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Ellipsoidal spline functions for gravity data interpolation and smoothing

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ABSTRACT

The aim of this paper is to study the theory of spline interpolation and smoothing problems on the surface of a triaxial ellipsoid for the Consecutive Iterated Helmholtz operator and a set of linearly independent evaluation functionals. Spline functions were introduced based on the minimization of a semi-norm in the context of a semi-Hilbert space whose domain was the surface of the ellipsoid. The semi-Hilbert space was decomposed into two different subspaces, a particular Hilbert space and the null space of the desired operator. Using surface Green's functions for the Consecutive Iterated Helmholtz operator, the reproducing kernel for the Hilbert subspace was constructed. Spline and smoothing functions were explicitly represented based on the reproducing kernel and the evaluation functionals. An approximation formula was derived to facilitate the potential use in Earth's gravity field data interpolation and smoothing. An application of this technique was presented to show the interpolation of potential fields over Iran. Ellipsoidal and spherical splines were compared as well. It revealed the ellipsoidal splines to be more accurate than the spherical counterparts.

KEYWORDS

Spline Interpolant

Consecutive Iterated-Helmholtz operator

Minimization problem

Surface Green's functions

Evaluation functionals

1. Introduction

The focus of this paper is mainly on the derivation of spline interpolant and smoothing functions defined on the surface of an ellipsoid. The fundamental role of norm minimization is evident in the definition of spline and smoothing functions. Spline and smoothing functions are merely based on a particular reproducing kernel Hilbert space and underlying evaluation functionals, which ensure the uniqueness of the solution. The given evaluation functionals have to be linearly independent at points scattered on the ellipsoid's surface.

The application of this study is to Earth's gravity data interpolation, where one seeks a function with which the gravity data can be best interpolated. The criterion that determines this property is L^2-semi-norm minimization on a manifold which is performed in the sense of generalized and Green's functions. This leads to a function that does not vary much from one point to another and thus, the function satisfies the optimum continuity condition in terms of the interpolant's derivatives. This is something following Earth's gravity field nature.

The domain of interest qualifies the nature of the problem that has to be solved. The notion of Earth's spherical shape has long been supported since the sphere is the simplest and most ideal shape by which Earth's geometric shape is represented. The spline interpolation for sphere has been investigated in several different works, including Freeden et al. (2018a), Freeden (1981), Wahba (1981), and Wahba (1990).

Spherical splines and zonal kernels were defined in the context of the Sobolev space and the reproduction of kernel

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Hilbert spaces by Freeden et al. (2018a). Spherical spline approximation of discrete boundary value problems for smooth or regular surfaces as well as the Cauchy-Navier splines were introduced by Freeden et al. (2013). For a Sobolev space defined on a spherical surface, spline interpolation and (exact)integration for the Beltrami-Laplace operator have been discussed by Freeden et al. (2018b). A simple spline interpolation problem on the sphere for the Beltrami-Laplace and Helmholtz operators and their iterations was investigated by Freeden (1982) who introduced and derived surface Green's functions based on the eigenvalue expansion method. Freeden et al. (1998) discussed the computational aspects of the problems to facilitate their potential applications. In a comprehensive discussion of spherical splines, Green's functions were presented for Iterated Beltrami-Laplace and Consecutive Iterated Helmholtz operators, pseudodifferential operators, and radial basis functions. Freeden (1984) and Freeden (1981) discussed the most fundamental theory of spline functions through the distribution theory as well as surface Green's functions for Consecutive Iterated Helmholtz operator and their existence, uniqueness, and computational procedures. Green's functions and their underlying integral formulas for their iterations are defined in Freeden (2009). Spline functions and zonal kernels were introduced based on Green's function. Wahba (1981) discussed the spherical interpolation and splines smoothing for a set of linearly independent evaluation functionals. Also, kernels reproduction was given in an integral analytical representation.

Smoothing problems are investigated when the samples on the surface are obtained by observation and are inevitably erroneous. Compared to error-free interpolation, smoothing problems have more widespread uses. For instance, the gravity data on the sea surface obtained via shipborne gravimetry are highly noisy that should be smoothed. Another important domain is spheroid or an ellipsoid of revolution. The physical explanation for its importance can be given if one considers the interpolation of gravity data in Earth's gravity field, with its geometry being better defined with a spheroid. Earth's shape is affected by several different internal and external forces which make it irregular. One can use a sphere to approximate this rough surface. However, in the second approximation, observations have confirmed the spheroidal shape of the Earth. Therefore, having a mathematical framework for interpolating this unique surface is highly necessary. Although many efforts have been made on the spherical (spline) interpolation, the spheroidal case still requires more attention. An important work on an outer spheroidal spline, namely Abel-Poisson kernel spline, investigated by Akhtar et al. (2012) in the context of a Sobolev space. However, when the data are given on the surface, the problem has to be solved again in a different way. In this paper, by deriving the generalized triaxial ellipsoid spline and smoothing functions, we derive the surface spheroidal splines as a special case, once the data are referenced to the surface of the spheroid. To obtain a better framework for interpolation and smoothing for gravity data, we need to define triaxial ellipsoid splines. This is a generalization of the spherical and spheroidal case, which means spherical and spheroidal splines are special cases of the ellipsoidal spline.

This paper studies the spline interpolation and smoothing problems in triaxial ellipsoid geometry. Section 2 sets the preliminaries and states the minimization problem. Then, Section 3 chooses the surface Green's functions approach and derives the Consecutive Iterated surface Green's function. Section 4 defines the spline and smoothing functions. We used Green's functions to find the reproducing kernel of the Hilbert space defined on the surface of ellipsoid and the spline interpolation, which is the solution of the minimization problem. Section 5 presents an application example to a potential interpolation (a case study from Iran). Finally, Section 6 provides the concluding remarks.

2. Preliminaries and minimization problem

In this paper, $\mathcal{E}$ denotes the surface of the triaxial ellipsoid. We consider some essential definitions which will be necessary throughout the paper.

Let (x,y,z) be three-dimensional cartesian coordinates of an ellipsoid with axes $a>b>c>0$, respectively. Then, the ellipsoidal coordinate system (σ,τ,μ) is defined as

$$(x,y,z)=\left(\frac{\sigma\tau\mu}{e_1e_2},\frac{\sqrt{(\sigma^2-e_1^2)(\tau^2-e_1^2)(e_1^2-\mu^2)}}{e_1e_3},\frac{\sqrt{(\sigma^2-e_2^2)(e_2^2-\tau^2)(e_2^2-\mu^2)}}{e_2e_3}\right) \quad (1)$$

The linear eccentricities e_1, e_2, and e_3 are determined as the following

$$e_1=\sqrt{a^2-b^2},\ e_2=\sqrt{a^2-c^2}, e_3=\sqrt{b^2-c^2} \quad (2)$$

where

$$\mu^2\le e_1^2\le\tau^2\le e_3^2\le\sigma^2<\infty \quad (3)$$

Note that $\sigma=\sigma_0$ where σ_0 being an arbitrary constant with the same role as a sphere's radius that represents $\mathcal{E}$.

Let $\mathcal{M}$ (here $\mathcal{M}=\mathcal{E}$) be a differentiable manifold, of dimension r with a Metric Tensor whose elements and determinant are, respectively, $g_{ij}, i, j=1,\dots,r$. and The Iterated Beltrami-Laplace operator on $\mathcal{M}$ is defined as

$$\Delta_B^v=\frac{1}{\sqrt{g}}\partial_{i_v j_v}\left(\frac{\sqrt{g}}{g_{i_v j_v}}\partial^{i_v j_v}\left(\dots\left(\frac{1}{\sqrt{g}}\partial_{i_1 j_1}\left(\frac{\sqrt{g}}{g_{i_1 j_1}}\partial^{i_1 j_1}\right)\right)\dots\right)\right) \quad (4)$$

For the case $v=1$, which is called the Beltrami-Laplace operator, we simply have

$$\Delta_B=\frac{\sigma_0^2-\mu^2}{\mu^2-\tau^2}\left(\left(\tau^2-e_1^2\right)\left(\tau^2-e_2^2\right)\frac{\partial^2}{\partial\tau^2}+\tau\left(2\tau^2-e_1^2-e_2^2\right)\frac{\partial}{\partial\tau}\right)$$
$$+\frac{\sigma_0^2-\tau^2}{\tau^2-\mu^2}\left(\left(\mu^2-e_1^2\right)\left(\mu^2-e_2^2\right)\frac{\partial^2}{\partial\mu^2}+\tau\left(2\mu^2-e_1^2-e_2^2\right)\frac{\partial}{\partial\mu}\right) \quad (5)$$

The i-th Helmholtz operator is defined as the sum of the Beltrami-Laplace operator and the negative of its i-th eigenvalue $p_{i,m}$, i.e.,

$$\Delta_{H_i}=\frac{\sigma_0^2-\mu^2}{\mu^2-\tau^2}\left(\left(\tau^2-e_1^2\right)\left(\tau^2-e_2^2\right)\frac{\partial^2}{\partial\tau^2}+\tau\left(2\tau^2-e_1^2-e_2^2\right)\frac{\partial}{\partial\tau}\right)+$$
$$\frac{\sigma_0^2-\tau^2}{\tau^2-\mu^2}\left(\left(\mu^2-e_1^2\right)\left(\mu^2-e_2^2\right)\frac{\partial^2}{\partial\mu^2}+\tau\left(2\mu^2-e_1^2-e_2^2\right)\frac{\partial}{\partial\mu}\right)-p_{i,m} \quad (6)$$

Remark 2.1 The solution of the homogeneous Helmholtz equation (6) in ellipsoidal geometry leads to the surface Lame′ functions of the first kind $Y_{i,m}$ where

$$\Delta_{H_i}Y_{i,m}=0, \quad i=0,1,2,\cdots,m=1,\cdots,2i+1 \quad (7)$$

Based on Lame' numbers $l_{i,m}$, eigenvalue $p_{i,m}$ can be obtained by

$$p_{i,m}=\left(e_1^2+e_2^2\right)l_{i,m}-n\left(n+1\right)\sigma_0^2 \quad (8)$$

In this paper, surface Lame' functions of the first kind play the most important role in the definition of Green's functions and the spline interpolation.

Regarding Beltrami-Laplace operator, we consider the following operators and Hilbert space

- The i-th Iterated Helmholtz operator of degree v is defined as

$$\Delta_{H_i}^v=(\Delta_B-p_{i,m})^v \quad (9)$$

- The Consecutive Iterated Helmholtz operator to degree v, with its i-th element acting q_i times, is defined as

$$\Delta_{H_{C_{q_0,\dots,q_v}}}=\Delta_{H_0}^{q_1}\dots\Delta_{H_v}^{q_v} \quad (10)$$

- The semi-Hilbert space of all infinitely differentiable functions for the operator $\mathcal{L}$ of the form (5), (6), (9) and (10) is defined as

$$\mathcal{H}\left(\mathcal{E}\right)=\{F\,|\,F\in C^\infty\left(\mathcal{E}\right)\text{and}\mathcal{L}F\in L^2\left(\mathcal{E}\right)\} \quad (11)$$

Definition 2.1 The linearly independent evaluation functionals $\mathcal{L}_j\left(j=1,\dots,n\right)$ constitute a unisolvent system if for an arbitrary function $f\in\mathcal{H}\left(\mathcal{E}\right)$, the following condition holds for the Gramian determinant

$$|[\mathcal{L}_i f\left(\xi_j\right)]_{i,j=1,\cdots,n}|\neq 0 \quad (12)$$

The goal is to find the solution to a minimization problem. It is often performed for the norm of a differentiated function (i.e., a differential operator has acted on it), as in the spherical case (see Baramidze et al. (2006)).

Definition 2.2 $S\in\mathcal{H}\left(\mathcal{E}\right)$ is called a spline interpolant if it is the solution to the following minimization problem

$$S(\xi)=\arg(\min_{f\in\mathcal{H}(\mathcal{E})}\left\|\Delta_{H_0}^{q_1}\dots\Delta_{H_v}^{q_v}f\right\|_{L^2(\mathcal{E})}) \quad (13)$$

As an interpolation problem, it is needed to consider a given set $\mathcal{D}\subset\mathcal{E}$ with $\text{measure}\left(\mathcal{D}\right)=0$ and the cardinality condition $\text{card}\left(\mathcal{D}\right)<\infty$. E.g., $\mathcal{D}=\{\eta_i\in\mathcal{E}\,|\,i=1,\cdots,J\}$ so that the spline interpolant S satisfies

$$\mathcal{L}_i S\left(\eta_i\right)=u_i, \quad i=1,\cdots,J \quad (14)$$

where $u_i\in\mathbb{R}, i=1,\cdots,J,$ are given.

Smoothing splines are the functions in which the observations get the specific values in Eq. (14). This means that stochastic errors are inevitably included in them. Hence, we have a random-error with normal distribution $\{\epsilon_i \mid i=1,\cdots,J\}$ that are included in the value observed as u_i, namely,

$$\mathcal{L}_i S(\eta_i) = u_i + \epsilon_i, \quad i=1,\cdots,J \tag{15}$$

Using the least square method, a smoothing spline is the solution of the following minimization problem

$$S(\xi) = \arg\Big(\min_{f\in\mathcal{H}(\mathcal{E})} \sum_{i=1}^{J} (\mathcal{L}_i f(\eta) - U_i)^2 + \lambda \left\| \Delta_{H_0}^{q_1} \ldots \Delta_{H_v}^{q_v} f \right\|_{L^2(\mathcal{E})}\Big) \tag{16}$$

where λ denotes the smoothing parameter.

To obtain the reproducing kernel, one has to consider the Hilbert space $\mathcal{H}_0(\mathcal{E})$ in the following definition.

Definition 2.3 The Hilbert space $\mathcal{H}_0(\mathcal{E})$ is the set of all infinitely differentiable functions with the homogeneous discrete conditions as the following

$$\mathcal{H}_0(\mathcal{E}) = \{W \in \mathcal{H}(\mathcal{E}) \mid \mathcal{L}_i W(\eta_i) = 0, i = 1,\ldots,J \tag{17}$$

Remark 2.2 From the known results about L^2 semi-norm (see Davis (1975), Freeden (1981), and Kreyszig (1978) for more details) as well as specific conditions in (14), the existence and uniqueness of the solution of problem (13) are guaranteed. We emphasize that the uniqueness of solution (16) depends on the smoothing parameter λ.

3. Surface Green's functions

In order to define reproducing kernels, a proposed approach is to find surface Green's functions (see for example Freeden (1982) and Freeden (1981)). According to the definition of the reproducing kernel, for a function $f(\xi) \in \mathcal{H}_0(\mathcal{E})$ and a differential operator $\mathcal{L}$ of the form (5), (6), (9) and (10), it is proven that

$$f(\xi) = \iint_{\mathcal{E}} f\mathcal{L}\ (\eta) \mathcal{K}_{\mathcal{H}_0(\mathcal{E})}(\xi,\eta)\, d\eta \tag{18}$$

where later we introduce the reproducing kernel $\mathcal{K}_{\mathcal{H}_0(\mathcal{E})}$. Acting $\mathcal{L}$ on both sides (18) and taking into account the general theory of Green's functions (see Freeden (2009), Freeden (1981), and Szmytkowski (2006)) leads to the differential equation for the reproducing kernels and Green's functions

$$\mathcal{L}G(\xi,\eta) = \delta(\xi-\eta) - \sum_i \phi_i(\xi)\phi_i(\eta) \tag{19}$$

where $\{\phi_i \in \mathcal{H}(\mathcal{E}) \mid i=1,2,\cdots\}$ is a basis for the null space of the applied operator $\mathcal{L}$.

In this section, we derive the surface Green's function for the Consecutive Iterated Helmholtz differential operator. For this purpose, we use the general method of eigenvalue expansion (which is described fully in Greenberg (2015)). To use this method, one has to find the orthonomalized eigenfunctions of the differential operator. In order to orthonormalize Lame′ functions in (7), we have the following proposition.

Proposition 3.1 The norm of the suface Lame' function is

$$|Y_{i,m}|^2 = \frac{1}{4}\iint_{\mathcal{E}} (Y_{i,m}(\tau,\mu))^2 \frac{1}{|\mu\tau|} \times \sqrt{\frac{(\tau^2-\mu^2)(\tau^2-\sigma_0^2)(\mu^2-\sigma_0^2)(\mu^2-\tau^2)}{(\tau^2-e_1^2)(\tau^2-e_2^2)(\mu^2-e_1^2)(\mu^2-e_2^2)}}\, d\tau\, d\mu \tag{20}$$

The corresponding orthonormalized Lame′ function $Q_{i,m}$ is

$$Q_{i,m} = \frac{Y_{i,m}}{|Y_{i,m}|} \tag{21}$$

Based on these orthonormalized functions, we have the following definition of the generalized surface Green's function.

Definition 3.1 If ξ and η are two points on $\mathcal{E}$, the generalized surface Green's function for Consecutive Iterated Helmholtz operator, according to the theory of generalized Green's functions (Szmytkowski (2006)) is defined as

$$\Delta_{H_{C_{q_0,\ldots,q_v}}} G_{H_{C_{q_0,\ldots,q_v}}}(\xi,\eta) = \delta(\xi-\eta) - \sum_{k=0}^{v}\sum_{m=1}^{2k+1} Q_{k,m}(\xi) Q_{k,m}(\eta) \tag{22}$$

According to Hilbert's general theory of Green's functions (Freeden (2009)), the following convolution holds

$$G_{H_{C_{q0,\ldots,q_v}}}(\xi,\eta)=\iint_E G_{H_{C_{q0,\ldots,q_{v-1}}}}(\xi,\zeta)G_{H_v}(\zeta,\eta)dS_\zeta \quad (23)$$

where G_{H_v} is Green's function of v-th degree Helmholtz operator that it is defined as

$$\Delta_{H_v}G_{H_v}(\xi,\eta)=\delta(\xi-\eta)-\sum_{m=1}^{2v+1}Q_{v,m}(\xi)Q_{v,m}(\eta) \quad (24)$$

Using the eigenvalue expansion procedure in Freeden (2009) and the convolution form (23), one gets the generalized surface Green's function representation formula in the following lemma.

Lemma 3.1 The generalized surface Green's function representation formula reads as

$$G_{H_{C_{q0,\ldots,q_v}}}(\xi,\eta)=\sum_{k=v+1}^{\infty}\sum_{m=1}^{2k+1}\left(\frac{Q_{k,m}(\xi)Q_{k,m}(\eta)}{\left(\left(e_1^2+e_2^2\right)l_{k,m}-k(k+1)\sigma_0^2\right)^{q_0}\ldots\left(\left(e_1^2+e_2^2\right)\left(l_{v,m}-l_{k,m}\right)-\left(v(v+1)-k(k+1)\right)\sigma_0^2\right)^{q_v}}\right) \quad (25)$$

4. Spline smoothing and interpolant functions

We have derived the surface Green's function for the generalized Helmholtz operator. This Green's function will be used in the definition of the reproducing kernel for the Hilbert space $\mathcal{H}(\mathcal{E})$. According to Freeden (1982), Freeden (1984), and Freeden (1981), we have the following definitions:

Definition 4.1 For a unisolvent system of functionals $\mathcal{L}_i$, $i=1,\ldots,J$, regrding the specific conditions in (14), the unique orthonormal Lagrange basis B_k, $k=1,\ldots,J$, satisfies (see Davis (1975))

$$B_k(\eta_i)=\delta_{k,i} \quad k,i=1,\ldots,J \quad (26)$$

where δ denotes the Kronecker symbol.

Definition 4.2 In the Hilbert space $\mathcal{H}_0(\mathcal{E})$. and for the unisolvent system of functionals $\mathcal{L}_i$ ($i=1,\ldots,J$), the reproducing kernel for the conditions (14) is

$$\mathcal{K}_{\mathcal{H}_0(\mathcal{E})}(\xi,\eta)=G^i_{H_{C_{q0,\ldots,q_i}}}(\xi,\eta)-\sum_{j=1}^{J}G^i_{H_{C_{q0,\ldots,q_i}}}(\xi,\eta_j)B_j(\eta)+\sum_{j=1}^{J}\sum_{i=1}^{J}B_j(\xi)G^i_{H_{C_{q0,\ldots,q_i}}}(\eta_j,\eta_i)B_i(\eta) \quad (27)$$

Following the remarkable results given in Freeden (1984) and Wahba (1981), the definition of the smoothing spline reads as

$$S(\xi)=\sum_{j=1}^{J_1}c_jB_j(\xi)+\sum_{j=J_1+1}^{J}c_j\mathcal{L}_j^\eta\mathcal{K}_{\mathcal{H}_0(\mathcal{E})}(\xi,\eta) \quad (28)$$

where J_1 is the number of first elements of points that constitute a unisolvent system. The proof of uniqueness is the same as that of spherical splines in Freeden (1981).

Remark 4.1 One special case of (25) is the Iterated Beltrami-Laplace operator, achieved by setting $v=0$ and q as the iteration number, as the following

$$G_B^q(\xi,\eta)=\sum_{k=1}^{\infty}\sum_{m=1}^{2k+1}\frac{Q_{k,m}(\xi)Q_{k,m}(\eta)}{\left(\left(e_1^2+e_2^2\right)l_{k,m}-k(k+1)\sigma_0^2\right)^q} \quad (29)$$

Under Mercer's reproducing kernel theorem and direct sum decomposition of $\mathcal{H}(\mathcal{E})$ in the form of (see Wahba (1990))

$$\mathcal{H}(\mathcal{E})=\mathcal{H}_1(\mathcal{E})\oplus\{1\} \quad (30)$$

we have the following definition for the smoothing and spline functions on the ellipsoid.

Definition 4.3 In the Hilbert space $\mathcal{H}(\mathcal{E})$, the function of the following form, with the constants being determined by the given data on $\mathcal{E}$ (using the methods for finding the suitable smoothing parameter, such as Generalized Cross-Validation method) is called a smoothing spline

$$S(\xi)=c_0+\sum_{j=1}^{J}c_j\sum_{k=1}^{\infty}\sum_{m=1}^{2k+1}\frac{Q_{k,m}(\xi)\mathcal{L}_j^\eta}{\left(\left(e_1^2+e_2^2\right)l_{k,m}-k(k+1)\sigma_0^2\right)^q} \quad (31)$$

where $G_B^q(\xi,\eta)$ is used as the reproducing kernel.

Ithe following lemma, we have derived the second-degree approximation of the surface spline and smoothing functions for the Iterated Beltrami-Laplace operator of degree q.

Lemma 4.1 The spline interpolant and smoothing functions explicit representation is as follows

$$s(\xi) = \frac{c_0}{4\pi} + \sum_{j=1}^{J} c_j \frac{\mathcal{L}_j A_j}{\|Y_{1,1}\|((e_1^2+e_2^2)-2\sigma_0^2)^q} + \frac{\mathcal{L}_j B_j}{\|Y_{1,2}\|(e_2^2-2\sigma_0^2)^q} + \frac{\mathcal{L}_j C_j}{\|Y_{1,3}\|(e_1^2-2\sigma_0^2)^q}$$
$$+ \frac{\mathcal{L}_j D_j}{\|Y_{2,1}\|(l_{2,1}-6\sigma_0^2)^q} + \frac{\mathcal{L}_j E_j}{\|Y_{2,2}\|(l_{2,2}-6\sigma_0^2)^q}$$
$$+ \frac{\mathcal{L}_j(A_j \times B_j)}{|Y_{2,3}\|(l_{2,3}-6\sigma_0^2)^q} + \frac{\mathcal{L}_j(A_j \times C_j)}{|Y_{2,4}|(l_{2,4}-6\sigma_0^2)^q} + \frac{\mathcal{L}_j(B_j \times C_j)}{|Y_{2,5}|(l_{2,5}-6\sigma_0^2)^q} \tag{32}$$

where

$$A = \tau_\xi \tau_{\eta_j} \mu_\xi \mu_{\eta_j} \tag{33}$$

$$B = \sqrt{\tau_\xi^2 - e_1^2}\sqrt{\tau_{\eta_j}^2 - e_1^2}\sqrt{e_1^2 - \mu_\xi^2}\sqrt{e_1^2 - \mu_{\eta_j}^2} \tag{34}$$

$$C = \sqrt{e_2^2 - \tau_\xi^2}\sqrt{e_2^2 - \tau_{\eta_j}^2}\sqrt{e_2^2 - \mu_\xi^2}\sqrt{e_2^2 - \mu_{\eta_j}^2} \tag{35}$$

$$D = \left(\tau_\xi^2 + \frac{\sqrt{e_3^4 + e_1^2 e_2^2}}{3} - \frac{e_1^2+e_2^2}{3}\right)\left(\tau_{\eta_j}^2 + \frac{\sqrt{e_3^4 + e_1^2 e_2^2}}{3} - \frac{e_1^2+e_2^2}{3}\right) \times$$
$$\left(\mu_\xi^2 + \frac{\sqrt{e_3^4 + e_1^2 e_2^2}}{3} - \frac{e_1^2+e_2^2}{3}\right)\left(\mu_{\eta_j}^2 + \frac{\sqrt{e_3^4 + e_1^2 e_2^2}}{3} - \frac{e_1^2+e_2^2}{3}\right) \tag{36}$$

$$E = \left(\tau_\xi^2 - \frac{\sqrt{e_3^4 + e_1^2 e_2^2}}{3} - \frac{e_1^2+e_2^2}{3}\right)\left(\tau_{\eta_j}^2 - \frac{\sqrt{e_3^4 + e_1^2 e_2^2}}{3} - \frac{e_1^2+e_2^2}{3}\right) \times$$
$$\left(\mu_\xi^2 - \frac{\sqrt{e_3^4 + e_1^2 e_2^2}}{3} - \frac{e_1^2+e_2^2}{3}\right)\left(\mu_{\eta_j}^2 - \frac{\sqrt{e_3^4 + e_1^2 e_2^2}}{3} - \frac{e_1^2+e_2^2}{3}\right) \tag{37}$$

$$l_{2,1} = \left(e_1^2 + e_2^2\right)\left(2 + 2\sqrt{1 - 3\frac{e_1^2 e_2^2}{(e_1^2+e_2^2)^2}}\right) \tag{38}$$

$$l_{2,2} = \left(e_1^2 + e_2^2\right)\left(2 - 2\sqrt{1 - 3\frac{e_1^2 e_2^2}{(e_1^2+e_2^2)^2}}\right) \tag{39}$$

$$l_{2,3} = \left(e_1^2 + 4e_2^2\right) \tag{40}$$

$$l_{2,4} = \left(4e_1^2 + e_2^2\right) \tag{41}$$

$$\|Y_{1,i}\| = \frac{4\pi\left(e_1^2 e_2^2 e_3^2\right)}{3\left(e_i^2\right)}, i = 1,2,3 \tag{42}$$

$$|Y_{2,1}| = -\frac{16\pi}{15}\left(\sqrt{e_3^4 + e_1^2 e_2^2}\right)\left(z_1 - a^2\right)\left(z_1 - b^2\right)\left(z_1 - c^2\right) \tag{43}$$

$$z_1 = a^2 - \frac{e_1^2 + e_2^2}{3} + \frac{\sqrt{e_3^4 + e_1^2 e_2^2}}{3} \tag{44}$$

$$|Y_{2,2}| = \frac{16\pi}{15}\left(\sqrt{e_3^4 + e_1^2 e_2^2}\right)\left(z_2 - a^2\right)\left(z_2 - b^2\right)\left(z_2 - c^2\right) \tag{45}$$

$$z_2 = a^2 - \frac{e_1^2 + e_2^2}{3} - \frac{\sqrt{e_3^4 + e_1^2 e_2^2}}{3} \tag{46}$$

$$\|Y_{2,6-i}\| = \frac{14\pi}{5} e_1^2 e_2^2 e_3^2 e_i^2, i = 1,2,3 \tag{47}$$

with the condition

$$\sum_{j=1}^{J} c_j \mathcal{L}_j^{\eta} Q_{0^1}(\eta) = 0 \tag{48}$$

Proof. The norm of the surface Lamé functions and the second separation constants are given in Dassios (2012) and Dassios et al. (2012). The above formulae are obtained using the relation given in (31) and regarding Freeden (1984).

5. Application in potential interpolation: a case study for Iran

In this section, we present an application of the results obtained in the previous sections. One of the most important applications of interpolation problems on spherical and ellipsoidal surfaces is in the gravity field interpolation. Many references and papers such as Freeden (2009) and Kiani et al. (2019) have used the spline interpolant to interpolate gravity data globally. However, the application of spline interpolant is not confined to the gravity field. In Keller et al. (2019), the proposed method for spherical thin-plate spline interpolation is used to interpolate the Total Electron Content (TEC) and the location of GRACE satellites. The scattered data interpolation by spherical splines for geopotential values by satellite CHAMP is given in Baramidze et al. (2006). In this section, we use the EGM2008 global geopotential model to calculate the potential on the surface of the reference ellipsoid, which is a symmetrical ellipsoid ($e2 = e3 = 0$), in a rectangular $1^\circ \times 1^\circ$ grid. Then, the data are interpolated on this grid to produce a denser, $0.5^\circ \times 0.5^\circ$ grid and the result is compared with the actual $0.5^\circ \times 0.5^\circ$ grid, derived from the potential analytical formula, to infer the accuracy of interpolation. In order to compare the spherical and ellipsoidal splines, steps **(a1)**–**(a5)** are performed.

(a1) The potential values are computed from the EGM2008 geopotential model, from degree $n = 2$, by converting the coefficients from the spherical to ellipsoidal mode, using the relations in Jekeli (1988). These values are shown in Figure 1.

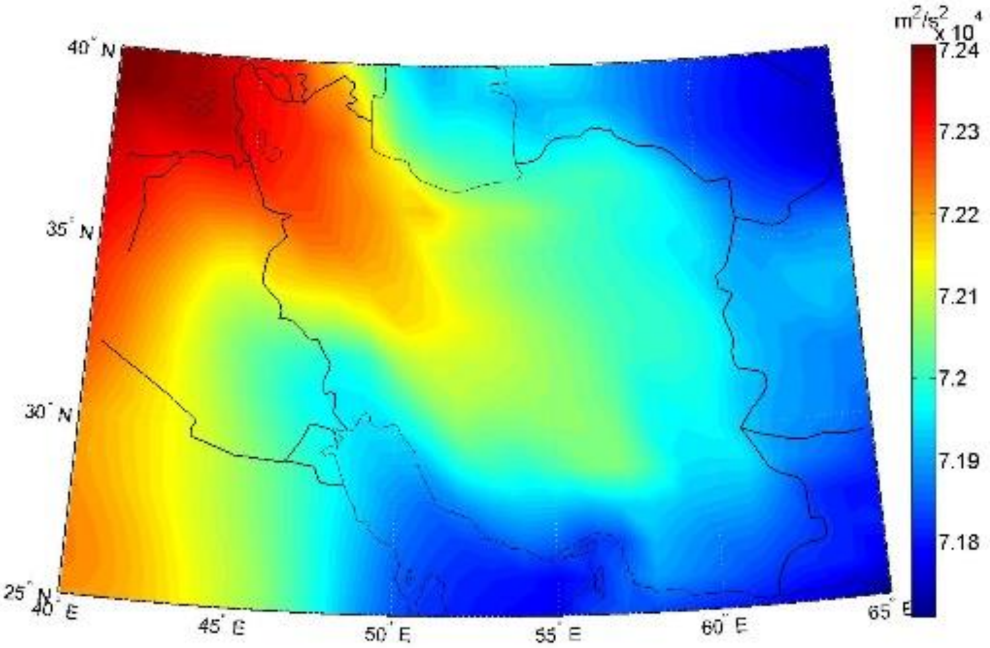
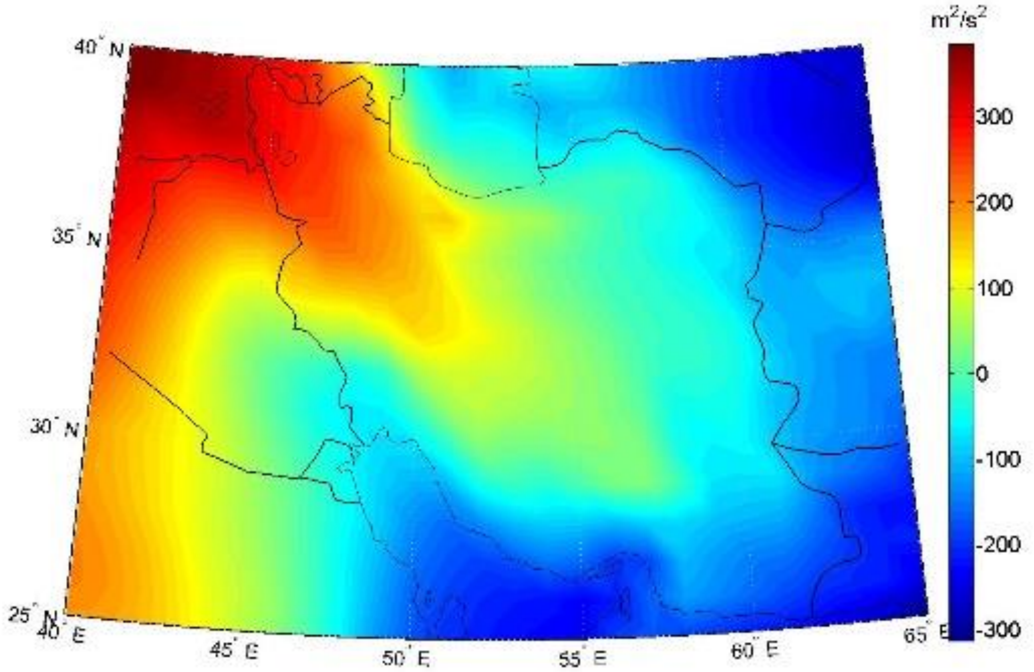


Figure 1: Potential values over Iran from degree $n = 2$ in ellipsoidal harmonic expansion formula for potential values, $1^\circ \times 1^\circ$ – grid

(a2) According to the requirement in (17), the mean value of this grid is subtracted from the values from a1. These "residual" values are shown in Figure 2.



(a3) The data in step **(a2)** are interpolated using the spline interpolant in (32) to produce a $0.5^{\circ} \times 0.5^{\circ}$ grid, and then the removed mean is added. Figure 3 shows these interpolated values.

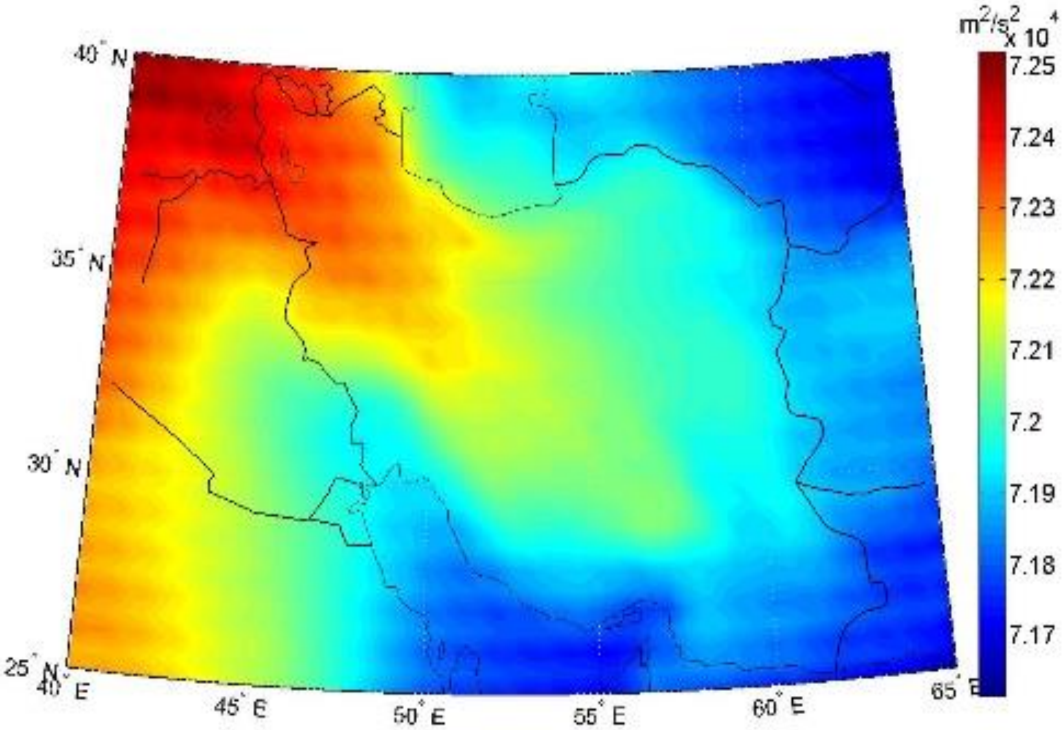


Figure 3: Potential over Iran, interpolated by spline interpolation in (32), $0.5^{\circ} \times 0.5^{\circ}$ grid

(a4) Subtract the interpolated values from those derived from the actual grid by the potential formula. The differences are shown in Figure 4.

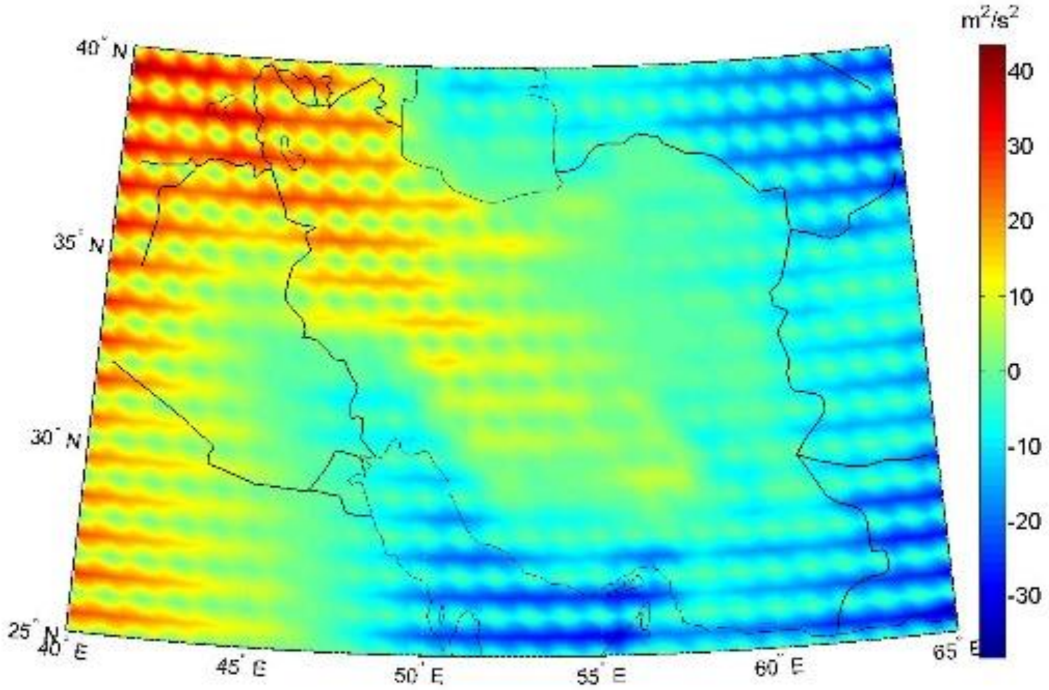


Figure 4: Difference between actual grid, produced by ellipsoidal potential formula, and those derived from interpolation of potential values, $0.5^\circ \times 0.5^\circ$-grid

(a5) Steps a1–a4 are repeated for the spherical spline interpolation to compare spherical and ellipsoidal splines. Figure 4 shows the final result of the difference between the values from the actual grid and those interpolated by the spherical spline.

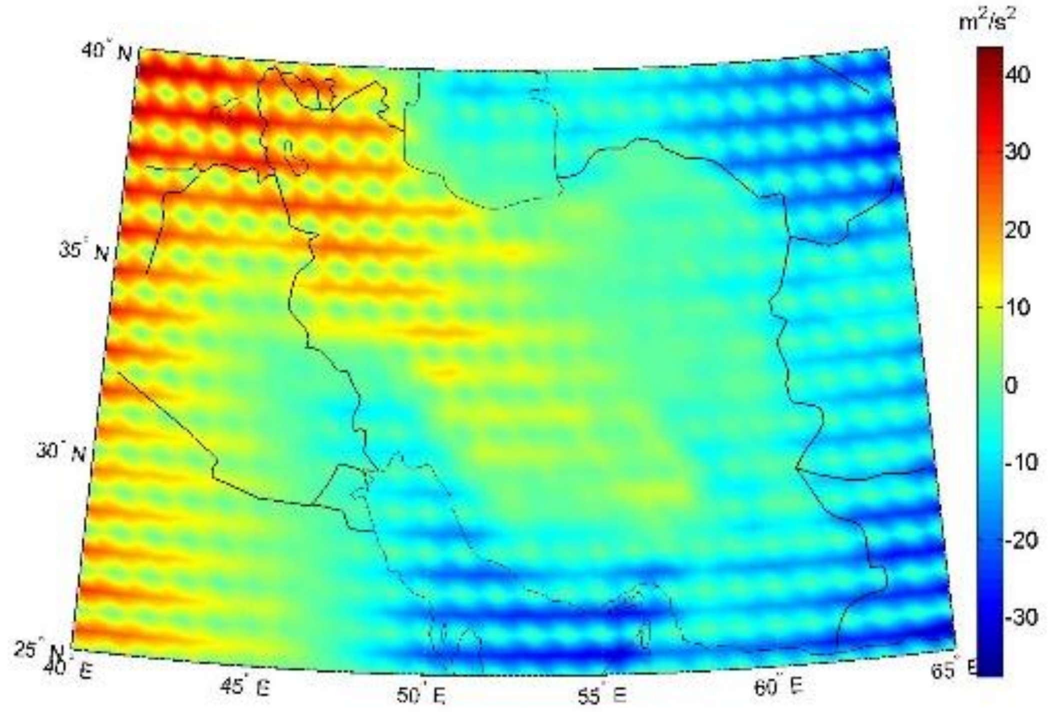


Figure 5: Difference between actual grid, produced by ellipsoidal potential formula, and those derived from the interpolation of potential values by spherical splines, $0.5^\circ \times 0.5^\circ$-grid

Table 1 shows the RMSE of the differences between spherical and ellipsoidal splines

Table 1: Analysis of the difference between the actual grid values and those interpolated by ellipsoidal and spherical splines

Method	RMSE($\frac{m^2}{s^2}$)
Ellipsoidal splines	12.4631
Spherical splines	12.4847

Table 1 conveys that ellipsoidal splines are more accurate than spherical splines. The improvement value is 0.0216 $\frac{m^2}{s^2}$.

6. Conclusions

This paper provided an introduction of ellipsoidal spline and smoothing functions for a set of linearly independent evaluation functionals. The minimization of a given differential operator was performed. Surface Green's functions for different operators in the (semi) Hilbert space of all indefinitely differentiable functions play a crucial role in defining the reproducing kernels. Spline and smoothing functions were defined based on the minimization problem and the corresponding reproducing kernel. This work is a generalization of the spherical and spheroidal cases, i.e., tending some or all of the linear eccentricities to zero. We recovered the spheroidal or spherical spline functions. An application of potential interpolation over Iran was presented in this paper. In this application, a $1^\circ \times 1^\circ$. grid of potentials was interpolated to produce a $0.5^\circ \times 0.5^\circ$ grid, for which both the spherical and ellipsoidal splines were used. The ellipsoidal splines were revealed to be more accurate than the spherical splines since their RMSE is 0.0216 $\frac{m^2}{s^2}$ smaller than that of the spherical splines. This work can be used in different study areas, including Earth's gravity field, where the geometrical structure of the Earth is better modeled with an ellipsoid. In future research works, we intend to extend the concept of spline functions to other manifolds, deal with numerical calculations, and use the results in modeling Earth's gravity field.

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