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Analysis of the nonlinear axial vibrations of a cantilevered pipe conveying pulsating two-phase flow

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ABSTRACT

The parametric resonance of the axial vibrations of a cantilever pipe conveying harmonically perturbed two-phase flow is investigated using the method of multiple scale perturbation. The nonlinear coupled and uncoupled planar dynamics of the pipe are examined for a scenario when the axial vibration is parametrically excited by the pulsating frequencies of the two phases conveyed by the pipe. Away from the internal resonance condition, the stability regions are determined analytically. The stability boundaries are found to reduce as the void fraction is increasing. With the amplitude of the harmonic velocity fluctuations of the phases taken as the control parameters, the presence of internal resonance condition results in the occurrence of both axial and transverse resonance peaks due to the transfer of energy between the planar directions. However, an increase in the void fraction is observed to reduce the amplitude of oscillations due to the increase in mass content in the pipe and which further dampens the motions of the pipe.

1. Introduction

The vibration of pipes due to the dynamic interaction between the fluid and the pipe is known to be a result of either instability or resonance. The earlier which is because of the decrease in the effective pipe stiffness with the flow speed Ibrahim [1], and when the flow velocity attains a critical value, the stiffness vanishes, and the instability occurs. However, the latter occurs when the pipe conveys a pulsatile flow resulting in parametric resonance.

Ginsberg [2] is about the earliest publication on the dynamic instability of pipes conveying pulsatile flow for a pinned-pinned pipe. Chen [3] investigated the effect of small displacements of a pipe conveying a pressurized flow with pulsating velocity. Equations of motion were derived for general end conditions and the Eigenfunction expansion method was used to obtain solutions for the case of simple supports. It was discovered that in the presence of pulsatile flow, the pipe has regions of dynamic instability whose boundaries increase with the increased magnitude of fluctuations. Paidoussis and Issid [4] investigated the dynamics and stability of flexible pipes-conveying fluid where the flow velocity is either constant or with a small harmonic component superposed. For the harmonically varying velocity, stability maps were presented for parametric instabilities using the Eigenfunction expansion method for pinned or clamped ends pipes, and also for cantilevered pipes. It was found that as the flow velocity increases for both clamped and pinned end pipes, instability regions increase, while a more complex behavior was obtained for the cantilevered pipes. For all cases, dissipation reduces or eliminates zones of parametric instability. Paidoussis and Sundararajan [5] worked on a pipe clamped at both ends and revealed that the parametric and combination resonance is exhibited by the pipe when is conveys single-phase flow at a velocity that is harmonically perturbed. However, Neyfeh and Mook [6] highlighted that nonlinearities are responsible for various unusual phenomena in the presence of internal and/or external resonance. Sequel to these early studies on the linear dynamics of the system, many studies were also published on the nonlinear dynamics of the subject, notable among these, are the works of Semler and Paidoussis [7] on the nonlinear analysis of parametric resonance of a planar fluid-conveying cantilevered pipe. Namachchivaya and Tien [8] on the nonlinear behaviour of supported pipes conveying pulsating fluid examined in the vicinity of subharmonic and combination resonance using the method of averaging. Pranda and Kar [9] studied the nonlinear dynamics of a hinged-hinged pipe conveying pulsating flow with combination, principal parametric and internal resonance, using the method of multiple scales. Mohammadi and Rastagoo [10] investigated the primary and secondary resonance phenomenon in an FG/lipid nanoplate considering porosity distribution based on the nonlinear elastic medium. Asemi, Mohammadi and Farajpour [11]

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considered nonlocality and geometric nonlinearity due to nanosized effect and mid-plane stretching in the study of the nonlinear stability of orthotropic single-layered graphene sheet. Mohammadi and Rastagoo [12] studied the primary, superharmonic and subharmonic resonances as a result of the presence of nonlinearities in the modeling of the vibrations of a viscoelastic composite nanoplate with three directionally imperfect porous FG core using the Bubno-Galerkin method and the multiple scale method. Danesh, Farajpour and Mohammadi [13] investigated the axial vibrations of a tapered nanorod considering elasticity theory and adopted the differential quadrature method in solving the governing equations for various boundary conditions. As demonstrated by various publications, the effect of nonlinearities is known to be highly crucial in the understanding of the dynamics and stability of pipes and porous media conveying fluid.

Leaving aside the much more established analysis of the dynamics of pipes conveying single-phase flows, the question remains as to how a pulsatile two-phase through a pipe will influence the dynamic behaviour of the pipe. As seen in the review of literature, most of the existing publications focused on the transverse vibrations, but the axial oscillations of the pipe can be of interest also when considering pulsatile flow due to possible amplification of oscillation amplitude as a result of resonance phenomenon. This present study investigates the nonlinear axial vibrations of a cantilever pipe conveying pulsating two-phase flow with the pulsating frequencies of the two phases parametrically exciting the axial vibrations of the pipe. An approximate analytical approach will be used to resolve the governing equations by imposing the method of multiple scales perturbation technique directly to the systems equations (direct-perturbation method).

2. Problem formulation and modeling

2.1. Assumptions

Considering a cantilever pipe of length (L), with a cross-sectional area (A), mass per unit length (m) and flexural rigidity (EI), conveying multiphase flow; flowing parallel to the pipe's centre line. The flow is assumed to have a velocity profile can be represented as a plug flow, the diameter of the pipe is small compared to its length so that the pipe behaves like a Euler-Bernoulli beam, the motion is planar, deflections of the pipe are large, but the strains are small, rotatory inertia and shear deformation are neglected and pipe centerline is assumed to be extensible.

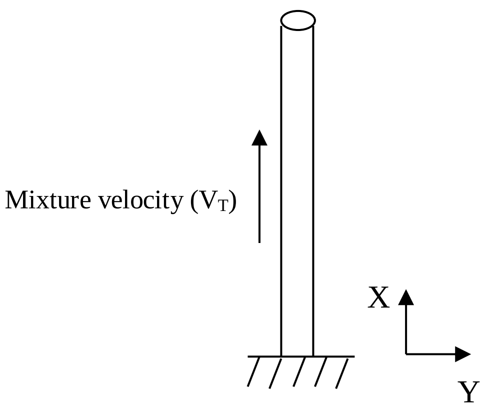


Figure 1. System's Schematics.

2.2. Equation of motion of an extensible cantilever pipe conveying Two-phase flow

The coupled nonlinear equations of motion of a cantilevered pipe conveying multiphase flow including the nonlinearity due to midline stretching is giving by Adegoke and Oyediran [14]:

$$\left(m+\sum_{j=1}^{n} M_j\right)\ddot{u} + \sum_{j=1}^{n} M_j\dot{U}_j + \sum_{j=1}^{n} 2M_jU_j\dot{u}' + \sum_{j=1}^{n} M_j U_j^2 u'' + \sum_{j=1}^{n} M_j\dot{U}_j u' - EAu'' - EI(v''''v' + v''v''') + (T_0 - P - EA(\alpha\Delta T) - EA)v'v'' - \left(T_0 - P - EA(\alpha\Delta T)\right)' + \left(m+\sum_{j=1}^{n} M_j\right)g = 0, \quad (1)$$

$$\left(m+\sum_{j=1}^{n} M_j\right)\ddot{v} + \sum_{j=1}^{n} 2M_jU_j\dot{v}' + \sum_{j=1}^{n} M_j U_j^2 v'' - \sum_{j=1}^{n} aM_j U_j^2 v'' + \sum_{j=1}^{n} M_j\dot{U}_j v' + EIv'''' - \left(T_0 - P - EA(\alpha\Delta T)\right)v'' - EI\left(3u'''v'' + 4v'''u'' + 2u'v'''' + v'u'''' + 2v'^2v'''' + 8v'v''v''' + 2v''^3\right) + \left(T_0 - P - EA(\alpha\Delta T) - EA\right)\left(u'v'' + v'u'' + \frac{3}{2}v'^2v''\right) = 0 \quad (2)$$

With the associated boundary condition

$$u(0) = u'(L) = 0 \quad (3)$$

$$v(0) = v'(0) \text{ and } v''(L) = v'''(L) = 0 \quad (4)$$

Where *x* is the longitudinal axis, *v* is the transverse deflection, *u* is the axial deflection, *n* is the number of phases, *m* is the mass of the pipe, M_j is the masses of the internal fluid phases, *EA* is axial stiffness, *EI* is Bending stiffness, T_0 is the tension term, *P* is the pressure term, α is the thermal expansivity term, ΔT relates to the temperature difference and *"a"* relates to the Poisson ratio (r) as *a=1-2r.*

2.3. Dimensionless Equation of motion for two-phase Flow

The equation of motion may be reduced to that of two-phase flow by considering n to be 2 and rendered dimensionless by introducing the following non-dimensional quantities;

$$\bar{u} = \frac{u}{L}, \quad \bar{v} = \frac{v}{L}, \quad \bar{t} = \left[\frac{EI}{M_1+M_2+m}\right]^{1/2}\frac{t}{L^2},$$

$$\bar{\boldsymbol{U}}_{\boldsymbol{1}} = \left[\frac{M_1+M_2}{EI}\right]^{1/2} U_1 L, \ \bar{\boldsymbol{U}}_{\boldsymbol{2}} = \left[\frac{M_1+M_2}{EI}\right]^{1/2} U_2 L$$

$$\gamma = \frac{M_1+M_2+m}{EI}L^3 g, \ \beta_1 = \frac{M_1}{M_1+M_2+m},$$

$$\Psi_1 = \frac{M_1}{M_1+M_2}, \ \beta_2 = \frac{M_2}{M_1+M_2+m}, \quad \Psi_2 = \frac{M_2}{M_1+M_2},$$

$$Tension: \Pi_0 = \frac{T_0L^2}{EI}, \ Flexibility: \Pi_1 = \frac{EAL^2}{EI},$$

$$Presure: \Pi_2 = \frac{PL^2}{EI}$$

$$\ddot{\bar{u}} + \dot{\bar{\boldsymbol{U}}}_{\boldsymbol{1}}\sqrt{\Psi_1}\sqrt{\beta_1} + \dot{\bar{\boldsymbol{U}}}_{\boldsymbol{2}}\sqrt{\Psi_2}\sqrt{\beta_2} + 2\bar{\boldsymbol{U}}_{\boldsymbol{1}}\sqrt{\Psi_1}\sqrt{\beta_1}\dot{\bar{u}}' + 2\bar{\boldsymbol{U}}_{\boldsymbol{2}}\sqrt{\Psi_2}\sqrt{\beta_2}\dot{\bar{u}}' + \Psi_1\bar{\boldsymbol{U}}_{\boldsymbol{1}}^2\bar{u}'' + \Psi_2\bar{\boldsymbol{U}}_{\boldsymbol{2}}^2\bar{u}'' + \dot{\bar{\boldsymbol{U}}}_{\boldsymbol{1}}\sqrt{\Psi_1}\sqrt{\beta_1}\bar{u}' + \dot{\bar{\boldsymbol{U}}}_{\boldsymbol{2}}\sqrt{\Psi_2}\sqrt{\beta_2}\bar{u}' - \Pi_1\bar{u}'' - (\bar{v}''''\bar{v}' + \bar{v}''\bar{v}''') + (\Pi_0 - \Pi_2 - \Pi_1(\alpha\Delta T) - \Pi_1)\bar{v}'\bar{v}'' - \left(\Pi_0 - \Pi_2 - \Pi_1(\alpha\Delta T)\right)' + \gamma = 0 \quad (5)$$

$$\ddot{\bar{v}} + 2\bar{\boldsymbol{U}}_{\boldsymbol{1}}\sqrt{\Psi_1}\sqrt{\beta_1}\dot{\bar{v}}' + 2\bar{\boldsymbol{U}}_{\boldsymbol{2}}\sqrt{\Psi_2}\sqrt{\beta_2}\dot{\bar{v}}' + \Psi_1\bar{\boldsymbol{U}}_{\boldsymbol{1}}^2\bar{v}'' + \Psi_2\bar{\boldsymbol{U}}_{\boldsymbol{2}}^2\bar{v}'' - a\Psi_1\bar{\boldsymbol{U}}_{\boldsymbol{1}}^2\bar{v}'' - a\Psi_2\bar{\boldsymbol{U}}_{\boldsymbol{2}}^2\bar{v}'' + \dot{\bar{\boldsymbol{U}}}_{\boldsymbol{1}}\sqrt{\Psi_1}\sqrt{\beta_1}\bar{v}' + \dot{\bar{\boldsymbol{U}}}_{\boldsymbol{2}}\sqrt{\Psi_2}\sqrt{\beta_2}\bar{v}' - \left(\Pi_0 - \Pi_2 - \Pi_1(\alpha\Delta T)\right)\bar{v}'' + \bar{v}'''' - \left(3\bar{u}'''\bar{v}'' + 4\bar{v}'''\bar{u}'' + 2\bar{u}'\bar{v}'''' + \bar{v}'\bar{u}'''' + 2\bar{v}'^2\bar{v}'''' + 8\bar{v}'\bar{v}''\bar{v}''' + 2\bar{v}''^3\right) + \left(\Pi_0 - \Pi_2 - \Pi_1(\alpha\Delta T) - \Pi_1\right)\left(\bar{u}'\bar{v}'' + \bar{v}'\bar{u}'' + \frac{3}{2}\bar{v}'^2\bar{v}''\right) = 0 \quad (6)$$

In these equations, $\bar{u}$ and $\bar{v}$ respectively, are the dimensionless displacements in the longitudinal and transverse direction, $\bar{\boldsymbol{U}}_{\boldsymbol{1}}$ and $\bar{\boldsymbol{U}}_{\boldsymbol{2}}$ are the flow velocities of the constituent phases, β_1 and β_2 are the mass ratios for each phase which are the same as in single-phase flows as derived by Ghayesh, Païdoussis and

Amabili [15], Ψ_1 and Ψ_2 are new mass ratios that are unique to two-phase flow, relating the fluid masses independent of the mass of the pipe.

Assuming that the velocities are harmonically fluctuating about their constant mean velocities, the velocities of the phases can be expressed as:

$$\overline{\boldsymbol{U_1}} = \overline{U_1}\,(1 + \mu_1 \sin(\Omega_1 T0)) \tag{7}$$

$$\overline{\boldsymbol{U_2}} = \overline{U_2}\,(1 + \mu_2 \sin(\Omega_2 T0)) \tag{8}$$

Using these notations,

$C11 = \sqrt{\Psi_1}\sqrt{\beta_1}$, $C12 = \sqrt{\Psi_2}\sqrt{\beta_2}$, $C21 = 2\sqrt{\Psi_1}\sqrt{\beta_1}$, $C22 = 2\sqrt{\Psi_2}\sqrt{\beta_2}$, $C31 = \Psi_1$, $C32 = \Psi_2$, $C5 = \Pi_1$, $C6 = (\Pi_0 - \Pi_2 - \Pi_1(\alpha\Delta T) - \Pi_1)$, $C7 = \Pi_0 - \Pi_2 - \Pi_1(\alpha\Delta T)$

The equations are reduced as:

$$\ddot{\bar{u}} + \dot{\overline{\boldsymbol{U_1}}}C11 + \dot{\overline{\boldsymbol{U_2}}}C12 + \overline{\boldsymbol{U}}_{\boldsymbol{1}}C21\dot{\bar{u}}' + \overline{\boldsymbol{U}}_{\boldsymbol{2}}C22\dot{\bar{u}}' + C31\overline{\boldsymbol{U_1}}^2\bar{u}'' + C32\overline{\boldsymbol{U_2}}^2\bar{u}'' + \dot{\overline{\boldsymbol{U_1}}}\,C11\bar{u}' + \dot{\overline{\boldsymbol{U_2}}}\,C12\bar{u}' - C5\bar{u}'' - (\bar{v}''''\bar{v}' + \bar{v}''\bar{v}''') + C6\bar{v}'\bar{v}'' - C7' + \gamma = 0, \tag{9}$$

$$\ddot{\bar{v}} + \overline{\boldsymbol{U}}_{\boldsymbol{1}}C21\dot{\bar{v}}' + \overline{\boldsymbol{U}}_{\boldsymbol{2}}C22\dot{\bar{v}}' + C31\overline{\boldsymbol{U_1}}^2\bar{v}'' + C32\overline{\boldsymbol{U_2}}^2\bar{v}'' - aC31\overline{\boldsymbol{U_1}}^2\bar{v}'' - aC32\overline{\boldsymbol{U_2}}^2\bar{v}'' + \dot{\overline{\boldsymbol{U_1}}}\,C11\bar{v}' + \dot{\overline{\boldsymbol{U_2}}}\,C12\bar{v}' - C8\bar{v}'' + \bar{v}'''' - \left(3\bar{u}'''\bar{v}'' + 4\bar{v}'''\bar{u}'' + 2\bar{u}'\bar{v}'''' + \bar{v}'\bar{u}'''' + 2\bar{v}'^2\bar{v}'''' + 8\bar{v}'\bar{v}''\bar{v}''' + 2\bar{v}''^3\right) + C6\left(\bar{u}'\bar{v}'' + \bar{v}'\bar{u}'' + \tfrac{3}{2}\bar{v}'^2\bar{v}''\right) = 0. \tag{10}$$

2.4. The empirical gas-liquid two-phase flow model

The component's velocities in terms of the superficial velocities are expressed as:

$$V_g = U_g vf, \quad V_l = U_l(1 - vf) \tag{11}$$

Where U_g and U_l are the superficial flow velocities. Adopting the Chisholm empirical relations as presented in [16],

Void fraction:

$$vf = \left[1 + \sqrt{1 - \mathrm{x}\left(1 - \frac{\rho_l}{\rho_g}\right)}\left(\frac{1-\mathrm{x}}{\mathrm{x}}\right)\left(\frac{\rho_g}{\rho_l}\right)\right]^{-1} = \frac{\text{Volume of gas}}{\text{Volume of gas+Volume of Liquid}} \tag{12}$$

Slip Ratio:

$$\mathrm{S} = \frac{\mathrm{V_g}}{\mathrm{V_l}} = \left[1 - \mathrm{x}\left(1 - \frac{\rho_l}{\rho_g}\right)\right]^{1/2} \tag{13}$$

Where: (x) is the vapor quality and (ρ_l and ρ_g) are the densities of the liquid and gas phases respectively.

The mixture velocity can be expressed as:

$$\mathrm{V_T} = \mathrm{U_g vf} + \mathrm{U_l}(1 - \mathrm{vf}) \tag{14}$$

Individual Velocities:

$$\mathrm{V_l} = \frac{\mathrm{V_T}}{\mathrm{S+1}}, \mathrm{V_g} = \frac{\mathrm{SV_T}}{\mathrm{S+1}} \tag{15}$$

For various void fractions (0.1, 0.3, and 0.5) and a series of mixture velocities, the corresponding slip ratio and individual velocities are estimated and used for numerical calculations.

3. Method of Solution

Multiple-time scale perturbation technique is used to seek an approximate solution; this approach is applied directly to the partial differential equations (9) and (10).

$$\ddot{\bar{u}} + \dot{\overline{\mathbf{U_1}}}C11 + \dot{\overline{\mathbf{U_2}}}C12 + \overline{\mathbf{U}}_{\mathbf{1}}C21\dot{\bar{u}}' + \overline{\mathbf{U}}_{\mathbf{2}}C22\dot{\bar{u}}' + C31\overline{\mathbf{U_1}}^2\bar{u}'' + C32\overline{\mathbf{U_2}}^2\bar{u}'' + \dot{\overline{\mathbf{U_1}}}\,C11\bar{u}' + \dot{\overline{\mathbf{U_2}}}\,C12\bar{u}' - C5\bar{u}'' + \varepsilon(-(\bar{v}''''\bar{v}' + \bar{v}''\bar{v}''') + C6\bar{v}'\bar{v}'' - C7' + \gamma) = 0, \tag{16}$$

$$\ddot{\bar{v}} + \overline{\mathbf{U}}_{\mathbf{1}}C21\dot{\bar{v}}' + \overline{\mathbf{U}}_{\mathbf{2}}C22\dot{\bar{v}}' + C31\overline{\mathbf{U_1}}^2\bar{v}'' + C32\overline{\mathbf{U_2}}^2\bar{v}'' - aC31\overline{\mathbf{U_1}}^2\bar{v}'' - aC32\overline{\mathbf{U_2}}^2\bar{v}'' + \dot{\overline{\mathbf{U_1}}}\,C11\bar{v}' + \dot{\overline{\mathbf{U_2}}}\,C12\bar{v}' - C7\bar{v}'' + \bar{v}''''\varepsilon\left(-\left(3\bar{u}'''\bar{v}'' + 4\bar{v}'''\bar{u}'' + 2\bar{u}'\bar{v}'''' + \bar{v}'\bar{u}'''' + 2\bar{v}'^2\bar{v}'''' + 8\bar{v}'\bar{v}''\bar{v}''' + 2\bar{v}''^3\right) + C6\left(\bar{u}'\bar{v}'' + \bar{v}'\bar{u}'' + \tfrac{3}{2}\bar{v}'^2\bar{v}''\right)\right) = 0 \tag{17}$$

Also, perturbing the harmonically fluctuation of the velocity about the constant mean velocity;

$$\overline{\mathbf{U_1}} = \overline{\mathrm{U}_1}\,(1 + \varepsilon\mu_1 \sin(\Omega_1 \mathrm{T0})) \tag{18}$$

$$\overline{\mathbf{U_2}} = \overline{\mathrm{U}_2}\,(1 + \varepsilon\mu_2 \sin(\Omega_2 \mathrm{T0})) \tag{19}$$

We seek approximate solutions in the form:

$$\bar{u} = \bar{u}_0(T_0, T_1) + \varepsilon\bar{u}_1(T_0, T_1) + \varepsilon^2\bar{u}_2(T_0, T_1) + O(\varepsilon) \tag{20}$$

$$\bar{v} = \bar{v}_0(T_0, T_1) + \varepsilon\bar{v}_1(T_0, T_1) + \varepsilon^2\bar{v}_2(T_0, T_1) + O(\varepsilon) \tag{21}$$

Two-time scales are needed $T_0 = t$ and $T_1 = \varepsilon t$. Where ε is a small dimensionless measure of the amplitude of $\bar{u}$ and $\bar{v}$, used as a book-keeping parameter.
The time derivatives are:

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + O(\varepsilon) \tag{22}$$

$$\frac{d^2}{dt^2} = D_0{}^2 + 2\varepsilon D_0 D_1 + \varepsilon^2(D_1{}^2 + 2D_0 D_2) + O(\varepsilon) \tag{23}$$

$Where\ D_n = \frac{\partial}{\partial T_n}$

Substituting Equation (20), Equation (21), Equation (22) and Equation (23) into Equation (16) and Equation (17) and equating the coefficients of (ε) to zero and one respectively:

<u>U-Equation:</u>

$$O(\varepsilon^0).\quad D_0{}^2\bar{u}_0 + C21D_0\bar{u}_0{}'\overline{U}_1 + C22D_0\bar{u}_0{}''\overline{U}_2 + C31\bar{u}_0{}''\overline{U}_1{}^2 + C32\bar{u}_0{}''\overline{U}_2{}^2 - C5\bar{u}_0{}'' = 0 \tag{24}$$

$$O(\varepsilon^1).\quad D_0{}^2\bar{u}_1 + C21D_0\bar{u}_1{}'\overline{U}_1 + C22D_0\bar{u}_1{}'\overline{U}_2 + 2D_0D_1\bar{u}_0 + C31\bar{u}_1{}''\overline{U}_1{}^2 + C32\bar{u}_1{}''\overline{U}_2{}^2 + C21D_0\bar{u}_1{}'\overline{U}_1 + C22D_0\bar{u}_1{}'\overline{U}_2 - C5\bar{u}_1{}'' - \bar{v}_0{}''''\bar{v}_0{}' - C7' + \gamma - \bar{v}_0{}''\bar{v}_0{}''' + C6\bar{v}_0{}'\bar{v}_0{}'' + C21D_1\bar{u}_0{}'\overline{U}_1 + C22D_1\bar{u}_0{}'\overline{U}_2 + C11\Omega_1\mu_1\cos(\Omega_1 T0)\overline{U}_1 + C12\Omega_2\mu_2\cos(\Omega_2 T0)\overline{U}_2 + 2C31\mu_1\sin(\Omega_1 T0)\overline{U}_1{}^2\,\bar{u}_0{}'' + 2C32\mu_2\sin(\Omega_2 T0)\overline{U}_2{}^2\,\bar{u}_0{}'' + C21\mu_1\sin(\Omega_1 T0)D_0\overline{U}_1\,\bar{u}_0{}' + C22\mu_2\sin(\Omega_2 T0)D_0\overline{U}_2\,\bar{u}_0{}' + C41\Omega_1\mu_1\cos(\Omega_1 T0)\overline{U}_1\bar{u}_0{}' + C42\Omega_2\mu_2\cos(\Omega_2 T0)\overline{U}_2\bar{u}_0{}' = 0 \tag{25}$$

V-Equation:

$$O(\varepsilon^0). \quad {D_0}^2\bar{v}_0 - C7\bar{v}_0'' + \bar{v}_0'''' + C21D_0\bar{v}_0'\bar{U}_1 + C22D_0\bar{v}_0'\bar{U}_2 + C31\bar{v}_0''{\bar{U}_1}^2 + C32\bar{v}_0''{\bar{U}_2}^2 - aC31\bar{v}_0''{\bar{U}_1}^2 - aC32 = 0 \tag{26}$$

$$O(\varepsilon^1). \quad {D_0}^2\bar{v}_1 - C7\bar{v}_1'' + \bar{v}_1'''' - \bar{u}_0''''\bar{v}_0' - 2\bar{u}_0'\bar{v}_0'''' - 4\bar{u}_0''\bar{v}_0''' - 3\bar{v}_0''\bar{v}_0''' - 2{\bar{v}_0}^{3''} - 2\bar{v}_0''''{\bar{v}_0}^{2'} + 2D_0D_1\bar{v}_0 + C31\bar{v}_1''{\bar{U}_1}^2 + C32\bar{v}_1''{\bar{U}_2}^2 - 8\bar{v}_0'\bar{v}_0''\bar{v}_0''' + C6\bar{u}_0'\bar{v}_0'' + C6\bar{u}_0''\bar{v}_0' + \frac{3}{2}C6{\bar{v}_0}^{2'}\bar{v}_0'' + C21D_0\bar{v}_0'\bar{U}_1 + C22D_0\bar{v}_0'\bar{U}_2 + C21D_1\bar{v}_0'\bar{U}_1 + C22D_1\bar{v}_0'\bar{U}_2 - aC31\bar{v}_1''{\bar{U}_1}^2 - aC32\bar{v}_1''{\bar{U}_2}^2 + 2C31\mu_1\sin(\Omega_1T0){\bar{U}_1}^2\bar{v}_0'' + 2C32\mu_2\sin(\Omega_2T0){\bar{U}_2}^2\bar{v}_0'' + C21\mu_1\sin(\Omega_1T0)D_0\bar{U}_1\bar{v}_0' + C22\mu_2\sin(\Omega_2T0)D_0\bar{U}_2\bar{v}_0' - 2aC31\mu_1\sin(\Omega_1T0){\bar{U}_1}^2\bar{v}_0'' - 2aC32\mu_2\sin(\Omega_2T0){\bar{U}_2}^2\bar{v}_0'' + C41\Omega_1\mu_1\cos(\Omega_1T0)\bar{U}_1\bar{v}_0' + C42\Omega_2\mu_2\cos(\Omega_2T0)\bar{U}_2\bar{v}_0' = 0 \tag{27}$$

The homogeneous solution of the leading order equations Equation (24) and Equation (27) can be expressed as:

$$\bar{u}(x,T_0,T_1)_0 = \phi(x)_n\exp(i\omega_nT_0) + CC \tag{28}$$
$$\bar{v}(x,T_0,T_1)_0 = \eta(x)_n\exp(i\lambda_nT_0) + CC \tag{29}$$

Where (CC) is the complex conjugate, $\phi(x)_n$ and $\eta(x)_n$ are the complex modal functions for the axial and transverse vibrations for each mode (n) and, ω_n and λ_n are the eigenvalues for the axial and transverse vibrations for each mode (n).

3.1. Axial natural frequencies and mode shape

The analytical expression for the axial frequencies is obtained as:

$$\omega_n = \frac{2\pi n - i.ln\left(\frac{b}{a}\right)}{(a-b)L}, \; n = 1,2,3,\ldots \tag{30}$$

Where:

$$a = \frac{\frac{C21\bar{U}_1}{2} + \frac{C22\bar{U}_2}{2} + \frac{\sqrt{C21^2{\bar{U}_1}^2 + 2C21C22\bar{U}_1\bar{U}_2 + C22^2{\bar{U}_2}^2 - 4C31{\bar{U}_1}^2 - 4C32{\bar{U}_2}^2 + 4C5}}{2}}{C5 - C31{\bar{U}_1}^2 - C32{\bar{U}_2}^2},$$

$$b = \frac{\frac{C21\bar{U}_1}{2} + \frac{C22\bar{U}_2}{2} - \frac{\sqrt{C21^2{\bar{U}_1}^2 + 2C21C22\bar{U}_1\bar{U}_2 + C22^2{\bar{U}_2}^2 - 4C31{\bar{U}_1}^2 - 4C32{\bar{U}_2}^2 + 4C5}}{2}}{C5 - C31{\bar{U}_1}^2 - C32{\bar{U}_2}^2}$$

With the modal shape expressed as:

$$\phi(x)_n = G_n(exp(ik_1x) + exp(ik_2x)) \tag{31}$$

The constant G_n can be obtained using the orthogonality relationship.

3.2. Transverse natural frequencies and mode shape

Conversely to the axial vibrations, direct analytical estimation is not possible for the natural frequencies of the transverse vibrations. However, the natural frequencies can be estimated by solving the quartic equation (32) and the condition of obtaining a non-trivial solution of the boundary condition matrix (33) simultaneously with a nonlinear numerical routine:

$$z^4{}_{jn} + \left(C7 - C31{\bar{U}_1}^2 - C32{\bar{U}_2}^2 + aC31{\bar{U}_1}^2 + aC32{\bar{U}_2}^2\right)z^2{}_{jn} - (C21\bar{U}_1 + C22\bar{U}_2)z_{jn}\lambda_n - \lambda^2{}_n = 0 \quad j = 1,2,3,4 \; and \; n = 1,2,3,4,5\ldots \tag{32}$$

Boundary condition matrix:

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ z_{1n} & z_{2n} & z_{3n} & z_{4n} \\ (z_{1n})^2.\exp(i.z_{1n}) & (z_{2n})^2.\exp(i.z_{2n}) & (z_{3n})^2.\exp(i.z_{3n}) & (z_{4n})^2.\exp(i.z_{4n}) \\ (z_{1n})^3.\exp(i.z_{1n}) & (z_{2n})^3.\exp(i.z_{2n}) & (z_{3n})^3.\exp(i.z_{3n}) & (z_{4n})^3.\exp(i.z_{4n}) \end{bmatrix}}_{G} \begin{bmatrix} 1 \\ H2_n \\ H3_n \\ H4_n \end{bmatrix} H1_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

For a non-trivial solution, the determinant of (G) must varnish, that is:

$$DET(G) = 0 \tag{33}$$

Where (λ_n), are the natural frequencies and (Z_n) are the eigenvalues. The mode function of the transverse vibration corresponding to the nth eigenvalue is expressed as:

$$\eta(x)_n = H1_n.[e^{x.z_{1n}.i} - (A + B + C + D) - E] \tag{34}$$

$$A = \frac{e^{x.z_{4n}.i}.\,[e^{z_{1n}.i}.(z_{1n})^3.z_{2n} - e^{z_{1n}.i}.(z_{1n})^3.\,z_{3n} - e^{z_{1n}.i}.\,z_{4n}.(z_{1n})^2.z_{2n}}{(z_{2n} - z_{4n}).(z_{3n} - z_{4n}).[e^{z_{2n}.i}.(z_{2n})^2 - e^{z_{3n}.i}.(z_{3n})^2]}$$

$$B = \frac{e^{x.z_{4n}.i}.\,[e^{z_{1n}.i}.z_{4n}.(z_{1n})^2.z_{3n} - e^{z_{2n}.i}.z_{1n}.(z_{2n})^3 + e^{z_{2n}.i}.\,z_{4n}.\,z_{1n}.(z_{2n})^2}{(z_{2n} - z_{4n}).(z_{3n} - z_{4n}).[e^{z_{2n}.i}.(z_{2n})^2 - e^{z_{3n}.i}.(z_{3n})^2]}$$

$$C = \frac{e^{x.z_{4n}.i}.\,[e^{z_3.i}.z_{1n}.(z_{3n})^3 - e^{z_3.i}.z_{4n}.z_{1n}.(z_{3n})^2 + e^{z_{2n}.i}.(z_{2n})^3.z_{3n}}{(z_{2n} - z_{4n}).(z_{3n} - z_{4n}).[e^{z_{2n}.i}.(z_{2n})^2 - e^{z_{3n}.i}.(z_{3n})^2]}$$

$$D = \frac{e^{x.z_{4n}.i}.\,[-e^{z_{2n}.i}.z_{4n}.(z_{2n})^2.z_{3n} - e^{z_3.i}.z_{2n}.(z_{3n})^3 + e^{z_3.i}.z_{4n}.z_{2n}.(z_{3n})^2}{(z_{2n} - z_{4n}).(z_{3n} - z_{4n}).[e^{z_{2n}.i}.(z_{2n})^2 - e^{z_{3n}.i}.(z_{3n})^2]}$$

$$E = \frac{e^{x.z_{2n}.i}.(z_{1n} - z_{4n}).[e^{z_1.i}.\,(z_{1n})^2 - e^{z_3.i}.(z_{3n})^2]}{(z_{2n} - z_{4n}).\,[e^{z_2.i}.(z_{2n})^2 - e^{z_3.i}.(z_{3n})^2]} + \frac{e^{x.z_3.i}.(z_{1n} - z_{4n}).[e^{z_{1n}.i}.\,(z_{1n})^2 - e^{z_2 i}.(z_{2n})^2]}{(z_{3n} - z_{4n}).\,[e^{z_2.i}.(z_{2n})^2 - e^{z_3.i}.(z_{3n})^2]}$$

The constant H1 can be obtained using the orthogonality relationship.

3.3. Axial principal parametric resonance

Substituting equation (28) and equation (29) into the equations (25) and (27) gives;

$$\begin{aligned} &{D_0}^2\bar{u}_1 - C5\bar{u}_1'' + C21D_0\bar{u}_1'\bar{U}_1 + C22D_0\bar{u}_1'\bar{U}_2 + C31\bar{u}_1''{\bar{U}_1}^2 + C32\bar{u}_1''{\bar{U}_2}^2 = -\Big(C21\frac{\partial X(T1)}{\partial T1}\frac{\partial\phi(x)}{\partial x}\bar{U}_1 + C22\frac{\partial X(T1)}{\partial T1}\frac{\partial\phi(x)}{\partial x}\bar{U}_2 + \\ &2i\frac{\partial X(T1)}{\partial T1}\omega\Big)exp(i\omega T_0) + Y(T1)^2\left(\frac{\partial\eta(x)}{\partial x}\frac{\partial^4\eta(x)}{\partial x^4} + \frac{\partial^2\eta(x)}{\partial x^2}\frac{\partial^3\eta(x)}{\partial x^3} - C6\frac{\partial\eta(x)}{\partial x}\frac{\partial^2\eta(x)}{\partial x^2}\right)exp(2i\lambda T_0) + \Big[C32\mu_2\frac{\partial^2\phi(x)}{\partial x^2}exp(i\Omega_2T_0)\,{\bar{U}_2}^2 i - \\ &\frac{1}{2}\left(C21\mu_1\frac{\partial\phi(x)}{\partial x}exp(-i\Omega_1T_0)\bar{U}_1\,\omega\right) + \\ &\frac{1}{2}\left(C21\mu_1\frac{\partial\phi(x)}{\partial x}exp(i\Omega_1T_0)\bar{U}_1\,\omega\right) - \\ &\frac{1}{2}\left(C22\mu_2\frac{\partial\phi(x)}{\partial x}exp(-i\Omega_2T_0)\bar{U}_2\,\omega\right) + \\ &\frac{1}{2}\left(C22\mu_2\frac{\partial\phi(x)}{\partial x}exp(i\Omega_2T_0)\bar{U}_2\,\omega\right) - \\ &\frac{1}{2}\left(C41\Omega_1\mu_1\frac{\partial\phi(x)}{\partial x}exp(-i\Omega_1T_0)\bar{U}_1\right) - \\ &\frac{1}{2}\left(C41\Omega_1\mu_1\frac{\partial\phi(x)}{\partial x}exp(i\Omega_1T_0)\bar{U}_1\right) - \end{aligned}$$

$$\frac{1}{2}\left(C42\Omega_2\mu_2\frac{\partial\phi(x)}{\partial x}exp(-i\Omega_2T_0)\bar{U}_2\right)-\frac{1}{2}\left(C42\Omega_2\mu_2\frac{\partial\phi(x)}{\partial x}exp(i\Omega_2T_0)\bar{U}_2\right)-C32\mu_2\frac{\partial^2\phi(x)}{\partial x^2}exp(-i\Omega_2T_0)\,\bar{U}_2{}^2i-C31\mu_1\frac{\partial^2\phi(x)}{\partial x^2}exp(-i\Omega_1T_0)\,\bar{U}_1{}^2i+C31\mu_1\frac{\partial^2\phi(x)}{\partial x^2}exp(i\Omega_1T_0)\,\bar{U}_1{}^2i\Big]X(T1)exp(i\omega T_0)+\Big[C32\mu_2\frac{\partial^2\overline{\phi(x)}}{\partial x^2}exp(i\Omega_2T_0)\,\bar{U}_2{}^2i+\frac{1}{2}\left(C21\mu_1\frac{\partial\overline{\phi(x)}}{\partial x}exp(i\Omega_1T_0)\bar{U}_1\,\omega\right)+\frac{1}{2}\left(C22\mu_2\frac{\partial\overline{\phi(x)}}{\partial x}exp(i\,\Omega_2T_0)\bar{U}_2\,\omega\right)-\frac{1}{2}\left(C41\Omega_1\mu_1\frac{\partial\overline{\phi(x)}}{\partial x}exp(i\Omega_1T_0)\bar{U}_1\right)-\frac{1}{2}\left(C42\Omega_2\mu_2\frac{\partial\overline{\phi(x)}}{\partial x}exp(i\Omega_2T_0)\bar{U}_2\right)+C31\mu_1\frac{\partial^2\overline{\phi(x)}}{\partial x^2}exp(i\Omega_1T_0)\,\bar{U}_1{}^2i\Big]\overline{X(T1)}exp(-i\omega T_0)+NST+CC=0 \quad (35)$$

$$D_0{}^2\bar{v}_1-C7\bar{v}_1''+\bar{v}_1''''+C21D_0\bar{v}_1'\bar{U}_1+C22D_0\bar{v}_1'\bar{U}_2+C31\bar{v}_1''\bar{U}_1{}^2+C32\bar{v}_1''\bar{U}_2{}^2-aC31\bar{v}_1''\bar{U}_1{}^2-aC32\bar{v}_1''\bar{U}_2{}^2=\left(-\frac{\partial Y(T1)}{\partial T1}\left(C21\frac{\partial\eta(x)}{\partial x}\bar{U}_1+C22\frac{\partial\eta(x)}{\partial x}\bar{U}_2+2\eta(x)\lambda i\right)+6Y(T1)^2\overline{Y(T1)}\left(\frac{\partial\eta(x)}{\partial x}\right)^2\frac{\partial\overline{\eta(x)}}{\partial x}+2Y(T1)^2\overline{Y(T1)}\left(\frac{\partial\eta(x)}{\partial x}\right)^2\frac{\partial^4\overline{\eta(x)}}{\partial x^4}+4Y(T1)^2\overline{Y(T1)}\frac{\partial\eta(x)}{\partial x}\frac{\partial\overline{\eta(x)}}{\partial x}\frac{\partial^4\eta(x)}{\partial x^4}+8Y(T1)^2\overline{Y(T1)}\frac{\partial\eta(x)}{\partial x}\frac{\partial^2\overline{\eta(x)}}{\partial x^2}\frac{\partial^3\eta(x)}{\partial x^3}+8Y(T1)^2\overline{Y(T1)}\frac{\partial\overline{\eta(x)}}{\partial x}\frac{\partial^2\eta(x)}{\partial x^2}\frac{\partial^3\eta(x)}{\partial x^3}-3C6.Y(T1)^2\overline{Y(T1)}\frac{\partial\eta(x)}{\partial x}\frac{\partial\overline{\eta(x)}}{\partial x}\frac{\partial^2\eta(x)}{\partial x^2}+8Y(T1)^2\overline{Y(T1)}\frac{\partial\eta(x)}{\partial x}\frac{\partial^2\eta(x)}{\partial x^2}\frac{\partial^3\overline{\eta(x)}}{\partial x^3}-\frac{3}{2}C6.Y(T1)^2\overline{Y(T1)}\left(\frac{\partial\eta(x)}{\partial x}\right)^2\frac{\partial^2\eta(x)}{\partial x^2}\right)exp(i\lambda T_0)+\left(2X(T1)\overline{Y(T1)}\frac{\partial\Phi(x)}{\partial x}\frac{\partial^4\overline{\eta(x)}}{\partial x^4}+4X(T1)\overline{Y(T1)}\frac{\partial^2\Phi(x)}{\partial x^2}\frac{\partial^3\overline{\eta(x)}}{\partial 3}+3X(T1)\overline{Y(T1)}\frac{\partial^2\overline{\eta(x)}}{\partial 2}\frac{\partial^3\Phi(x)}{\partial 3}\right)exp(i\omega T_0)\,exp(-i\lambda T_0)-\left(C6X(T1)\overline{Y(T1)}\frac{\partial\Phi(x)}{\partial x}\frac{\partial^2\overline{\eta(x)}}{\partial x^2}+C6X(T1)\overline{Y(T1)}\frac{\partial\overline{\eta(x)}}{\partial x}\frac{\partial^2\Phi(x)}{\partial x^2}\right)exp(i\omega T_0)\,exp(-i\lambda T_0)+\Big[\left(\frac{1}{2}\left(C22\mu_2\frac{\partial\eta(x)}{\partial x}\bar{U}_2\lambda\right)-\frac{1}{2}\left(C42\Omega_2\mu_2\frac{\partial\eta(x)}{\partial x}\bar{U}_2\right)+aC32\mu_2\frac{\partial^2\eta(x)}{\partial x^2}\bar{U}_2{}^2i-C32\mu_2\frac{\partial^2\eta(x)}{\partial x^2}\bar{U}_2{}^2i\right)exp(-i\Omega_2T_0)+\left(\frac{1}{2}\left(C21\mu_1\frac{\partial\eta(x)}{\partial x}\bar{U}_1\lambda\right)-\frac{1}{2}\left(C41\Omega_1\mu_1\frac{\partial\eta(x)}{\partial x}\bar{U}_1\right)+aC31\mu_1\frac{\partial^2\eta(x)}{\partial x^2}\bar{U}_1{}^2i-C31\mu_1\frac{\partial^2\eta(x)}{\partial x^2}\bar{U}_1{}^2i\right)exp(-i\Omega_1T_0)-\left(\frac{1}{2}\left(C21\mu_1\frac{\partial\eta(x)}{\partial x}\bar{U}_1\lambda\right)-\frac{1}{2}\left(C41\Omega_1\mu_1\frac{\partial\eta(x)}{\partial x}\bar{U}_1\right)+aC31\mu_1\frac{\partial^2\eta(x)}{\partial x^2}\bar{U}_1{}^2i-C31\mu_1\frac{\partial^2\eta(x)}{\partial x^2}\bar{U}_1{}^2i\right)exp(i\Omega_1T_0)-\left(\frac{1}{2}\left(C22\mu_2\frac{\partial\eta(x)}{\partial x}\bar{U}_2\lambda\right)-\frac{1}{2}\left(C42\Omega_2\mu_2\frac{\partial\eta(x)}{\partial x}\bar{U}_2\right)+aC32\mu_2\frac{\partial^2\eta(x)}{\partial x^2}\bar{U}_2{}^2i-C32\mu_2\frac{\partial^2\eta(x)}{\partial x^2}\bar{U}_2{}^2i\right)exp(i\Omega_2T_0)\Big]Y(T1)exp(i\lambda T_0)+\Big[\left(\frac{1}{2}\left(C21\mu_1\frac{\partial\overline{\eta(x)}}{\partial x}\bar{U}_1\lambda\right)-\frac{1}{2}\left(C41\Omega_1\mu_1\frac{\partial\overline{\eta(x)}}{\partial x}\bar{U}_1\right)-aC31\mu_1\frac{\partial^2\overline{\eta(x)}}{\partial x^2}\bar{U}_1{}^2i+C31\mu_1\frac{\partial^2\overline{\eta(x)}}{\partial x^2}\bar{U}_1{}^2i\right)exp(i\Omega_1T_0)+\left(\frac{1}{2}\left(C22\mu_2\frac{\partial\overline{\eta(x)}}{\partial x}\bar{U}_2\lambda\right)-\frac{1}{2}\left(C42\Omega_2\mu_2\frac{\partial\overline{\eta(x)}}{\partial x}\bar{U}_2\right)-aC32\mu_2\frac{\partial^2\overline{\eta(x)}}{\partial x^2}\bar{U}_2{}^2i+C32\mu_2\frac{\partial^2\overline{\eta(x)}}{\partial x^2}\bar{U}_2{}^2i\right)exp(i\Omega_2T_0)\Big]\overline{Y(T1)}exp(-i\lambda T_0)+NST+CC \quad (36)$$

Here NST denotes non-secular terms. The next task is to determine the requirements for X(T1) and Y(T1), that permits the solutions of $\bar{u}_1$ and $\bar{v}_1$ to be independent of secular terms. However, examining equations (35) and (36), it can be observed that various scenarios exist. With a focus on the principal parametric resonance cases when the pulsation frequencies of the phases Ω1 and Ω2 are close to 2ω but far from 2λ. Also, consider the possibility of having internal resonance (ω = 2λ) and away from internal resonance (ω ≠ 2λ) relationship between the axial and the transverse natural frequencies.

The proximity of the nearness can be expressed as:

$$\Omega 1 = 2\omega + \varepsilon\sigma_2 \; and \; \Omega 2 = 2\omega + \varepsilon\sigma_2 \quad (37)$$

Therefore, $\Omega 1 \equiv \Omega 2$
Where σ_2 is the detuning parameter.

Substituting the equations of nearness to resonance (37) into equations (35) and (36), replacing εT0 with T1 and collating the secular terms, the principal parametric resonance conditions are identified and assessed as follows:

3.4. When $\boldsymbol{\omega}$ is far from $\mathbf{2\lambda}$

If ω is far from 2λ then none of the coupled nonlinear terms will generate secular terms, therefore resulting in the uncoupled response.

The two equations will have bounded solutions only if the solvability condition holds. The solvability condition demands that the coefficient of $exp(i\omega T_0)$ and $exp(i\lambda T_0)$ vanishes [17-19], that is, X (T1) and Y (T1) should satisfy the following relation:

$$-\left(C21\frac{\partial X(T1)}{\partial T1}\frac{\partial\phi(x)}{\partial x}\bar{U}_1+C22\frac{\partial X(T1)}{\partial T1}\frac{\partial\phi(x)}{\partial x}\bar{U}_2+2\phi(x)\omega i\frac{\partial X(T1)}{\partial T1}\right)+\Big[C32\mu_2\frac{\partial^2\overline{\phi(x)}}{\partial x^2}exp(i\sigma_2T_1)\,\bar{U}_2{}^2i+\frac{1}{2}\left(C21\mu_1\frac{\partial\overline{\phi(x)}}{\partial x}exp(i\sigma_2T_1)\bar{U}_1\,\omega\right)+\frac{1}{2}\left(C22\mu_2\frac{\partial\overline{\phi(x)}}{\partial x}exp(i\sigma_2T_1)\bar{U}_2\,\omega\right)-\left(C41\mu_1\frac{\partial\overline{\phi(x)}}{\partial x}exp(i\sigma_2T_1)\bar{U}_1\,\omega\right)-\left(C42\mu_2\frac{\partial\overline{\phi(x)}}{\partial x}exp(i\sigma_2T_2)\bar{U}_2\,\omega\right)-\frac{1}{2}\left(C41\varepsilon\sigma_2\mu_1\frac{\partial\overline{\phi(x)}}{\partial x}exp(i\sigma_2T_1)\bar{U}_1\right)-\frac{1}{2}\left(C42\varepsilon\sigma_2\mu_2\frac{\partial\overline{\phi(x)}}{\partial x}exp(i\sigma_2T_1)\bar{U}_2\right)+C31\mu_1\frac{\partial^2\overline{\phi(x)}}{\partial x^2}exp(i\sigma_2T_1)\,\bar{U}_1{}^2i\Big]\overline{X(T1)}=0 \quad (38)$$

$$\left(-\frac{\partial Y(T1)}{\partial T1}\left(C21\frac{\partial\eta(x)}{\partial x}\bar{U}_1+C22\frac{\partial\eta(x)}{\partial x}\bar{U}_2+2\eta(x)\lambda i\right)+6Y(T1)^2\overline{Y(T1)}\left(\frac{\partial\eta(x)}{\partial x}\right)^2\frac{\partial\overline{\eta(x)}}{\partial x}+2Y(T1)^2\overline{Y(T1)}\left(\frac{\partial\eta(x)}{\partial x}\right)^2\frac{\partial^4\overline{\eta(x)}}{\partial x^4}+4Y(T1)^2\overline{Y(T1)}\frac{\partial\eta(x)}{\partial x}\frac{\partial\overline{\eta(x)}}{\partial x}\frac{\partial^4\eta(x)}{\partial x^4}+8Y(T1)^2\overline{Y(T1)}\frac{\partial\eta(x)}{\partial x}\frac{\partial^2\overline{\eta(x)}}{\partial x^2}\frac{\partial^3\eta(x)}{\partial x^3}+8Y(T1)^2\overline{Y(T1)}\frac{\partial\overline{\eta(x)}}{\partial x}\frac{\partial^2\eta(x)}{\partial x^2}\frac{\partial^3\eta(x)}{\partial x^3}-3C6.Y(T1)^2\overline{Y(T1)}\frac{\partial\eta(x)}{\partial x}\frac{\partial\overline{\eta(x)}}{\partial x}\frac{\partial^2\eta(x)}{\partial x^2}+8Y(T1)^2\overline{Y(T1)}\frac{\partial\eta(x)}{\partial x}\frac{\partial^2\eta(x)}{\partial x^2}\frac{\partial^3\overline{\eta(x)}}{\partial x^3}-$$

$$\frac{3}{2}C6.Y(T1)^2\overline{Y(T1)}\left(\frac{\partial\eta(x)}{\partial x}\right)^2\frac{\partial^2\eta(x)}{\partial x^2}\Big) = 0 \tag{39}$$

With the inner product defined for complex functions on [0, 1] as:

$$\langle f, g\rangle = \int_0^1 f\bar{g}\,dx \tag{40}$$

Equations (52) and (53) can be cast as:

$$\frac{\partial X(T1)}{\partial T1} + Q\overline{X(T1)}\,exp(i\sigma_2 T_1) = 0 \tag{41}$$

$$\frac{\partial Y(T1)}{\partial T1} + SY(T1)^2\overline{Y(T1)} = 0 \tag{42}$$

Where:

$$Q = \frac{\int_0^1 -\overline{\phi(x)}\left[C32\mu_2\frac{\partial^2\overline{\phi(x)}}{\partial x^2}\bar{U}_2{}^2 i+\frac{1}{2}\left(C21\mu_1\frac{\partial\overline{\phi(x)}}{\partial x}\bar{U}_1\omega\right)\right]dx}{(C21\bar{U}_1+C22\bar{U}_2)\int_0^1\frac{\partial\phi(x)}{\partial x}\overline{\phi(x)}dx+2i\omega\int_0^1\phi(x)\overline{\phi(x)}dx} + \frac{\int_0^1 -\overline{\phi(x)}\left[\frac{1}{2}\left(C22\mu_2\frac{\partial\overline{\phi(x)}}{\partial x}\bar{U}_2\omega\right)-\left(C41\mu_1\frac{\partial\overline{\phi(x)}}{\partial x}\bar{U}_1\omega\right)\right]dx}{(C21\bar{U}_1+C22\bar{U}_2)\int_0^1\frac{\partial\phi(x)}{\partial x}\overline{\phi(x)}dx+2i\omega\int_0^1\phi(x)\overline{\phi(x)}dx} + \frac{\int_0^1 -\overline{\phi(x)}\left[-\left(C42\mu_2\frac{\partial\overline{\phi(x)}}{\partial x}\bar{U}_2\omega\right)-\frac{1}{2}\left(C41\varepsilon\sigma_2\mu_1\frac{\partial\overline{\phi(x)}}{\partial x}\bar{U}_1\right)\right]dx}{(C21\bar{U}_1+C22\bar{U}_2)\int_0^1\frac{\partial\phi(x)}{\partial x}\overline{\phi(x)}dx+2i\omega\int_0^1\phi(x)\overline{\phi(x)}dx} + \frac{\int_0^1 -\overline{\phi(x)}\left[-\frac{1}{2}\left(C42\varepsilon\sigma_2\mu_2\frac{\partial\overline{\phi(x)}}{\partial x}\bar{U}_2\right)+C31\mu_1\frac{\partial^2\overline{\phi(x)}}{\partial x^2}\bar{U}_1{}^2 i\right]dx}{(C21\bar{U}_1+C22\bar{U}_2)\int_0^1\frac{\partial\phi(x)}{\partial x}\overline{\phi(x)}dx+2i\omega\int_0^1\phi(x)\overline{\phi(x)}dx}$$

$$S = \frac{\int_0^1\left[6\left(\frac{\partial\eta(x)}{\partial x}\right)^2\frac{\partial\overline{\eta(x)}}{\partial x}+2\left(\frac{\partial\eta(x)}{\partial x}\right)^2\frac{\partial^4\overline{\eta(x)}}{\partial x^4}\right]\overline{\eta(x)}dx}{-\left((C21\bar{U}_1+C22\bar{U}_2)\int_0^1\frac{\partial\eta(x)}{\partial x}\overline{\eta(x)}dx+2i\lambda\int_0^1\eta(x)\overline{\eta(x)}dx\right)} + \frac{\int_0^1\left[4\frac{\partial\eta(x)}{\partial x}\frac{\partial\overline{\eta(x)}}{\partial x}\frac{\partial^4\eta(x)}{\partial x^4}+8\frac{\partial\eta(x)}{\partial x}\frac{\partial^2\overline{\eta(x)}}{\partial x^2}\frac{\partial^3\eta(x)}{\partial x^3}\right]\overline{\eta(x)}dx}{-\left((C21\bar{U}_1+C22\bar{U}_2)\int_0^1\frac{\partial\eta(x)}{\partial x}\overline{\eta(x)}dx+2i\lambda\int_0^1\eta(x)\overline{\eta(x)}dx\right)} + \frac{\int_0^1\left[8\frac{\partial\overline{\eta(x)}}{\partial x}\frac{\partial^2\eta(x)}{\partial x^2}\frac{\partial^3\eta(x)}{\partial x^3}-3C6\frac{\partial\eta(x)}{\partial x}\frac{\partial\overline{\eta(x)}}{\partial x}\frac{\partial^2\eta(x)}{\partial x^2}\right]\overline{\eta(x)}dx}{-\left((C21\bar{U}_1+C22\bar{U}_2)\int_0^1\frac{\partial\eta(x)}{\partial x}\overline{\eta(x)}dx+2i\lambda\int_0^1\eta(x)\overline{\eta(x)}dx\right)} + \frac{\int_0^1\left[8\frac{\partial\eta(x)}{\partial x}\frac{\partial^2\eta(x)}{\partial x^2}\frac{\partial^3\overline{\eta(x)}}{\partial x^3}-\frac{3}{2}C6\left(\frac{\partial\eta(x)}{\partial x}\right)^2\frac{\partial^2\overline{\eta(x)}}{\partial x^2}\right]\overline{\eta(x)}dx}{-\left((C21\bar{U}_1+C22\bar{U}_2)\int_0^1\frac{\partial\eta(x)}{\partial x}\overline{\eta(x)}dx+2i\lambda\int_0^1\eta(x)\overline{\eta(x)}dx\right)}$$

Where Q and S are complex numbers such that:

$$Q = Q^R + iQ^I and\ S = S^R + iS^I \tag{43}$$

To estimate the stability region for the axial vibration due to the principal parametric resonance, X (T1) is expressed in polar form as:

$$X(T1) = B(T1)e^{\left(\frac{T1\sigma_2 i}{2}\right)}\ and\ \overline{X(T1)} = \overline{B(T1)}e^{\left(\frac{-T1\sigma_2 i}{2}\right)} \tag{44}$$

Substituting equation (44) into equation (41)

$$\frac{dB(T1)}{dT1} + Q\overline{B(T1)} + \frac{\sigma_2\overline{X(T1)}i}{2} = 0 \tag{45}$$

With complex amplitudes expressed as;

$$b = b^R + ib^I \tag{46}$$

$$B(T1) = be^{\gamma T1}\ and\ \overline{B(T1)} = \bar{b}e^{\gamma T1} \tag{47}$$

Substituting (46) into (47), (47) into (45) and separate to real and imaginary components;

$$\begin{pmatrix}\gamma + Q^R & Q^I - \frac{\sigma_2}{2}\\ Q^I + \frac{\sigma_2}{2} & \gamma - Q^R\end{pmatrix}\begin{pmatrix}b^R\\ b^I\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \tag{48}$$

To find a non-trivial solution, the determinant of the matrix must varnish. Therefore;

$$\gamma = \pm\frac{\sqrt{4Q^{R^2}+4Q^{I^2}-\sigma_2{}^2}}{2} \tag{49}$$

Stable solutions require that $\gamma = 0$. Therefore, the stability boundaries can be expressed as:

$$\sigma_2 = \mp 2\sqrt{Q^{R^2} + Q^{I^2}} \tag{50}$$

Therefore; the stability regions can be expressed as:

$$\Omega1, \Omega2 = 2\omega \mp 2\varepsilon\sqrt{Q^{R^2} + Q^{I^2}} \tag{51}$$

3.5. When $\boldsymbol{\omega}$ is close to $\boldsymbol{2\lambda}$

However, to examine the coupled nonlinear dynamics of the system, which is the scenario when $\omega = 2\lambda$, another detuning parameter σ_1 is introduced.

$$\omega = 2\lambda + \varepsilon\sigma_1 \tag{52}$$

$$2\lambda T0 = \omega T0 - \sigma_1 T1\ and\ (\omega - \lambda)T0 = \lambda T0 + \sigma_1 T1 \tag{53}$$

The two equations (50) and (51) will have bounded solutions only if the solvability condition holds. The solvability condition demands that the coefficient of $\exp(i\omega T_0)$ and $\exp(i\lambda T_0)$ vanishes [17-19], that is, with $\varepsilon T0 = T1$, X (T1) and Y (T1) should satisfy the following relation:

$$-\Big(C21\frac{\partial X(T1)}{\partial T1}\frac{\partial\phi(x)}{\partial x}\bar{U}_1 + C22\frac{\partial X(T1)}{\partial T1}\frac{\partial\phi(x)}{\partial x}\bar{U}_2 + 2\phi(x)\omega i\frac{\partial X(T1)}{\partial T1}\Big) + Y(T1)^2\Big(\frac{\partial\eta(x)}{\partial x}\frac{\partial^4\eta(x)}{\partial x^4} + \frac{\partial^2\eta(x)}{\partial x^2}\frac{\partial^3\eta(x)}{\partial x^3} - C6\frac{\partial\eta(x)}{\partial x}\frac{\partial^2\eta(x)}{\partial x^2}\Big)exp(-T1\sigma_1 i) + \Big[C32\mu_2\frac{\partial^2\overline{\phi(x)}}{\partial x^2}exp(i\sigma_2 T_1)\,\bar{U}_2{}^2 i + \frac{1}{2}\Big(C21\mu_1\frac{\partial\overline{\phi(x)}}{\partial x}exp(i\sigma_2 T_1)\bar{U}_1\,\omega\Big) + \frac{1}{2}\Big(C22\mu_2\frac{\partial\overline{\phi(x)}}{\partial x}exp(i\sigma_2 T_1)\bar{U}_2\,\omega\Big) - \Big(C41\mu_1\frac{\partial\overline{\phi(x)}}{\partial x}exp(i\sigma_2 T_1)\bar{U}_1\,\omega\Big) - \Big(C42\mu_2\frac{\partial\overline{\phi(x)}}{\partial x}exp(i\sigma_2 T_2)\bar{U}_2\,\omega\Big) - \frac{1}{2}\Big(C41\varepsilon\sigma_2\mu_1\frac{\partial\overline{\phi(x)}}{\partial x}exp(i\sigma_2 T_1)\bar{U}_1\Big) - \frac{1}{2}\Big(C42\varepsilon\sigma_2\mu_2\frac{\partial\overline{\phi(x)}}{\partial x}exp(i\sigma_2 T_1)\bar{U}_2\Big) + C31\mu_1\frac{\partial^2\overline{\phi(x)}}{\partial x^2}exp(i\sigma_2 T_1)\,\bar{U}_1{}^2 i\Big]\overline{X(T1)} = 0 \tag{54}$$

$$\Big(-\frac{\partial Y(T1)}{\partial T1}\Big(C21\frac{\partial\eta(x)}{\partial x}\bar{U}_1 + C22\frac{\partial\eta(x)}{\partial x}\bar{U}_2 + 2\eta(x)\lambda i\Big) + 6Y(T1)^2\overline{Y(T1)}\left(\frac{\partial\eta(x)}{\partial x}\right)^2\frac{\partial\overline{\eta(x)}}{\partial x} + 2Y(T1)^2\overline{Y(T1)}\left(\frac{\partial\eta(x)}{\partial x}\right)^2\frac{\partial^4\overline{\eta(x)}}{\partial x^4} + 4Y(T1)^2\overline{Y(T1)}\frac{\partial\eta(x)}{\partial x}\frac{\partial\overline{\eta(x)}}{\partial x}\frac{\partial^4\eta(x)}{\partial x^4} + 8Y(T1)^2\overline{Y(T1)}\frac{\partial\eta(x)}{\partial x}\frac{\partial^2\overline{\eta(x)}}{\partial x^2}\frac{\partial^3\eta(x)}{\partial x^3} +$$

$$8Y(T1)^2\overline{Y(T1)}\frac{\partial\overline{\eta(x)}}{\partial x}\frac{\partial^2\eta(x)}{\partial x^2}\frac{\partial^3\eta(x)}{\partial x^3}-$$
$$3C6.Y(T1)^2\overline{Y(T1)}\frac{\partial\eta(x)}{\partial x}\frac{\partial\overline{\eta(x)}}{\partial x}\frac{\partial^2\eta(x)}{\partial x^2}+$$
$$8Y(T1)^2\overline{Y(T1)}\frac{\partial\eta(x)}{\partial x}\frac{\partial^2\eta(x)}{\partial x^2}\frac{\partial^3\overline{\eta(x)}}{\partial x^3}-$$
$$\frac{3}{2}C6.Y(T1)^2\overline{Y(T1)}\left(\frac{\partial\eta(x)}{\partial x}\right)^2\frac{\partial^2\eta(x)}{\partial x^2}\Bigg)+$$
$$\Bigg(2X(T1)\overline{Y(T1)}\frac{\partial\Phi(x)}{\partial x}\frac{\partial^4\overline{\eta(x)}}{\partial x^4}+X(T1)\overline{Y(T1)}\frac{\partial^4\Phi(x)}{\partial x^4}\frac{\partial\overline{\eta(x)}}{\partial x}+$$
$$4X(T1)\overline{Y(T1)}\frac{\partial^2\Phi(x)}{\partial x^2}\frac{\partial^3\overline{\eta(x)}}{\partial 3}+$$
$$3X(T1)\overline{Y(T1)}\frac{\partial^2\overline{\eta(x)}}{\partial 2}\frac{\partial^3\Phi(x)}{\partial 3}\Bigg)exp(T1\sigma_1 i)-$$
$$\Bigg(C6X(T1)\overline{Y(T1)}\frac{\partial\Phi(x)}{\partial x}\frac{\partial^2\overline{\eta(x)}}{\partial x^2}+$$
$$C6X(T1)\overline{Y(T1)}\frac{\partial\overline{\eta(x)}}{\partial x}\frac{\partial^2\Phi(x)}{\partial x^2}\Bigg)exp(T1\sigma_1 i)=0 \qquad (55)$$

With the inner product defined for complex functions on [0, 1] as:

$$\langle f,g\rangle=\int_0^1 f\bar{g}\,dx \qquad (56)$$

The equations can be cast as:

$$\frac{\partial X(T1)}{\partial T1}-J2\,Y(T1)^2exp(-i\sigma_1T_1)+J3\overline{X(T1)}\exp(-i\sigma_2T_1)=0 \qquad (57)$$

$$\frac{\partial Y(T1)}{\partial T1}+K2\left(Y(T1)^2\overline{Y(T1)}\right)+K3\left(X(T)\overline{Y(T1)}\exp(i\sigma_1T_1)\right)=0 \qquad (58)$$

Where:

$$J2=\frac{\int_0^1\left[\frac{\partial\eta(x)}{\partial x}\frac{\partial^4\eta(x)}{\partial x^4}+\frac{\partial^2\eta(x)}{\partial x^2}\frac{\partial^3\eta(x)}{\partial x^3}-C6\frac{\partial\eta(x)}{\partial x}\frac{\partial^2\eta(x)}{\partial x^2}\right]\overline{\phi(x)}dx}{\int_0^1\left[(C21\bar{U}_1+C22\bar{U}_2)\frac{d\phi(x)}{dx}+2i\omega\,\phi(x)\right]\overline{\phi(x)}dx}$$

$$J3=\frac{\int_0^1-\overline{\phi(x)}\left[C32\mu_2\frac{\partial^2\overline{\phi(x)}}{\partial x^2}\bar{U}_2{}^2i+\frac{1}{2}\left(C21\mu_1\frac{\partial\overline{\phi(x)}}{\partial x}\bar{U}_1\omega\right)\right]dx}{(C21\bar{U}_1+C22\bar{U}_2)\int_0^1\frac{\partial\phi(x)}{\partial x}\overline{\phi(x)}dx+2i\omega\int_0^1\phi(x)\overline{\phi(x)}dx}+$$
$$\frac{\int_0^1-\overline{\phi(x)}\left[\frac{1}{2}\left(C22\mu_2\frac{\partial\overline{\phi(x)}}{\partial x}\bar{U}_2\omega\right)-\left(C41\mu_1\frac{\partial\overline{\phi(x)}}{\partial x}\bar{U}_1\omega\right)\right]dx}{(C21\bar{U}_1+C22\bar{U}_2)\int_0^1\frac{\partial\phi(x)}{\partial x}\overline{\phi(x)}dx+2i\omega\int_0^1\phi(x)\overline{\phi(x)}dx}+$$
$$\frac{\int_0^1-\overline{\phi(x)}\left[-\left(C42\mu_2\frac{\partial\overline{\phi(x)}}{\partial x}\bar{U}_2\omega\right)-\frac{1}{2}\left(C41\varepsilon\sigma_2\mu_1\frac{\partial\overline{\phi(x)}}{\partial x}\bar{U}_1\right)\right]dx}{(C21\bar{U}_1+C22\bar{U}_2)\int_0^1\frac{\partial\phi(x)}{\partial x}\overline{\phi(x)}dx+2i\omega\int_0^1\phi(x)\overline{\phi(x)}dx}+$$
$$\frac{\int_0^1-\overline{\phi(x)}\left[-\frac{1}{2}\left(C42\varepsilon\sigma_2\mu_2\frac{\partial\overline{\phi(x)}}{\partial x}\bar{U}_2\right)+C31\mu_1\frac{\partial^2\overline{\phi(x)}}{\partial x^2}\bar{U}_1{}^2i\right]dx}{(C21\bar{U}_1+C22\bar{U}_2)\int_0^1\frac{\partial\phi(x)}{\partial x}\overline{\phi(x)}dx+2i\omega\int_0^1\phi(x)\overline{\phi(x)}dx}$$

$$K2=\frac{\int_0^1\left[6\left(\frac{\partial\eta(x)}{\partial x}\right)^2\frac{\partial\overline{\eta(x)}}{\partial x}+2\left(\frac{\partial\eta(x)}{\partial x}\right)^2\frac{\partial^4\overline{\eta(x)}}{\partial x^4}\right]\overline{\eta(x)}dx}{-\left((C21\bar{U}_1+C22\bar{U}_2)\int_0^1\frac{\partial\eta(x)}{\partial x}\overline{\eta(x)}dx+2i\lambda\int_0^1\eta(x)\overline{\eta(x)}dx\right)}+$$
$$\frac{\int_0^1\left[4\frac{\partial\eta(x)}{\partial x}\frac{\partial\overline{\eta(x)}}{\partial x}\frac{\partial^4\eta(x)}{\partial x^4}+8\frac{\partial\eta(x)}{\partial x}\frac{\partial^2\overline{\eta(x)}}{\partial x^2}\frac{\partial^3\eta(x)}{\partial x^3}\right]\overline{\eta(x)}dx}{-\left((C21\bar{U}_1+C22\bar{U}_2)\int_0^1\frac{\partial\eta(x)}{\partial x}\overline{\eta(x)}dx+2i\lambda\int_0^1\eta(x)\overline{\eta(x)}dx\right)}+$$
$$\frac{\int_0^1\left[8\frac{\partial\overline{\eta(x)}}{\partial x}\frac{\partial^2\eta(x)}{\partial x^2}\frac{\partial^3\eta(x)}{\partial x^3}-3C6\frac{\partial\eta(x)}{\partial x}\frac{\partial\overline{\eta(x)}}{\partial x}\frac{\partial^2\eta(x)}{\partial x^2}\right]\overline{\eta(x)}dx}{-\left((C21\bar{U}_1+C22\bar{U}_2)\int_0^1\frac{\partial\eta(x)}{\partial x}\overline{\eta(x)}dx+2i\lambda\int_0^1\eta(x)\overline{\eta(x)}dx\right)}+$$
$$\frac{\int_0^1\left[8\frac{\partial\eta(x)}{\partial x}\frac{\partial^2\eta(x)}{\partial x^2}\frac{\partial^3\overline{\eta(x)}}{\partial x^3}-\frac{3}{2}C6\left(\frac{\partial\eta(x)}{\partial x}\right)^2\frac{\partial^2\overline{\eta(x)}}{\partial x^2}\right]\overline{\eta(x)}dx}{-\left((C21\bar{U}_1+C22\bar{U}_2)\int_0^1\frac{\partial\eta(x)}{\partial x}\overline{\eta(x)}dx+2i\lambda\int_0^1\eta(x)\overline{\eta(x)}dx\right)}$$

$$K3=\frac{\int_0^1\left[2\frac{\partial\Phi(x)}{\partial x}\frac{\partial^4\overline{\eta(x)}}{\partial x^4}+4\frac{\partial^2\Phi(x)}{\partial x^2}\frac{\partial^3\overline{\eta(x)}}{\partial x^3}+\frac{\partial^4\Phi(x)}{\partial x^4}\frac{\partial\overline{\eta(x)}}{\partial x}\right]\overline{\eta(x)}dx}{-\left((C21\bar{U}_1+C22\bar{U}_2)\int_0^1\frac{\partial\eta(x)}{\partial x}\overline{\eta(x)}dx+2i\lambda\int_0^1\eta(x)\overline{\eta(x)}dx\right)}+$$
$$\frac{\int_0^1\left[3\frac{\partial^2\eta(x)}{\partial x^2}\frac{\partial^3\Phi(x)}{\partial x^3}-C6\frac{\partial\Phi(x)}{\partial x}\frac{\partial^2\overline{\eta(x)}}{\partial x^2}-C6\frac{\partial\overline{\eta(x)}}{\partial x}\frac{\partial^2\Phi(x)}{\partial x^2}\right]\overline{\eta(x)}dx}{-\left((C21\bar{U}_1+C22\bar{U}_2)\int_0^1\frac{\partial\eta(x)}{\partial x}\overline{\eta(x)}dx+2i\lambda\int_0^1\eta(x)\overline{\eta(x)}dx\right)}$$

To determine X(T1) and Y(T1), the solution of equations (57) and (58) is expressed in polar form:

$$Y(T1)=\frac{1}{2}\alpha y(T1)e^{i\beta y(T1)}\ \ and\ \ \overline{Y(T1)}=\frac{1}{2}\alpha y(T1)e^{-i\beta y(T1)} \qquad (59)$$

$$X(T1)=\frac{1}{2}\alpha x(T1)e^{i\beta x(T1)}\ \ and\ \ \overline{Y(T1)}=\frac{1}{2}\alpha x(T1)e^{-i\beta x(T1)} \qquad (60)$$

Substituting into the solvability condition and separating real and imaginary parts. The following set of modulation equation is formed:

$$0=\frac{d\alpha x(T1)}{dT1}+J3R\alpha x(T1)\cos(\psi 2)-J3I\alpha x(T1)\sin(\psi 2)-\frac{J2R\alpha y(T1)^2}{2}\cos(\psi 1)-\frac{J2I\alpha y(T1)^2}{2}\sin(\psi 1) \qquad (61)$$

$$0=\frac{d\alpha y(T1)}{dT1}+\frac{K2R\alpha y(T1)^3}{4}+\frac{K3R\alpha y(T1)\alpha x(T1)}{2}\cos(\psi 1)-\frac{K3I\alpha y(T1)\alpha x(T1)}{2}\sin(\psi 1) \qquad (62)$$

$$0=\alpha x(T1)\frac{d\beta x(T1)}{dT1}-\frac{J2I\alpha y(T1)^2}{2}\cos(\psi 1)+\frac{J2R\alpha y(T1)^2}{2}\sin(\psi 1)+J3I\alpha x(T1)\cos(\psi 2)-J3R\alpha x(T1)\sin(\psi 2) \qquad (63)$$

$$0=\alpha y(T1)\frac{d\beta y(T1)}{dT1}+\frac{K2I\alpha y(T1)^3}{4}+\frac{K3I\alpha y(T1)\alpha x(T1)}{2}\cos(\psi 1)+\frac{K3R\alpha y(T1)\alpha x(T1)}{2}\sin(\psi 1) \qquad (64)$$

Where:

$$\psi 1=\beta x(T1)-2\beta y(T1)+\sigma_1 T1$$

and $\psi 2=\sigma_2 T1-2\beta x(T1)$

J2R, J3R, K2R, K3R are the real part of J2, J3, K2, and K3

J2I, J3I, K2I, K3I are the imaginary part of J2, J3, K2, and K3

Seeking for stationary solutions, $\alpha(x)'=\alpha(y)'=\psi 1'=\psi 2'=0$ in modulation equations (61) to (64),

$$0=2J3R\alpha x(T1)\cos(\psi 2)-2J3I\alpha x(T1)\sin(\psi 2)-J2R\alpha y(T1)^2\cos(\psi 1)-J2I\alpha y(T1)^2\sin(\psi 1) \qquad (65)$$

$$0=\frac{K2R\alpha y(T1)^3}{4}+\frac{K3R\alpha y(T1)\alpha x(T1)}{2}\cos(\psi 1)-\frac{K3I\alpha y(T1)\alpha x(T1)}{2}\sin(\psi 1) \qquad (66)$$

$$0=\alpha x(T1)\sigma_2-J2I\alpha y(T1)^2\cos(\psi 1)+J2R\alpha y(T1)^2\sin(\psi 1)+2J3I\alpha x(T1)\cos(\psi 2)-2J3R\alpha x(T1)\sin(\psi 2) \qquad (67)$$

$$0=\alpha y(T1)\left(\frac{2\sigma_1+\sigma_2}{4}\right)+\frac{K2I\alpha y(T1)^3}{4}+\frac{K3I\alpha y(T1)\alpha x(T1)}{2}\cos(\psi 1)+\frac{K3R\alpha y(T1)\alpha x(T1)}{2}\sin(\psi 1) \qquad (68)$$

The linear solutions can be obtained by setting the coefficient of the nonlinear terms to zero. Therefore,

$$\alpha x(T1) = \alpha y(T1) = 0 \quad (69)$$

The nonlinear solutions can be obtained by solving for αx(T1) and αy(T1) completely. Equation (62) and (64) can be rewritten as:

$$-\left(\frac{K2R\alpha y(T1)^2}{2}\right) = R1\cos(\psi 1 + \theta 1) \quad (70)$$

$$-\left(\frac{K2I\alpha y(T1)^2}{2} + \sigma_1 + \frac{\sigma_2}{2}\right) = R1\sin(\psi 1 + \theta 1) \quad (71)$$

Where,

$$\tan(\theta 1) = \frac{K3I}{K3R}, \quad (72)$$

$$R1 = \sqrt{\alpha x(T1)^2(K3I^2 + K3R^2)}) \quad (73)$$

From,

$$(\sin(\psi 1 + \theta 1))^2 + (\cos(\psi 1 + \theta 1))^2 = 1$$

We obtained;

$$K2I^2\alpha y(T1)^4 + 4K2I^2\sigma_1\alpha y(T1)^2 + 2K2I^2\sigma_2\alpha y(T1)^2 + K2R^2\alpha y(T1)^4 + 4{\sigma_1}^2 + 4\sigma_1\sigma_2 + {\sigma_2}^2 - 4\alpha x(T1)^2(K3I^2 + K3R^2) = 0 \quad (75)$$

And:

$$\psi 1 = \tan^{-1}\left[\frac{K2I\alpha y(T1)^2 + 2\sigma_1 + \sigma_2}{K2R\alpha y(T1)^2}\right] - \tan^{-1}\left(\frac{K3I}{K3R}\right)$$

Also, equations (61) and (63) can be rewritten as:

$$\left(\frac{J2R\alpha y(T1)^2\cos(\psi 1) + J2I\alpha y(T1)^2\sin(\psi 1)}{2}\right) = R2\cos(\psi 2 + \theta 2) \quad (77)$$

$$-\left(\frac{\alpha x(T1)^2\sigma_2}{2} - \frac{J2I\alpha y(T1)^2\cos(\psi 1)}{2} + \frac{J2R\alpha y(T1)^2\sin(\psi 1)}{2}\right) = R2\sin(\psi 2 + \theta 2) \quad (78)$$

Where:

$$\tan(\theta 2) = \frac{J3I}{J3R}, \quad (79)$$

$$R2 = \sqrt{\alpha x(T1)^2(J3I^2 + J3R^2)} \quad (80)$$

Transforming the equations (77) and (78) to;

$$\left(\frac{J2R\,G2 + J2I\,G1}{2}\right) = R2\cos(\psi 2 + \theta 2) \quad (81)$$

$$-\left(\frac{\alpha x(T1)^2\sigma_2}{2} - \frac{J2I\,G2}{2} + \frac{J2R\,G1}{2}\right) = R2\sin(\psi 2 + \theta 2) \quad (82)$$

Where:

$$G1 = \frac{2\sigma_1\cos(\theta 1) + \sigma_2\cos(\theta 1) + K2I\alpha y(T1)^2\cos(\theta 1) - K2R\alpha y(T1)^2\sin(\theta 1)}{K2R\sqrt{\frac{K2I^2\alpha y(T1)^4 + 4K2I\sigma_1\alpha y(T1)^2 + 2K2I\sigma_2\alpha y(T1)^2 + K2R^2\alpha y(T1)^4 + 4{\sigma_1}^2 + 4\sigma_1\sigma_2 + {\sigma_2}^2}{K2R^2\alpha y(T1)^4}}}$$

$$G2 = \frac{2\sigma_1\sin(\theta 1) + \sigma_2\sin(\theta 1) + K2I\alpha y(T1)^2\cos(\theta 1) + K2R\alpha y(T1)^2\sin(\theta 1)}{K2R\sqrt{\frac{K2I^2\alpha y(T1)^4 + 4K2I\sigma_1\alpha y(T1)^2 + 2K2I\sigma_2\alpha y(T1)^2 + K2R^2\alpha y(T1)^4 + 4{\sigma_1}^2 + 4\sigma_1\sigma_2 + {\sigma_2}^2}{K2R^2\alpha y(T1)^4}}}$$

From;

$$(\sin(\psi 2 + \theta 2))^2 + (\cos(\psi 2 + \theta 2))^2 = 1 \quad (83)$$

We have:

$$G1^2(J2I^2 + J2R^2) + 2G1J2R\sigma_2\alpha x(T1) + G2^2(J2I^2 + J2R^2) - 2G2J2I\sigma_2\alpha x(T1) + {\sigma_2}^2\alpha x(T1)^2 - 4\alpha x(T1)^2(J3I^2 + J3R^2) = 0 \quad (84)$$

Resolving equations (75) and (84) for real and positive values of $\alpha y(T1) and\ \alpha x(T1)$.

Consequently,

$$\psi 2 = -\tan^{-1}\left(\frac{J3I}{J3R}\right) - \tan^{-1}\left[\frac{\alpha x(T1)^2\sigma_2 - J2I\alpha y(T1)^2\cos(\psi 1) + J2R\alpha y(T1)^2\sin(\psi 1)}{J2R\alpha y(T1)^2\cos(\psi 1) + J2I\alpha y(T1)^2\sin(\psi 1)}\right] \quad (74)$$

Therefore, considering the n-th values of αx(T1), αy(T1), βx(T1) and βy(T1) corresponding to the n-th modal functions and the n-th natural frequencies, the n-th solution of the coupled problem is expressed as:

$$\bar{u}(x,t)_n = \alpha x(T1)_n\phi(x)_n\,cos(\omega_n T0 + \beta x(T1)_n) + O(\varepsilon) \quad (86)$$

Substituting into the equations (86);

$$T0 = t,\ T1 = \varepsilon t, \alpha x(T1)_n = \alpha x_n,\ \alpha y(T1)_n = \alpha y_n,\ \beta y(T1)_n = \frac{\beta x(T1)_n - \psi 1_n + \sigma_{1n}T1}{2},\ \beta x(T1)_n = \frac{\sigma_{2n}T1 - \psi 2_n}{2},\ \Omega 1 = 2\omega_n + \varepsilon\sigma_{2n},\ \Omega 2 = 2\omega_n + \varepsilon\sigma_{2n},\ \Omega 1 = \Omega 2 = \Omega\ and\ \sigma_{1n}T1 = \omega_n T0 - 2\lambda_n T0 \quad (76)$$

With the solvability condition fulfilled, the particular solution of equation (35) for the internal resonance condition is obtained as:

$$u_1 = \mathrm{C}1\alpha x(T_1)\,cos\big(\beta y(T_1) + T_0(\omega + \Omega)\big) + C2\alpha x(T_1)\,cos\big(\beta x(T_1) + T_0(\omega - \Omega)\big) + 2\mathrm{C}3\,cos(T_0\Omega) \quad (87)$$

Where:

$$C1 = \frac{\frac{C21\mu_1\frac{d\phi(x)}{dx}\bar{U}_1\omega_n}{2} - \frac{C41\Omega_1\mu_1\frac{d\phi(x)}{dx}\bar{U}_1}{2} - C31\frac{d^2\phi(x)}{dx^2}(\bar{U}_1)^2 i}{\int_0^{Le}\overline{\phi(x)}\left(C22\Omega\frac{d\phi(x)}{dx}\bar{U}_2 i + C21\Omega\frac{d\phi(x)}{dx}\bar{U}_1 i\right)dx - \Omega^2 + {\omega_n}^2[C5 - C32(\bar{U}_2)^2 - C31(\bar{U}_1)^2]} + \frac{\frac{C22\mu_2\frac{d\phi(x)}{dx}\bar{U}_1\omega_n}{2} - \frac{C42\Omega_2\mu_2\frac{d\phi(x)}{dx}\bar{U}_2}{2} - C32\mu_2\frac{d^2\phi(x)}{dx^2}(\bar{U}_2)^2 i}{\int_0^{Le}\overline{\phi(x)}\left(C22\Omega\frac{d\phi(x)}{dx}\bar{U}_2 i + C21\Omega\frac{d\phi(x)}{dx}\bar{U}_2 i\right)dx - \Omega^2 + {\omega_n}^2[C5 - C32(\bar{U}_2)^2 - C31(\bar{U}_1)^2]}$$

$$C2 = \frac{\frac{C21\mu_1\frac{d\phi(x)}{dx}\bar{U}_1\omega_n}{2} - C32\mu_2\frac{d^2\phi(x)}{dx^2}(\bar{U}_2)^2 i + \frac{C22\mu_2\frac{d\phi(x)}{dx}\bar{U}_2\omega_n}{2}}{\int_0^{Le}\overline{\phi(x)}\left(C22\Omega\frac{d\phi(x)}{dx}\bar{U}_2 i + C21\Omega\frac{d\phi(x)}{dx}\bar{U}_2 i\right)dx - \Omega^2 + {\omega_n}^2[C5 - C32(\bar{U}_2)^2 - C31(\bar{U}_1)^2]} - \frac{\frac{C41\Omega_1\mu_1\frac{d\phi(x)}{dx}\bar{U}_1}{2} - \frac{C42\Omega_2\mu_2\frac{d\phi(x)}{dx}\bar{U}_2}{2} - C31\mu_1\frac{d^2\phi(x)}{dx^2}(\bar{U}_1)^2 i}{\int_0^{Le}\overline{\phi(x)}\left(C22\Omega\frac{d\phi(x)}{dx}\bar{U}_2 i + C21\Omega\frac{d\phi(x)}{dx}\bar{U}_1 i\right)dx - \Omega^2 + {\omega_n}^2[C5 - C32(\bar{U}_2)^2 - C31(\bar{U}_1)^2]}$$

$$C3 = \frac{\frac{1}{2}C11\Omega_1\mu_1\bar{U}_1 + \frac{1}{2}C12\Omega_2\mu_2\bar{U}_2}{\int_0^{Le}\overline{\phi(x)}\left(C22\Omega\frac{d\phi(x)}{dx}\bar{U}_2 i + C21\Omega\frac{d\phi(x)}{dx}\bar{U}_1 i\right)dx - \Omega^2 + {\omega_n}^2[C5 - C32(\bar{U}_2)^2 - C31(\bar{U}_1)^2]}$$

The first order approximate solution is expressed as:

$$\bar{u}(x,t) = \sum_{\mathrm{n}=1}^{\infty}\alpha x_n|\phi(x)_n|\,cos\left(\frac{t\Omega}{2} - \frac{\psi 2_n}{2} + \varphi x_n\right) +$$

$O(\varepsilon)$ (88)

Where the phase angles φx_n are given by:

$$\tan(\varphi x_n) = \frac{Im\{\phi(x)_n\}}{Re\{\phi(x)_n\}},$$

4. Results and Discussion

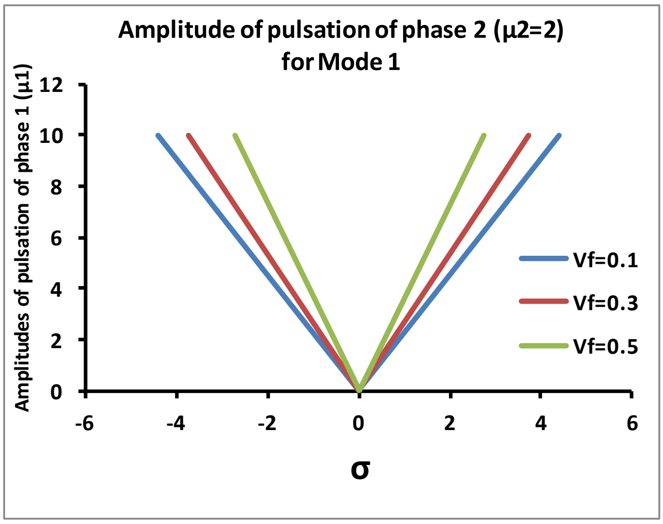
This section presents the numerical solutions of the nonlinear dynamics of a cantilever pipe, conveying steady pressurized air/water two-phase flow.

Table 1: Summary of pipe and flow parameter

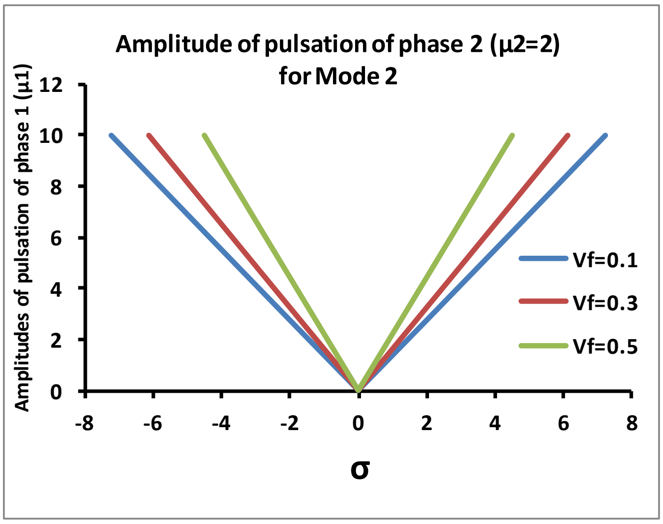
Parameter Name	Parameter Unit	Parameter Values
External Diameter	D_o (m)	0.0113772
Internal Diameter	D_i (m)	0.00925
Length	L (m)	0.1467
Pipe density	ρ_{pipe} (kg/m^3)	7800
Gas density	ρ_{Gas} (kg/m^3)	1.225
Water density	ρ_{Water} (kg/m^3)	1000
Tensile and compressive stiffness	EA (N)	7.24E+06
Bending stiffness	EI (N)	1.56E+03

4.1. Results for ω is far from 2λ (Uncoupled axial and transverse vibration)

Numerical examples are presented for the first two modes to examine the effects of the variation in the void fraction of the conveyed two-phase flow on the parametric stability boundaries based on equation (51).



(a)



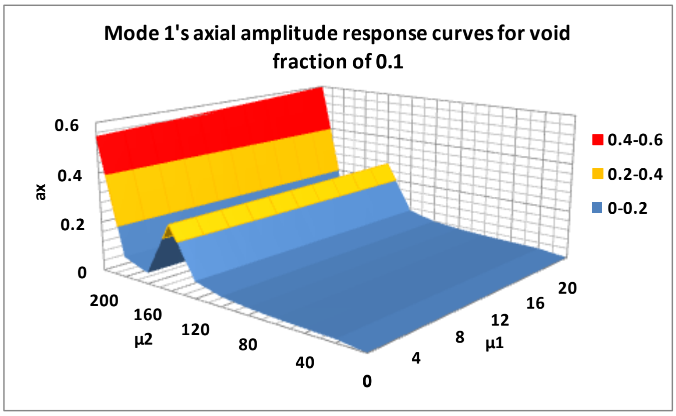
(b)

***Figure 2**: Parametric stability boundaries of mode 1 and mode 2 for varying void fractions.*

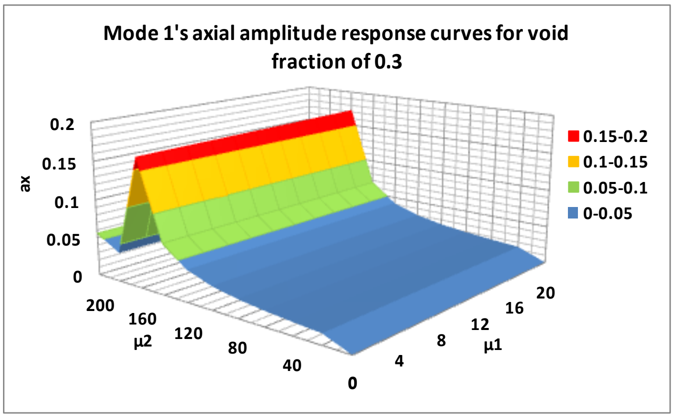
In Figures 2a and 2b, stable and unstable boundaries are plotted for the parametric resonance cases for the first and second mode for three different void fractions. In all the figures, the regions between the boundaries are unstable while other areas are stable. The stability boundaries are wider for the second mode as compared to the first mode. However, for the first and second modes, Increase in the void fraction is observed to reduce the stability boundaries.

4.2. Results for ω is close to 2λ (Internal resonance case)

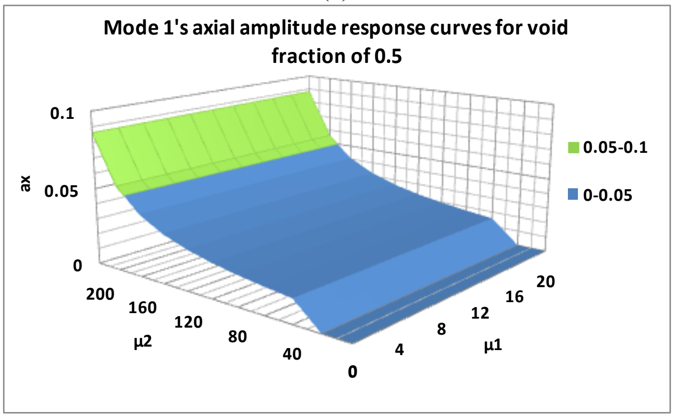
The internal resonance case presents a coupling between the axes which results in the transference of energy between the axes. For varying amplitude of pulsation for the two phases, both phases slightly detuned by 0.2 from the axial frequency and the axial and transverse frequencies also detuned by 0.2. The amplitude response curves are plotted for various void fractions.



(a)

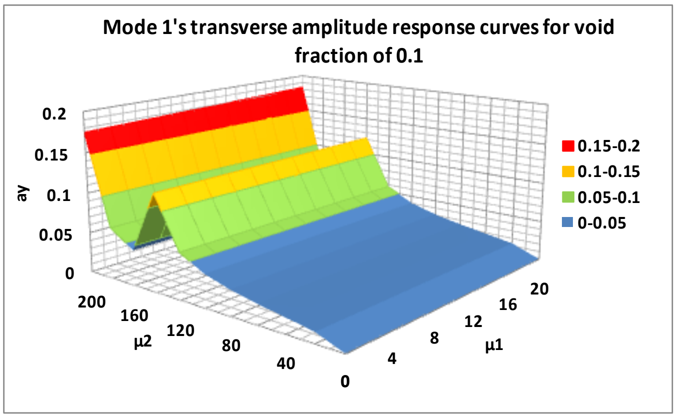


(b)

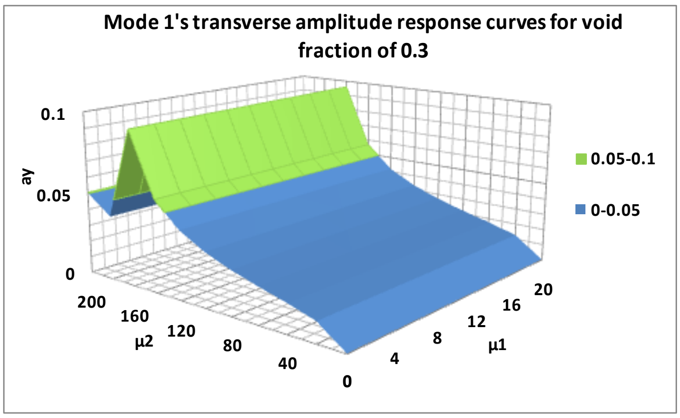


(c)

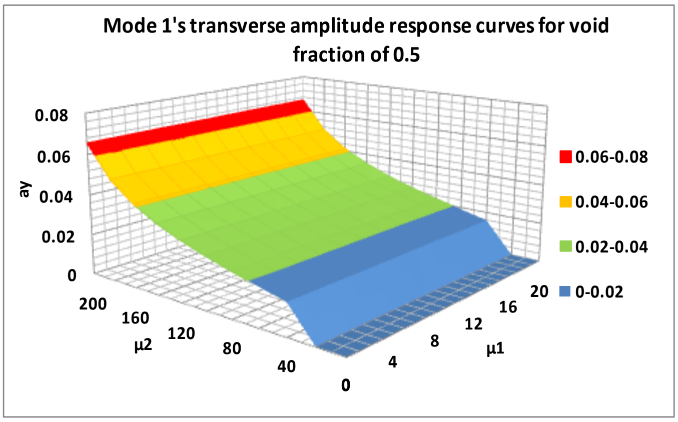
Figure 3: Amplitude response curves of the tip's axial vibration for mode 1



(a)

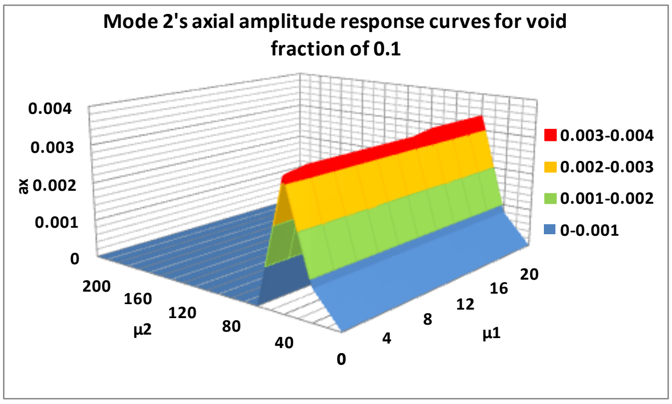


(b)

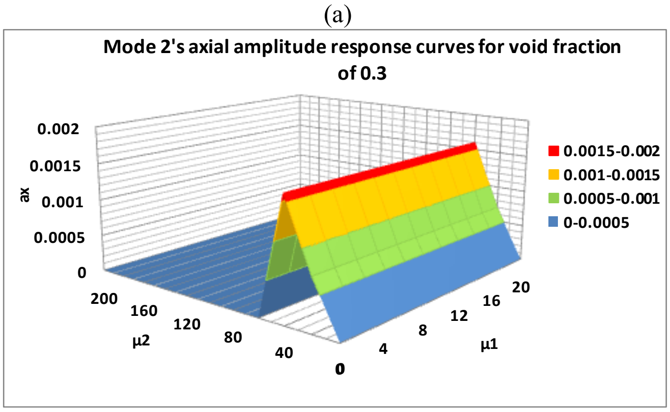


(c)

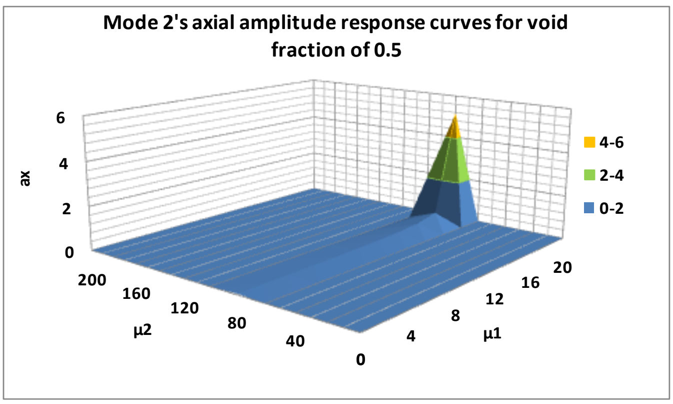
Figure 4: Amplitude response curves of the tip's transverse vibration for mode 1



(a)

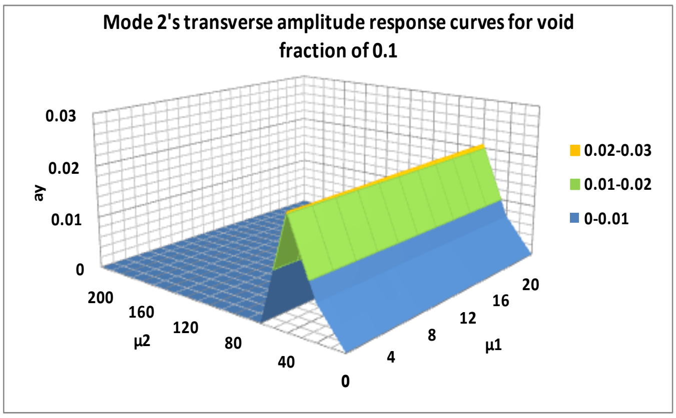


(b)

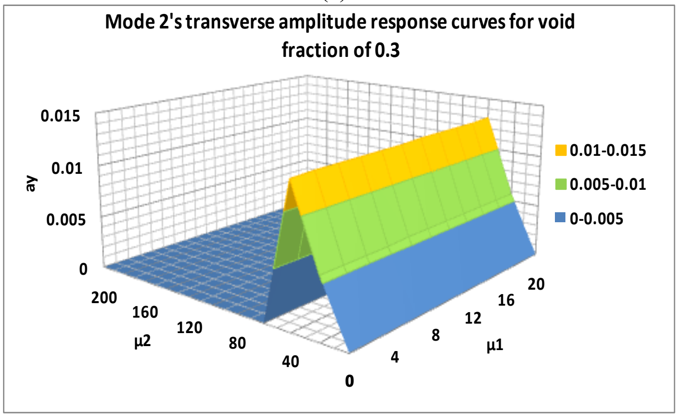


(c)

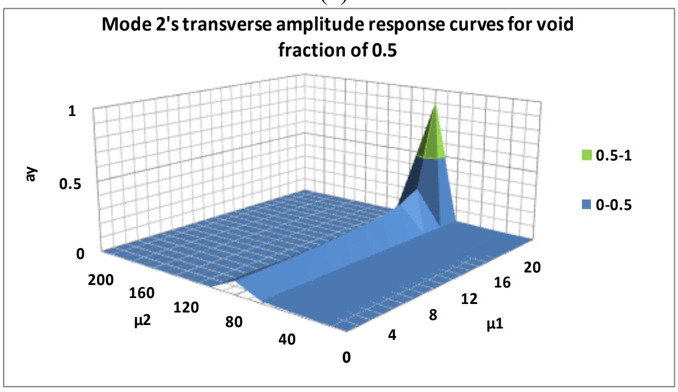
Figure 5: Amplitude response curves of the tip's axial vibration for mode 2



(a)



(b)



(c)

Figure 6: Amplitude response curves of the tip's transverse vibration for mode 2

As a result of the internal coupling between the planar axis, energy is transferred, oscillation is observed in both the axial and transverse directions as a result of the parametric axial vibrations. Figures 2, 3, 4, 5 and 6 shows that for both mode 1 and mode 2, an increase in the void fraction is observed to reduce the amplitude of oscillations due to increasing in mass content in the pipe and which further dampens the motions of the pipe. However, at a high void fraction of 0.5, Figures 5c and 6c show the occurrence of a resonance peak in the second mode in both axial and transverse oscillations.

5. Conclusion

In this study, the axial vibrations of a cantilevered pipe have been investigated. The velocity is assumed to be harmonically varying about a mean value. The method of multiple scales is applied to the equation of motion to determine the velocity-dependent frequencies and the study of the parametric resonance behavior of the system. Away from the internal resonance condition, the influence of small fluctuations of flow velocities of the phases on the stability of the system is examined. The boundaries separating stable and unstable regions are estimated and it was observed that an increase in the void fraction reduces the stability boundaries. With internal resonance, transverse oscillations will also be generated due to the transfer of energy from the resonated axial vibrations. However, an increase in void fraction dampens the motions of the pipe.

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