_k-magic labeling of $D_3(P_4)$

Theorem 2.10. The graph $CE(C_1, C_2)$ of two cycles C_1 and C_2 with a common edge is Z_k-magic if one of the cycle C_1 or C_2 is an even cycle.

Proof. Let $CE(C_1, C_2)$ be a graph of two cycles with a common edge. Let u and v be the end vertices of the common edge. Let u_1 and v_1 are the vertices adjacent to u and v in C_1 and u_2 and v_2 are the vertices adjacent to u and v in C_2. Let $\alpha_1, \alpha_2, \alpha_3$ be the distinct elements from $Z_k - \{0\}$ such that $\alpha_1 + \alpha_2 \not\equiv 0(mod\ k)$ and $\alpha_1 + \alpha_3 \not\equiv 0(mod\ k)$. Assign α_1 to the common edge uv.
We consider the following two cases.
Case (i): C_1 and C_2 are even cycles.
Label the edges of C_1 starting from uu_1 with α_2 and $\alpha_1 + \alpha_3$ alternately. Then the edge vv_1 receives the label α_2. Again label the edges of C_2 starting from uu_2 with α_3 and $\alpha_1 + \alpha_2$ alternately. Then the edge vv_2 receives the label α_3. Hence all the vertices of $CE(C_1, C_2)$ get the sum $\alpha_1 + \alpha_2 + \alpha_3$.
Case (ii): Either C_1 or C_2 is even cycle.
Suppose that C_1 is an odd cycle and C_2 is an even cycle. Assign the label α_1 to the common edge and α_2 to all the edges of C_1. Also label the edges of C_2 starting from uu_2 with $\alpha_2 - \alpha_1$ vv_1 and $\alpha_1 + \alpha_2$ alternately. Then vv_2 receives the label $\alpha_2 - \alpha_1$. Hence all the vertices of $CE(C_1, C_2)$ get the label $2\alpha_2$.
Then $CE(C_1, C_2)$ is Z_k-magic. □

An example of Z_k-magic labeling of $CE(C_1, C_2)$ are shown in Figure 10.

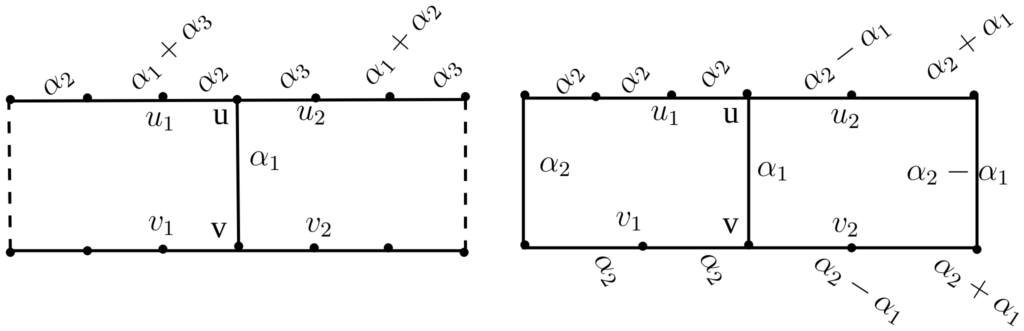


Figure 10: Z_k-magic labeling of $CE(C_1, C_2)$

References

[1] Jeyanthi, P. and Jeya Daisy, K. *Z_k-magic labeling of open star of graphs*, Bulletin of the International Mathematical Virtual Institute, 7 (2017), 243–255.

[2] Jeyanthi, P. and Jeya Daisy, K. *Z_k-magic labeling of subdivision graphs*, Discrete Math. Algorithm. Appl., 8(3) (2016), [19 pages] DOI: 10.1142/ S1793830916500464.

[3] Jeyanthi, P. and Jeya Daisy, K. *Certain classes of Z_k-magic graphs*, Journal of Graph Labeling, 4(1) (2018), 38–47.

[4] Jeyanthi, P. and Jeya Daisy, K. *Some results on Z_k-magic labeling*, Palestine Journal of Mathematics, to appear.

[5] Gallian, J.A. *A dynamic survey of graph labeling*, The Electronic Journal of Combinatorics, 2017, $\#DS6$.

[6] Kavitha, K. and Thirusangu, K. *Group magic labeling of cycles with a common vertex*, International Journal of Computing Algorithm, 2 (2013), 239–242.

[7] Low, R.M and Lee, S.M. *On the products of group-magic graphs*, Australas. J. Combin., 34 (2006), 41–48.

[8] Sedláček, J. *On magic graphs*, Math. Slov., 26 (1976), 329–335.

[9] Shiu, W.C., Lam, P.C.B. and Sun, P.K. *Construction of magic graphs and some A-magic graphs with A of even order*, Congr. Numer., 167 (2004), 97–107.

[10] Shiu, W.C. and Low, R.M. *Z_k-magic labeling of fans and wheels with magic-value zero*, Australas. J. Combin., 45 (2009), 309–316.