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On the outer-connected reinforcement and bondage problems in bipartite graphs: the algorithmic complexity

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ABSTRACT

An outer connected dominating(OCD) set of a graph $G = (V, E)$ is a set $\tilde{D} \subseteq V$ such that every vertex not in S is adjacent to a vertex in S, and the induced subgraph of G by $V \setminus \tilde{D}$, i.e. $G[V \setminus \tilde{D}]$, is connected. The OCD number of G is the smallest cardinality of an OCD set of G. The outer-connected bondage number of a nonempty graph G is the smallest number of edges whose removal from G results in a graph with a larger OCD number. Also, the outer-connected reinforcement number of G is the smallest number of edges whose addition to G results in a graph with a smaller OCD number. In 2018, Hashemi et al. demonstrated that the decision problems for the Outer-Connected Bondage and the

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1 Abstract continued

Outer-Connected Reinforcement numbers are all NP-hard in general graphs. In this paper, we improve these results and show their hardness for bipartite graphs. Also, we obtain bounds for the outer-connected bondage number.

2 Introduction

The set of terminology and notation in graph theory used in this paper follows the reference Xu [22]. Let $G = (V, E)$ be a simple undirected graph with vertex set V and edge set E. The degree of a vertex $x \in V$ is denoted by $d(x)$ and equals the number of its adjacent vertices. For every vertex $v \in V$, $N_G(v)$ is the *open neighborhood* of v which is defined as $N_G(v) = \{u \in V : \{u, v\} \in E(G)\}$. Similarly, the *closed neighborhood* of v, $N_G[v]$, is defined as $N_G[v] = N_G(v) \cup \{v\}$. A subset $S \subseteq V$ is a *dominating set* of G if every vertex in $V \setminus S$ has at least one neighbor in S. The minimum cardinality in all dominating sets is called the *domination number* of the graph G and is denoted by $\gamma(G)$. Moreover, a dominating set of cardinality $\gamma(G)$ is called a $\gamma - set$. Due to the widespread applications of dominating sets and its variants, it is one of the most studied topics in pure and applied mathematics, e.g. [5, 10, 20] and references therein.
In this paper, the *outer-connected domination*, as a variant of dominating sets is studied. Cyman in [4] introduced the notion of the OCD sets. Akhbari et. al. studied properties of OCD and obtained some bounds for OCD in [1]. Keil and Pardhan in [14] studied the problem of obtaining an OCD set for chordal graphs. A dominating set $\tilde{D} \subseteq V$ for a graph $G = (V, E)$ is called an *outer-connected dominating set* if the induced subgraph $G[V \setminus \tilde{D}]$ is connected. the outer-connected domination number of G, $\tilde{\gamma}_c(G)$, is the minimum cardinality among all outer-connected dominating sets of G [4]. An outer-connected dominating set $\tilde{D}$ is called a $\tilde{\gamma} - set$ of G if $|\tilde{D}| = \tilde{\gamma}_c(G)$. The outer-connected domination problem seeks to obtain a $\tilde{\gamma} - set$ of G which is shown to be **NP**-complete for arbitrary graphs by Cyman in [4].
There are many applications for the OCD problem in computer networks such as the following scenario: in a client-server network, each client needs to be directly connected to at least one server, in addition of being able to transmit data to other clients without interrupting any servers. Mathematically, such a network topology is an OCD. Finding a minimum cost network topology with the best number of required servers is equivalent to solving the OCD problem [17].
In this paper, we investigate the effect of the removal and addition of edges on the OCD number. The least number of edges whose removal from G causes an increase in the domination number of the graph is called the *bondage number* of G, introduced by Fink et al. in [6], and is denoted by $b(G)$. The least number of edges whose addition to G causes a decrease in the domination number of the graph is called the *reinforcement number* of G, introduced by Kok and Mynhardt in [15], and is denoted by $r(G)$. Moreover, Huang et al. in [13] studied the reinforcement number for direct graphs. Recently, Xu in [23] gave a review article on the bondage numbers. The **NP**-hardness of the reinforcement,

the bondage, the total reinforcement, and the total bondage problems is shown in the general case by Hu and Xu in [12]. Also, Hu and Sohn in [11] proved that these problems for bipartite graphs are NP-complete. Hattingh et al. in [9] showed that the problem of the restrained bondage is **NP**-complete, even for bipartite graphs. Also, for several classes of graphs, they have determined the exact values of the bondage number. Lu and et al. in [16] studied the complexity of paired bondage and p-reinforcement problems in general graphs. Jafari Rad in [19] showed that the problems of the independent bondage, the total restrained bondage, the k-rainbow bondage, and the paired bondage numbers are all **NP**-hard, even if they are restricted to bipartite graphs. Also, he in [18] showed that the problems of the p-reinforcement, the p-total reinforcement, the total restrained reinforcement, and the k-rainbow reinforcement are all **NP**-hard for bipartite graphs. Amjadi et al. in [2] initiated the study of the restrained k-rainbow bondage number in graphs and presented some sharp bounds for k-rainbow bondage number. Vaidya and Parner in [21] introduced the concept of total equitable bondage number and proved several results for it. A linear time algorithm is proposed by Hartnell et al. in [7] to compute the bondage number for a tree.

The least number of edges whose removal from G causes an increase in the OCD number of the graph is called the *outer-connected bondage number* of G, introduced by Hashemi et al. in [8], and is denoted by $b_{OCD}(G)$. The least number of edges whose addition to G causes a decrease in the domination number of the graph is called the *outer-connected reinforcement number* of G, introduced by Hashemi et al. in [8], and is denoted by $r_{OCD}(G)$. They showed these problems **NP**-hard for general graphs. Also, they have determined the exact values of the outer-connected bondage number for several classes of graphs.

The classes of median graphs, partial cubes, hypercube graphs, and grid graphs are examples of bipartite graph classes [11]. The reinforcement and the bondage problems for such graph classes are widely studied. So, we concentrate on the algorithmic complexity of the bondage and reinforcement problems in bipartite graphs and show that the Outer-Connected Bondage and the Outer-Connected Reinforcement problems are **NP**-hard for bipartite graphs, i.e. there are not polynomial-time algorithms to answer these problems unless $\mathbf{P} = \mathbf{NP}$.

The rest of the paper is organized as follows: In Section 2, we remind the 3 − Sat problem. The proofs for the hardness of the Outer-Connected Bondage and the Outer-Connected Reinforcement problems in bipartite graphs are given in Sections 3 and 4, respectively. Finally, in Section 5, we obtain bounds for outer-connected bondage number in general graphs.

3 3-Satisfiability Problem

Let $\mathcal{U} = \{u_1, u_2, \ldots, u_n\}$ be a set of boolean variables. A *clause* over $\mathcal{U}$ is the disjunction of a set of literals where each literal is either u_i or $\bar{u}_i$ for $1 \leq i \leq n$. A clause is satisfied with respect to a truth assignment if and only if at least one of its literals is true with respect to that truth assignment. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be a collection of clauses over $\mathcal{U}$. The objective of SAT problem is to determine whether there exists a truth assignment of boolean variables $\{u_1, u_2, \ldots, u_n\}$ such that all the clauses C_j for $1 \leq j \leq m$ are satisfied. Such a truth assignment, if exists, is called a satisfying truth assignment. Given these notations, the $3-\mathrm{SAT}$ problem is defined as follows.

$3-$ **SAT Problem.**
Input instance: A collection $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ of clauses over a finite set of variables $\mathcal{U}$ such that $|C_j| = 3$ for $1 \leq j \leq m$.
Question: Is there a satisfying truth assignment for $\mathcal{C}$?

Theorem 3.1. *The* $3-\mathrm{SAT}$ *problem is* **NP**-*complete [3].*

4 Complexity of the Outer-Connected Reinforcement Problem for Bipartite Graphs

In this section, the outer-connected reinforcement of bipartite graphs defined as follow, is shown to be an **NP**-hard problem.

Bi-Outer-Connected Reinforcement Problem.
Input instance: A positive integer k and a bipartite graph G with no isolated vertices.
Question: Does $r_{BOCD}(G) \leq k$ hold?

Theorem 4.1. *The* Bi-Outer-Connected Reinforcement *problem is* **NP**-*hard.*

Proof. We provide a polynomial time reduction from the $3-\mathrm{SAT}$ problem to the Bi-Outer-Connected Reinforcement problem to show its **NP**-hardness. Let $I = (\mathcal{U} = \{u_1, u_2, \ldots, u_n\}, \mathcal{C} = \{C_1, C_2, \ldots, C_m\})$ be a $3-\mathrm{SAT}$ problem instance. We construct a bipartite graph G such that for an arbitrary positive integer k, this instance of the $3-\mathrm{SAT}$ is satisfiable if and only if G has a Bi-outer-connected reinforcement of cardinality at most k, i.e. $r_{BOCD}(G) \leq k$. Next, we describe the construction of G.
We add a set of vertices $\mathcal{R}_i = \{u_i, v_i, \bar{u}_i, n_i, w_i, m_i\}$ as well as edges $(v_i, u_i), (v_i, \bar{u}_i)$,$(m_i, u_i), (n_i, \bar{u}_i)$, $(m_i, w_i), (n_i, w_i)$ in G in correspondence to each variable $u_i \in \mathcal{U}$ for $1 \leq i \leq n$. Also, for each clause C_j, a single vertex c_j is added to G as well as edge $\{c_j, u_i\}(\{c_j, \bar{u}_i\})$ if the literal $u_i(\bar{u}_i)$ appears in the clause C_j, for $1 \leq j \leq m$. Moreover, a path $P = x, y, t, r$ is added to G and the vertices t and x are connected to every vertex c_j by adding edges. Finally, we add edges (y, u_i), $(y, \bar{u}_i)$ and (y, w_i) to G for $1 \leq i \leq n$ and set $k = 1$.
This construction is applicable in polynomial time (which is easy to verify). Next we show that $r_{BOCD}(G) \leq k$ if and only if $I = (\mathcal{C}, \mathcal{U})$ is satisfiable by considering these three lemmas.

Lemma 4.2. *For any graph G constructed as above, we have $\tilde{\gamma}_c(G) = 2n + 2$.*

Proof. Let $\tilde{D}$ be a $\tilde{\gamma}$-set of G. It is clear that $|\tilde{D} \cap V(\mathcal{R}_i)| \geq 2$ for $1 \leq i \leq n$, $|\tilde{D} \cap N[r]| \geq 1$ and $|\tilde{D} \cap N[x]| \geq 1$. Then, we have $\tilde{\gamma}_c(G) = |\tilde{D}| \geq 2n + 2$. On the other hand, $\tilde{D}' = \{x, r, u_1, n_1, u_2, n_2, \ldots, u_n, n_n\}$ is an OCD set for G, which implies $\tilde{\gamma}_c(G) \leq |\tilde{D}'| = 2n + 2$. Thus, we have $\tilde{\gamma}_c(G) = 2n + 2$. □

Lemma 4.3. *Let $e \in E(\bar{G})$ be an edge where $\tilde{\gamma}_c(G + e) = 2n + 1$ and $\tilde{D}_e$ be a $\tilde{\gamma}$-set for $G + e$. Then, $|\tilde{D}_e \cap V(\mathcal{R}_i)| = 2$ for $1 \leq i \leq n$, while $c_j, x, y, t \notin \tilde{D}_e$ for $1 \leq j \leq m$. Moreover, exactly one of the vertices u_i or $\bar{u}_i$ are in $\tilde{D}_e$ and $r \in \tilde{D}_e$.*

Proof. Since the connection between the vertices in $V \setminus \tilde{D}_e$ is due to the vertex y, then $y \notin \tilde{D}_e$. Also, t is a cut-vertex which implies $t \notin \tilde{D}_e$. Hence, we have $r \in \tilde{D}_e$. On the contrary, suppose that $|\tilde{D}_e \cap V(\mathcal{R}_\ell)| < 2$ for some $1 \leq \ell \leq n$. Without loss of generality, suppose that $w_\ell \in \tilde{D}_e$. Since v_ℓ needs to be dominated by $\tilde{D}_e$ and $v_\ell, u_\ell, \bar{u}_\ell \notin \tilde{D}_e$, then, v_ℓ is dominated via the edge e in $G + e$ by the set $\tilde{D}_e$. So, one of the end-vertices of the edge e should be v_ℓ. Also, for every $i \neq \ell$, we have $|\tilde{D}_e \cap V(\mathcal{R}_i)| \geq 2$, because the set $\tilde{D}_e$ dominates all the vertices v_i. It is clear that u_ℓ and $\bar{u}_\ell$ are not in the same clause simultaneously. So, for no j, the vertex c_j is adjacent to both of them. because u_ℓ and $\bar{u}_\ell$ needs to be dominated by $\tilde{D}_e$, there are two vertices $c_j \neq c_\ell \in \tilde{D}_e$ such that c_j and c_ℓ dominate u_ℓ and $\bar{u}_\ell$. Thus, we have $|\tilde{D}_e| \geq 2n + 2$ which is a contradiction. So, $|V(\mathcal{R}_i) \cap \tilde{D}_e| = 2$ for $1 \leq i \leq n$ and $c_j, x \notin \tilde{D}_e$ for all j, because $|\tilde{D}_e| = 2n + 1$. □

Lemma 4.4. *$r_{BOCD}(G) = 1$ if and only if the $3-$SAT instance $I = (\mathcal{C}, \mathcal{U})$ is satisfiable.*

Proof. Let $f : \mathcal{U} \to \{T, F\}$ be a truth assignment which satisfies $\mathcal{C}$. Also, suppose that $\tilde{D}'$ is a subset of $V(G)$ constructed as follows. The vertices u_i, n_i and r are added to $\tilde{D}'$ if $f(u_i) = T$. If $f(u_i) = F$, then we put the vertices $\bar{u}_i, m_i, r$ in $\tilde{D}'$. Therefore, we have $|\tilde{D}'| = 2n+1$. At least one of the literals in the clause C_j is true under the assignment f for $1 \leq j \leq m$, since f is a satisfying truth assignment for $I = (\mathcal{C}, \mathcal{U})$. So, by the construction of graph G, the vertex c_j in G is adjacent to at least one vertex in $\tilde{D}'$. Without loss of generality, let $f(u_1) = T$. Hence, $\tilde{D}'$ is a dominating set for $G + \{x, u_1\}$. On the other hand, the induced graph $G[V \setminus \tilde{D}']$ is connected. Hence, $\tilde{D}'$ is an OCD set for $G + \{x, u_1\}$ and $\tilde{\gamma}_c(G + \{x, u_1\}) \leq |\tilde{D}'| = 2n + 1$.
By Lemma 4.2, we have $\tilde{\gamma}_c(G) = 2n+2$. Therefore, we obtain $\tilde{\gamma}_c(G+\{x, u_1\}) \leq 2n+1 < 2n + 2 = \tilde{\gamma}_c(G)$ which implies that $r_{BOCD} = 1$.
Conversely, suppose that $r_{BOCD} = 1$, which means that there is an edge e in $\bar{G}$ where $\tilde{\gamma}_c(G + e) = 2n + 1$. Let $\tilde{D}_e$ be a $\tilde{\gamma}$-set of $G + e$. Then, we have $|\tilde{D}_e \cap \{u_i, \bar{u}_i\}| = 1$ for $1 \leq i \leq n$ by Lemma 4.3. Assume that the mapping $f : \mathcal{U} \to \{T, F\}$ is defined as

$$f(u_i) = \begin{cases} T, & \text{if } \ u_i \in \tilde{D}_e, \\ F, & \text{if } \ \bar{u}_i \in \tilde{D}_e. \end{cases} \tag{1}$$

We show that the truth values assigned by the mapping f satisfy every clause in $\mathcal{C}$. Let $C_j \in \mathcal{C}$ be an arbitrarily clause. Since the vertex c_j in correspondence to clause C_j is not adjacent to any members of $\{v_i : 1 \leq i \leq n\}$, then there exists some index i such that c_j is dominated by either $u_i \in \tilde{D}_e$ or $\bar{u}_i \in \tilde{D}_e$. Without loss of generality, assume c_j is

dominated by $u_i \in \tilde{D}_e$. So, the vertex u_i is adjacent to c_j, namely u_i is in C_j. Because $u_i \in \tilde{D}_e$, we have $f(u_i) = T$ by Equation 1. So, f satisfies the clause C_j.
Now, Suppose that the vertex c_j is dominated by $\bar{u}_i \in \tilde{D}_e$. So, $\bar{u}_i$ is adjacent to c_j, namely, $\bar{u}_i$ is in C_j. Because $\bar{u}_i \in \tilde{D}_e$, by Equation 1 we have $f(u_i) = F$, which implies that $\bar{u}_i$ is assigned the truth value T by f. So, the clause C_j is satisfied by f. Since the clause C_j was chosen arbitrarily, so all the clauses in $\mathcal{C}$ are satisfied by f. Therefore, the $\mathcal{C}$ is satisfiable. □

These Lemmas conclude the proof. □

5 Complexity of the Outer-Connected Bondage problem for bipartite graphs

In this section, we will show that the OUTER-CONNECTED BONDAGE problem for bipartite graphs is an **NP**-hard problem. Consider the following decision problem.

> **Bi-Outer-Connected Bondage problem for bipartite graphs.**
> *Input Instance:* A positive integer k and a bipartite graph G with no isolated vertices.
> *Question:* Does $b_{BOCD}(G) \leq k$ hold?

Theorem 5.1. *The* BI-OUTER-CONNECTED BONDAGE *problem for bipartite graphs is* **NP***-hard.*

Proof. Let $I = (\mathcal{U} = \{u_1, u_2, \ldots, u_n\}\ , \mathcal{C} = \{C_1, C_2, \ldots, C_m\})$ be an arbitrary instance of the $3-\mathrm{SAT}$ problem. For an arbitrary positive integer k, we construct a bipartite graph G such that this instance of $3-\mathrm{SAT}$ is satisfiable if and only if G has an outer-connected bondage of cardinality of at most k, i.e. $b_{BOCD}(G) \leq k$. Next, we describe how the graph G is constructed.
We associate a set of vertices $\mathcal{H}_i = \{u_i, v_i, \bar{u}_i, x_i, y_i, m_i, w_i, n_i\}$ and edges $\{x_i, m_i\}, \{y_i, n_i\}$, $\{u_i, v_i\}$, $\{\bar{u}_i, v_i\}, \{m_i, w_i\}, \{u_i, m_i\}, \{\bar{u}_i, n_i\}$, $\{u_i, n_i\}, \{\bar{u}_i, m_i\}$ and $\{n_i, w_i\}$ for $1 \leq i \leq n$ to each $u_i \in \mathcal{U}$. Similarly, for each clause $C_j \in \mathcal{C}$, a single vertex c_j is associated and edge $\{c_j, u_i\}(\{c_j, \bar{u}_i\})$ is added if the literal $u_i(\bar{u}_i)$ appears in clause C_j where $1 \leq j \leq m$. Then, we add the set of vertices $S = \{s_1, s_2, s_3, s_4\}$ and join all the vertices s_1, s_3 and s_4 to the vertices c_j and s_2. Finally, we add a vertex t to the graph G and add edges $\{t, s_1\}, \{t, s_3\}, \{t, s_4\}, \{t, u_i\}$ and $\{t, \bar{u}_i\}$ for $1 \leq i \leq n$. Without losing generality, assume that $k = 1$. It is clear that the construction can be accomplished in polynomial time, since the graph G contains $8n+m+5$ vertices and $6m+12n+6$ edges. Next we show that $b_{BOCD}(G) \leq 1$ if and only if $I = (\mathcal{C}, \mathcal{U})$ is satisfiable by considering these five lemmas.

Lemma 5.2. *For graph G constructed as above, it is the case that $\tilde{\gamma}_c(G) \geq 4n + 1$.*

Proof. Assume that $\tilde{D}$ is a $\tilde{\gamma}$-set of G. So, $\tilde{\gamma}_c(G) = |\tilde{D}| \geq 4n+1$, because $|V(\mathcal{H}_i) \cap \tilde{D}| \geq 4$ for $1 \leq i \leq n$, i.e. to dominate vertices v_i and w_i for $1 \leq i \leq n$, we need to have at least one non-leaf vertex and leaf vertices x_i and y_i in $\tilde{D}$. Moreover, we have $|\tilde{D} \cap N[s_2]| \geq 1$. □

Lemma 5.3. *If $\tilde{\gamma}_c(G) = 4n+1$, then $c_j, t \notin \tilde{D}$ for $1 \leq j \leq m$, $\tilde{D} \cap V(S) = \{s_2\}$, $|\tilde{D} \cap V(\mathcal{H}_i)| = 4$ and $|\tilde{D} \cap \{u_i, \bar{u}_i\}| \leq 1$ for $1 \leq i \leq n$.*

Proof. Because the connection between $\mathcal{H}_i$ and S is due to the vertex t, then $t \notin \tilde{D}$. Suppose that we have $\tilde{\gamma}_c(G) = 4n+1$. Then, $|\tilde{D} \cap V(\mathcal{H}_i)| = 4$ for $1 \leq i \leq n$, while $|V(S) \cap \tilde{D}| = 1$. Therefore, $c_j \notin \tilde{D}$ for $1 \leq j \leq m$. Moreover, if $\tilde{D} \cap V(S) = \{s_1\}$, then s_3 and s_4 are not dominated. Hence, we have $s_1 \notin \tilde{D}$. Similarly, we have $s_3, s_4 \notin \tilde{D}$. Therefore, we have $\tilde{D} \cap V(S) = \{s_2\}$. Since $x_i, y_i \in \tilde{D}$ and w_i needs to be dominated by $\tilde{D}$, we have $|\tilde{D} \cap \{u_i, \bar{u}_i\}| \leq 1$ for $1 \leq i \leq n$. □

Lemma 5.4. *$\tilde{\gamma}_c(G) = 4n+1$ if and only if the $3-\mathrm{SAT}$ instance $I = (\mathcal{U}, \mathcal{C})$ is satisfiable.*

Proof. Let $f : \mathcal{U} \rightarrow \{T, F\}$ be a truth assignment which satisfies $\mathcal{C}$ and $\tilde{D}'$ be a subset of $V(G)$ constructed as follows. The vertices u_i and n_i are added to $\tilde{D}'$ if $f(u_i) = T$. If $f(u_i) = F$, then we put the vertices $\bar{u}_i$ and m_i in $\tilde{D}'$. Therefore, we have $|\tilde{D}'| = 2n$. At least one of the literals in C_j is assigned a satisfying value under the assignment of f for $1 \leq j \leq m$, since f is a satisfying truth assignment for $I = (\mathcal{U}, \mathcal{C})$. So, by the construction of G, the corresponding vertex to C_j in G is adjacent to at least one vertex in $\tilde{D}'$. Then, $D = \tilde{D}' \cup (\bigcup_{i=1}^{n} \{x_i, y_i\}) \cup \{s_2\}$ is a dominating set for G. On the other hand, the induced graph $G[V \setminus D]$ is connected. Hence, D is an OCD set for G and $\tilde{\gamma}_c(G) \leq |D| = 4n+1$. By Lemma 5.2, we have $\tilde{\gamma}_c(G) \geq 4n+1$. Therefore, we obtain $\tilde{\gamma}_c(G) = 4n+1$.
Conversely, suppose that $\tilde{\gamma}_c(G) = 4n+1$ and c_j is an arbitrary vertex. By Lemma 5.3, this vertex is adjacent to either $u_i \in \tilde{D}$ or $\bar{u}_i \in \tilde{D}$ because $c_j, s_1, s_3, s_4 \notin \tilde{D}$. By Lemma 5.3, we have $|\tilde{D} \cap \{u_i, \bar{u}_i\}| \leq 1$ for $1 \leq i \leq n$. Assume that the mapping $f : \mathcal{U} \rightarrow \{T, F\}$ is defined as

$$f(u_i) = \begin{cases} T, & \text{if } u_i \in \tilde{D}, \\ F, & \text{otherwise.} \end{cases} \tag{2}$$

We show that the truth values assigned by the mapping f satisfy all the clauses $I = (\mathcal{U}, \mathcal{C})$. We choose an arbitrary clause $C_j \in \mathcal{C}$. Since the corresponding vertex to the clause C_j is not adjacent to any vertices in correspondence with the set $\mathcal{H}_i \setminus \{u_i, \bar{u}_i : 1 \leq i \leq n\}$, there exists an index i such that c_j is dominated by either $u_i \in \tilde{D}$ or $\bar{u}_i \in \tilde{D}$. Without losing generality, assume that $u_i \in \tilde{D}$ dominates c_j. So, u_i is adjacent to c_j, namely u_i is in C_j. Since $u_i \in \tilde{D}$, we have $f(u_i) = T$ by Equation 2. So, f satisfies the clause C_j.
Next, suppose that the vertex c_j is dominated by the vertex $\bar{u}_i \in \tilde{D}$. So, $\bar{u}_i$ is adjacent to c_j, namely $\bar{u}_i$ is in C_j. Since $\bar{u}_i \in \tilde{D}$, we have $f(u_i) = F$ by Equation 2 which implies that $\bar{u}_i$ is assigned the truth value T by f, and the clause C_j is satisfied by f. Since C_j was chosen arbitrarily, all the clauses in $\mathcal{C}$ are satisfiable by f, which implies that, $I = (\mathcal{U}, \mathcal{C})$ is satisfiable.

□

Lemma 5.5. *For all edge $e \in E(G)$, we have $\tilde{\gamma}_c(G - e) \leq 4n+2$.*

Proof. Suppose that $E' = \{\{s_2, s_3\}, \{s_2, s_4\}, \{s_1, c_j\}, \{u_i, v_i\}, \{v_i, \bar{u}_i\}, \{t, s_1\}\}$ and $E'' = E \setminus E'$. Let $e \in E''$ be an edge. It is clear that the set $D' = (\bigcup_{i=1}^{n} \{x_i, y_i, w_i, v_i\}) \cup \{s_1, s_2\}$

is an OCD set for $G-e$, since every vertex in $V \setminus D'$ is adjacent to a vertex in D' due to an edge in E'. Also, the induced graph $(G-e) \setminus D'$ is connected. This connectedness is established by vertices t and s_i for $i \neq 1,2$. Given $|D'| = 4n+2$, then we have $\tilde{\gamma}_c(G-e) \leq 4n+2$.
We have four cases to consider:

Case 1: Either $e = \{s_2, s_3\}$, $e = \{s_1, c_j\}$ or $e = \{t, s_1\}$, which implies $D' = (\bigcup_{i=1}^{n} \{x_i, y_i, w_i, v_i\}) \cup \{s_3, s_2\}$ is an OCD set for $G-e$ and $\tilde{\gamma}_c(G-e) \leq |D'| = 4n+2$.

Case 2: Either $e = \{s_2, s_4\}$ which implies $D' = (\bigcup_{i=1}^{n} \{x_i, y_i, v_i, w_i\}) \cup \{s_4, s_2\}$ is an OCD set for $G-e$ and $\tilde{\gamma}_c(G-e) \leq |D'| = 4n+2$.

Case 3: If $e = \{v_i, \bar{u}_i\}$, then $D' = (\bigcup_{i=1}^{n} \{x_i, y_i, v_i, m_i\}) \cup \{s_1, s_2\}$ is an OCD set for $G-e$ and $\tilde{\gamma}_c(G-e) \leq |D'| = 4n+2$.

Case 4: If $e = \{u_i, v_i\}$, then $D' = (\bigcup_{i=1}^{n} \{x_i, y_i, v_i, n_i\}) \cup \{s_1, s_2\}$ is an OCD set for $G-e$ and $\tilde{\gamma}_c(G-e) \leq |D'| = 4n+2$.

□

Lemma 5.6. $b_{BOCD}(G) = 1$ *if and only if* $\tilde{\gamma}_c(G) = 4n+1$.

Proof. First, let $b_{BOCD}(G) = 1$. It follows by Lemma 5.2 that $\tilde{\gamma}_c(G) \geq 4n+1$. Suppose that e is an edge where $\tilde{\gamma}_c(G) < \tilde{\gamma}_c(G-e)$. By Lemma 5.5, we have $4n+1 \leq \tilde{\gamma}_c(G) < \tilde{\gamma}_c(G-e) \leq 4n+2$. So, $\tilde{\gamma}_c(G) = 4n+1$. Let $\tilde{\gamma}_c(G) = 4n+1$, $e = \{s_1, s_2\}$ and $\tilde{\gamma}_c(G-e) = \tilde{\gamma}_c(G)$. If $\tilde{D}$ is a $\tilde{\gamma}$-set of $G-e$, then $\tilde{D}$ is a $\tilde{\gamma}$-set for G of cardinality $4n+1$. By Lemma 5.3, we have $c_j, t \notin \tilde{D}$ for $1 \leq j \leq m$, and $\tilde{D} \cap V(S) = \{s_2\}$. So, $\tilde{D}$ does not dominate the vertex s_1, which is a contradiction. Therefore, we have $\tilde{\gamma}_c(G) < \tilde{\gamma}_c(G-e)$. Therefore, we obtain $b_{BOCD}(G) = 1$. □

So, by Lemmas 5.4 and 5.6, we have $b_{BOCD}(G) = 1$ if and only if I is satisfiable. □

6 Bounds in General

In this section, we obtain bounds on the outer-connected bondage number in general graphs.

Theorem 6.1. *Let G be a connected graph of order $n \geq 2$. Then, $b_{OCD}(G) \leq n-1$ and the bound is sharp.*

Proof. Suppose u and v are adjacent vertices with degrees $d(u)$ and $d(v)$, respectively and $d(u) \leq d(v)$. If $b_{OCD}(G) \leq d(u)$, then $b_{OCD}(G) \leq n-1$. Let $b_{OCD}(G) > d(u)$ and $E(u)$ is the set of incident edges with u. Since $b_{OCD}(G) > d(u)$, it follows that

$$\tilde{\gamma}_c(G - E(u)) = \tilde{\gamma}_c(G), \tag{3}$$

which means

$$\tilde{\gamma}_c(G \setminus u) = \tilde{\gamma}_c(G) - 1. \tag{4}$$

Now, suppose that $\tilde{D}$ is the union of all $\tilde{\gamma}_c$-sets for $G \setminus u$. It is clear that no vertices in $\tilde{D}$ are adjacent to u in G. So, we have

$$v \notin \tilde{D}, \quad |E(u)| \leq n - |\tilde{D}| - 1. \tag{5}$$

Next, consider that

$$F(v) = \{\{v, x\} \in E(G \setminus u) | x \in \tilde{D}\}. \tag{6}$$

Because $v \notin \tilde{D}$, we need to have

$$\tilde{\gamma}_c(G \setminus u - F(v)) > \tilde{\gamma}_c(G \setminus u), \tag{7}$$

which implies

$$\tilde{\gamma}_c(G \setminus u - F(v)) > \tilde{\gamma}_c(G) - 1. \tag{8}$$

Then,

$$\tilde{\gamma}_c(G - (E(u) \cup F(v)) = \tilde{\gamma}_c(G \setminus u - F(v)) - 1 > \tilde{\gamma}_c(G) - 1 > \tilde{\gamma}_c(G). \tag{9}$$

Therefore,

$$b_{OCD}(G) \leq |E(u) \cup F(v)| = |E(u)| + |F(v)| \leq n - |\tilde{D}| - 1 + |\tilde{D}| = n - 1. \tag{10}$$

□

Let $G = S_n$ be a star of order n. It is easy to see that $b_{OCD}(G) = n - 1$. So, the proposed bound is sharp.

Theorem 6.2. *Let G be a connected graph and u and v are adjucent vertices which $d(u) + d(v)$ is minimum and*

1. *$\tilde{D}$ is a $\tilde{\gamma}$-set for $G \setminus \{u, v\}$,*
2. *$\big(N_G(u) \cup N_G(v)\big) \setminus \{u, v\} \nsubseteq \tilde{D}$, and*
3. *$\delta(G) \geq 2$.*

Then, $b_{OCD}(G) \leq d(u) + d(v) - 1$.

Proof. The proof is by contradiction. Let $x = d(u) + d(v) - 1$ and $b_{OCD}(G) > x$. Let E' denote the set of edges incident with at least one of the vertices u and v. So, $|E|' = x$ and $\tilde{\gamma}_c(G - E') = \tilde{\gamma}_c(G)$. Hence, $\tilde{\gamma}_c(G \setminus \{u, v\}) = \tilde{\gamma}_c(G) - 2$. Since $\delta(G) \geq 2$, we have $N_G(u) \cup N_G(v) \setminus \{u, v\} \neq \varnothing$. Now, the following two cases need to be considered.

Case 1: $N_G(u) - \{v\} \nsubseteq \tilde{D}$ which implies that there is at least one vertex $w \neq v$ such that $w \in N_G(u)$ and $w \notin \tilde{D}$. So, $\tilde{D} \cup \{v\}$ is an set for G of cardinality $\tilde{\gamma}_c(G) - 1$. This is a contradiction.

Case 2: $N_G(v) - \{u\} \nsubseteq \tilde{D}$ which implies that there exists at least one vertex $w \neq u$ such that $w \in N_G(v)$ and $w \notin \tilde{D}$. So, $\tilde{D} \cup \{u\}$ is an OCD set for G of cardinality $\tilde{\gamma}_c(G) - 1$ which is again a contradiction.

Therefore, $b_{OCD}(G) \leq x$. □

Corollary 6.3. If G is the connected graph of Theorem 6.2. Then, $b_{OCD}(G) \leq \delta(G) + \Delta(G) - 1$.

Proof. Let $x, y \in G$ be vertices such that $d(x) = \delta(G)$ and $y \in N_G(x)$. Then, by Theorem 6.2, we have

$$b_{OCD}(G) \leq d(x) + d(y) - 1 = \delta(G) + d(y) - 1 \leq \delta(G) + \Delta(G) - 1. \tag{11}$$

□

Theorem 6.4. *Let G be a nonempty graph with $\tilde{\gamma}_c(G) \geq 2$. Then,*

$$b_{OCD}(G) \leq (\tilde{\gamma}_c(G) - 1)\Delta(G) + 1. \tag{12}$$

Proof. The proof is by the induction on the OCD number of G. Suppose that $\tilde{\gamma}_c(G) = 2$, and $b_{OCD}(G) > \Delta(G) + 1$. So, if $u \in V(G)$ is a vertex such that $d(u) = \Delta(G)$, we have $\tilde{\gamma}_c(G \setminus u) = \tilde{\gamma}_c(G) - 1 = 1$. Therefore, there exists a vertex $x \in V(G)$ where $x \neq u$ which is adjacent to all vertices in $V(G \setminus u)$. This implies $d(x) = \Delta(G)$, and the adjacency of u to every vertex in $V(G \setminus x)$. Let e be an arbitrary edge incident with x. Since $b_{OCD}(G \setminus u) \geq 2$, then $\tilde{\gamma}_c(G \setminus u) = 1$ if the edge e is removed from the induced graph $G \setminus u$. Hence there exists a vertex $y \neq x$ such that y is adjacent to all vertices in $G \setminus u$. Because x is the only vertex which is not in $N_G(u)$, we have $y \in N_G(u)$. So, we have by contradiction $\tilde{\gamma}_c(G) = 1$. So, $b_{OCD}(G) \leq \Delta(G) + 1$ for $\tilde{\gamma}_c(G) = 2$.
Now, suppose that H is a nonempty graph, $\tilde{\gamma}_c(H) = k \geq 2$ and $b_{OCD}(H) \leq (k-1)\Delta(H) + 1$. Suppose that G is a graph such that $\tilde{\gamma}_c(G) = k + 1$ and $b_{OCD}(G) > k\Delta(G) + 1$. Let x be an arbitrary vertex in G. Then, $\tilde{\gamma}_c(G \setminus x) = \tilde{\gamma}_c(G) - 1 = k$. Also, $b_{OCD}(G) \leq b_{OCD}(G \setminus x) + d(x)$. By the hypothesis of the induction, we obtain either

$$\begin{aligned} b_{OCD}(G) &\leq (k-1)\Delta(G \setminus x) + 1 + d(x) \\ &\leq (k-1)\Delta(G) + 1 + \Delta(G) = k\Delta(G) + 1. \end{aligned} \tag{13}$$

which is a contradictory to $b_{OCD}(G) > k\Delta(G) + 1$, which concludes the proof by induction. □

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