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Tenacity and rupture degree parameters for trapezoid graphs

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ABSTRACT

Reliability of networks is an important issue in the field of graph and network. Computation of network vulnerability parameters is NP-complete for popular network topologies such as tree, Mesh, Cube, etc. In this paper, we will show that the tenacity and rupture degree parameters for trapezoid graphs can be computed in polynomial time.

Keyword: Vulnerability parameters, Tenacity, rupture degree, Trapezoid graphs

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1 Introduction

The tenacity of an incomplete connected graph G is defined as:

$$T(G) = \min\{|S| + m(G-S)/c(G(V-S))\} \quad S \subseteq V$$

where $c(G-S)$ and $m(G-S)$, respectively, denote the number of components and the order of a largest component in $G-S$. This is a reasonable parameter to measure the vulnerability of networks, as it takes into account both the amount of work done to damage the network and how badly the network is damaged. Computing the tenacity of a graph is NP-hard in general.

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The rupture degree of a non complete connected graph G is defined as:

$$r(G) = \max\{c(G(V-S)) - |S| - m(G-S)\} \quad S \subseteq V, c(G(V-S)) \geq 2$$

where $c(G-S)$ and $m(G-S)$, respectively, denote the number of components and the order of a largest component in $G-S$. For a complete graph K_n, we define $r(K_n) = 1-n$. This parameter can be used to measure the vulnerability of a graph. To some extent, it represents a trade-off between the amount of work done to damage the network and how badly the network is damaged. The rupture degree of a graph is NP-complete.
The concept of tenacity of a graph G was introduced in [2], [3], as a useful measure of the "vulnerability" of G. In [3], Cozzens et al. calculated tenacity of the first and second case of the Harary Graphs but they didn't show the complete proof of the third case. In [13] we showed a new and complete proof for case three of the Harary Graphs. In [7], we compared integrity, connectivity, binding number, toughness, and tenacity for several classes of graphs. The results suggest that tenacity is the most suitable measure of stability or vulnerability in that for many graphs it is best able to distinguish between graphs that intuitively should have different levels of vulnerability. In [[7] - [14]] Moazzami studied more about this parameter. Conceptually graph vulnerability relates to the study of graph intactness when some of its elements are removed. The motivation for studying vulnerability measures is derived from design and analysis of networks under hostile environment. Graph tenacity has been an active area of research since the concept was introduced in 1992. Cozzens et al. in [2], introduced measure of network vulnerability termed the tenacity, $T(G)$.

The tenacity $T(G)$ of a graph G is defined as:

$$T(G) = \min\{|S| + m(G-S)/c(G(V-S))\} \quad S \subseteq V \tag{1}$$

where $c(G-S)$ and $m(G-S)$, respectively, denote the number of components and the order of a largest component in $G-S$.

In [14], Dadvand and Moazzami proved that computing the tenacity of a graph is NP-hard in general. So, it is an interesting problem to determine tenacity for some special graphs.

In [5], the rupture degree of a non complete connected graph G is defined as:

$$r(G) = \max\{c(G(V-S)) - |S| - m(G-S)\} \quad S \subseteq V, c(G(V-S)) \geq 2 \tag{2}$$

where $c(G-S)$ and $m(G-S)$, respectively, denote the number of components and the order of a largest component in $G-S$ and for a complete graph K_n, we have $r(K_n) = 1-n$. This parameter can be used to measure the vulnerability of a graph. To some extent, it represents a trade-off between the amount of work done to damage the network and how badly the network is damaged. The rupture degree of a graph is NP-complete.

In [4], some algorithms were introduced that proved toughness, scattering number and integrity of a graph are polynomial for trapezoid graphs. In this paper, it is shown that for these graphs, tenacity and rupture degree parameters can also be computed in polynomial time.

Preliminaries:

At first we present several notes about undirected graph [4]:
For a graph G, with n vertices, $G[W]$ is a subgraph of G induced by the vertex set $W \subseteq V$. The number of connected component of G is denoted by $c(G)$, and maximum order of a component of G by $n(G)$.a set $S \subseteq V$, is a separator of graph G if we have at least two components:

$$c(G(V - S)) > 1$$

We denote vertex connectivity of G by $\kappa(G)$ and an independent set of G by $\alpha(G)$.

Data Structures:

Our Algorithms fill arrays at each step:

Information about the fields of struct:

$n_{c,S}(G - S)$ is the largest component for minimum cut S and component number c.
$c_{n,S}(G - S)$ is the number of components for minimum cut S and largest component n.

Tenacity parameter defined in equation 1 can easily be derived from finding the minimum of whole T_Cell fields of our structs:

$$T(G) = \min\{\frac{|S| + n_{c,S}(G - S)}{c_{n,S}(G - S)}\} = \min\{T\_Cell\} \tag{3}$$

Rupture degree parameter defined in equation 2 can easily be derived from finding the maximum of whole R_Cell fields of our structs:

$$r(G) = \max\{c_{n,S}(G - S) - |S| - n_{c,S}(G - S)\} = \max\{R\_Cell\} \tag{4}$$

Trapezoid graphs:

Trapezoid graph is a class of intersection graph that is created by a finite collections of trapezoids between two parallel lines. Interval graph is a subclass of trapezoid graph. A trapezoid diagram consists of two parallel horizontal lines and series of trapezoids that have two corners on each horizontal line.
Graph $G = (V, E)$ is a trapezoid graph if there is a trapezoid diagram and an assignment of each vertex $v \in V$, to the trapezoid $td(v)$ such that $u, v \in V$ has an edge with each other if and only if intersection of $td(u)$ and $td(v)$ are non-empty.
There is a recognition algorithm for trapezoid graphs with $O(n^2)$ [6]. This algorithm also calculates the trapezoid diagram of it, if there exist.

In the rest of this paper, we assume that the trapezoid graph G, is given by its trapezoid diagram and the vertex v and trapezoid $td(v)$, is specified, we can assume that each point on the horizontal line, is one corner of at most one trapezoid in the diagram. So graph, is determined uniquely, by the sequence of vertices on each horizontal line.

Definition 1 A scanline in a trapezoid diagram, is each straight line segment with an end point on each horizontal line, such that these end points, dont coincide with any corners of the trapezoid $td(v)$.

Each scanline s, generates a set of $S(s)$ of the vertices of the graph, such that for each vertex v, in this set, trapezoid $td(v)$ has nonempty intersection with scanline s. For example

$$S(s1) = \{v_4, v_5, v_6\}$$

We say that two scanlines s and s', are equivalent together, if there is no corner of a trapezoid, between endpoints s and s', on the two horizontal lines. And also for two equivalent scanlines s and s', we have:

$$S(s) = S(s')$$

For example, two scanlines $s1$ and $s2$, are equivalent together.

Remk 1
A maximal set of pairwise non-equivalent scanlines, in the trapezoid diagram of a graph with n vertices, has $(2n + 1)^2$ scanlines.

Usefulness of scanlines:
Consider a scanline s, with at least one trapezoid in the left and at least one trapezoid in the right of it, we remove all trapezoids that their vertices have intersection with scanline s (removing all $S(s)$), Therefore between the vertices in the left and in the right of scanline s, in the new diagram, doesnt have any path. So scanlines have relationship with cuts.

Definition 2
scanlines $s1$, is in the left of scanline $s2$, If endpoints of $s1$, are in the left of endpoints of $s2$, on two horizontal lines.
Defintion 3
Let s_1 and s_2, be non-equivalent scanlines such that s_1 is in the left of scanline s_2, piece $\rho(s_1, s_2)$ consists of all vertices v such that trapezoid $td(v)$ is between s_1 and s_2 .the corners of $td(v)$ are completely between the end points of s_1 and s_2 on two horizontal lines. for example, in figure we have:

Definition 4
Two scanlines s_1 and s_2 are no crossing, if either s_1 is in the Left of s_2, or s_2 is in the Left of s_1.

Algorithms for trapezoid graphs:

Given a trapezoid diagram of a trapezoid graph $G(V, E)$, the algorithms computing $n_{c,S}(G - S)$ and $c_{n,S}(G - S)$ solve suitable shortest path problem on auxiliary directed acyclic graphs whose vertex set is a maximal set of pairwise nonequivalent scanlines in the diagram. Among these scanlines we denote by s_L and s_R the scanline totally to the left and totally to the right, respectively, of all trapezoids $td(v)$ of the trapezoid diagram.

Construction of the auxiliary graph $D^c(G)$: [4]

Construct the following auxiliary graph $D^c(G)$. The vertex set of $D^c(G)$ is a maximal set of pairwise nonequivalent scanlines in the diagram. There is an edge directed from s_1 to s_2 in D^c if $\rho(s_1, s_2)$ is non-empty and induces a connected subgraph of G. The weight of an edge (s_1, s_2) of D^c is $W(s_1, s_2) = |S(s_1) \backslash S(s_2)|$.

Lemma 1 [4] Let $w^c(t_1)$, for $c(G) + 1 <= t_1 <= \alpha(G)$, be the minimum weight of $\sum_{j=1}^{r} w(s_{j-1}, s_j)$ of a path $p_{t_1} = (s_0, s_1, \ldots, s_r)$ and $r >= t_1$, $s_0 = s_L$, $s_r = s_R$ among all paths in the graph $D^c(G)$ from s_L to s_R on at least t_1 edges, then $w^c(t_1) = \min\{i : c_i(G) >= t_1\}$.

Construction of the auxiliary graphs $D^n_{t_2}(G)$: [4]

Construct the following auxiliary graphs $D^n_{t_2}(G)$, $t_2 \in \{0, 1, \ldots, n\}$. The vertex set of $D^n_{t_2}(G)$ is a maximal set of pairwise nonequivalent scanlines in the diagram. There is an edge directed from s_1 to s_2 in $D^n_{t_2}(G)$ if $\rho(s_1, s_2) <= t_2$. The weight of an edge (s_1, s_2) of $D^n_{t_2}(G)$ is $W(s_1, s_2) = |S(s_1) \backslash S(s_2)|$.

Lemma 2 [4] We define $w^n_{t_2}$ to be the minimum weight of $\sum_{j=1}^{r} w(s_{j-1}, s_j)$ of a path $p = (s_0, s_1, \dots, s_r)$, $s_0 = s_L$ and $s_r = s_R$ among all paths in $D^n_{t_2}(G)$ from s_L to s_R. then $w^n_{t_2} = \min\{i : n_i(G) <= t_2\}$.

Observation 1
[4] The auxiliary directed graphs $D^c(G)$ and $D^n_{t_2}(G)$ have both $O(n^2)$ vertices and $O(n^4)$ edges for any trapezoid graph G of order n.

Construction of the auxiliary graphs $D^{nc}_{t_1,t_2}(G)$:

Construct the following auxiliary graphs $D^{nc}_{t_1,t_2}(G)$, $t_2 \in \{1, \dots, n\}$. The vertex set of $D^{nc}_{t_1,t_2}(G)$ is a maximal set of pairwise nonequivalent scanlines in the diagram. There is an edge directed from s_1 to s_2 in $D^{nc}_{t_1,t_2}(G)$ if $1 <= \rho(s_1, s_2) <= t_2$. The weight of an edge (s_1, s_2) of $D^{nc}_{t_1,t_2}(G)$ is $W(s_1, s_2) = |S(s_1) \backslash S(s_2)|$.

Considering lemma 1,1 in [4], we can conclude the following lemma:

Lemma 3 Let $w^{nc}_{t_2}(t_1)$, for $c(G) + 1 <= t_1 <= \alpha(G)$, be the minimum weight of $\sum_{j=1}^{r} w(s_{j-1}, s_j)$ of a path $p_{t_1} = (s_0, s_1, \dots, s_r)$ and $r >= t_1$, $s_0 = s_L$, $s_r = s_R$ among all paths in the graph $D^{nc}_{t_1,t_2}(G)$ from s_L to s_R on at least t_1 edges, then $w^{nc}_{t_2}(t_1) = \min\{|S| : c_{n,S}(G) >= t_1 \quad \&\& \quad n_{c,S}(G) <= t_2\}$ for $c(G)+1 <= t_1 <= \alpha(G)$ and $t_2 \in \{1, 2, \dots, n\}$.

Theorem 1
There are algorithms for given trapezoid graph G that compute tenacity parameter $T(G)$ and rupture degree parameter $r(G)$ in polynomial time $O(n^6)$.
Therefore the minimum weight of a path from s_L to s_R (or minimum cut S) can be computed in polynomial time for all cases. So for all polynomial cases, we fill the fields of our struct like: min cut S, $n_{c,S}$, $c_{n,S}$, T_cell and R_cell.

According to equations 1, 3, Tenacity parameter of a trapezoid graph $T(G)$ can be computed by finding the minimum of T_cell fields of all polynomial cases.

According to equations 2, 4, rupture degree parameter of a trapezoid graph $r(G)$ can be computed by finding the maximum of R_cell fields of all polynomial cases.

Conclusion:

Computing the tenacity of a graph is NP-hard and the rupture degree of a graph is NP-complete, in general. but in this paper, it was shown that for certain classes of intersection graphs they can be computed in polynomial time.

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References

[1] Bodlaender, H., Kloks, T., and Kratsch, D. *Treewidth and pathwidth of permutation graphs*, SIAM J. Disc. Math, (1995),606-616.

[2] Cozzens, M. B., Moazzami D., and Stueckle, S., *The Tenacity of a Graph*, Combinatorics, and Algorithms, Wiley, New York, (1995),1111-1112. 158

[3] Cozzens, M. B., Moazzami D., and Stueckle, S., *The Tenacity of the Harary Graphs*, Journal of Combinatorial Mathematics and Combinatorial Computing, 16(1994), 33-56. 158

[4] Kratsch, D., Kloks, T., Müller, H. *Measuring the vulnerability for classes of intersection graphs*, Discrete Applied Mathematics, Vol 77, 1997, 259 - 270. 159, 161, 162

[5] Li, F., Li, X., *Computing the Rupture Degrees of Graphs*, 7th International Symposium on Parallel Architectures, Algorithms and Networks, Proceedings. (2004), 368-373. 158

[6] Ma, Tze-Heng., Spinrad, J. P. *An $O(n^2)$ Algorithm for the 2-chain cover problem and related problems*, Proceeding of the 2nd Symposium on Discrete Algorithms, (1991),363-372. 160

[7] Moazzami D. Vulnerability in Graphs - a Comparative Survey, J. Combin. Math. Combin. Comput. 30 (1999), 23-31. 158

[8] Moazzami D., Stability Measure of a Graph - a Survey, Utilitas Mathematica, 57 (2000), 171-191.

[9] Moazzami, D., On Networks with Maximum Graphical Structure, Tenacity T and number of vertices p, J. Combin.Math. Combin. Comput. 39 (2001).

[10] D. Moazzami, A note on Hamiltonian properties of tenacity, Poceedings of the International conference," Paul Erdös and his Mathematics" Budapest, July 4 -11, (1999), 174-178.

[11] Moazzami, D., Salehian, S., On the edge-tenacity of graphs, Int. Math. Forum 3 (2008), no. 17-20, 929-936.

[12] Moazzami, D., Salehian, S.. Some results related to the tenacity and existence of k-trees, Discrete Applied Mathematics 8 (2009), 1794-1798.

[13] Moazzami, D.,Tenacity of a Graph with Maximum Connectivity, Disccrete Applied Mathematics, 159 (2011) 367-380. 158

[14] Moazzami, D., Dadvand, M., Moeini, A., *Recognizing Tenacious Graphs is NP-hard*, ARS combinatoria 115 (2014), pp. 163-174. 158

[15] Ray, S., Kannan, R., Zhang, D., Jiang, H. *The Weighted Integrity Problem is Polynomial for Interval Graphs*, Ars Combinatoria, vol. 79, April 2006, 77-95.

[16] Vahdat, N., Golshani, A. and Moazzami, D., *Tenacity and Rupture Degree Parameters of Interval Graphs can be computed in polynomial time.*, Submitted.

[17] Vahdat, N., Moazzami, D., *Computing some of the Vulnerability Parameters for classes of Intersection Graphs*, Submitted.