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# Axisymmetric Scaled Boundary Finite Element Formulation for Wave Propagation in Unbounded Layered Media

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**ABSTRACT:** Wave propagation in unbounded layered media with a new formulation of Axisymmetric Scaled Boundary Finite Element Method (AXI-SBFEM) is derived. Dividing the general three-dimensional unbounded domain into a number of independent two-dimensional ones, the problem could be solved by a significant reduction in required storage and computational time. The equations of the corresponding Axisymmetric Scaled Boundary Finite Element (AXI-SBFE) are derived in detail. For an arbitrary excitation frequency, the dynamic stiffness could be solved by a numerical integration method. The dynamic response of layered unbounded media has been verified with the literature. Numerical examples indicate the applicability and high accuracy of the new method.

**Keywords**: Axisymmetric Scaled Boundary Finite Element Method, Fourier Series, Layered System.

## INTRODUCTION

Correctly modeling of radiation damping in the infinite soil is one of the major challenges in wave propagation problem. Finding the solution of the wave propagation in heterogeneous layered systems are unreachable analytically. Almost all of the existing strategies to model propogation of waves in layered continuum can be categorized as the boundary element method (Karabalis and Mohammadi, 1998; Coulier et al., 2014, Morshedifard and Eskandari-Ghadi, 2017), the thin layer method (TLM) (Lysmer and Waas, 1972; Kausel and Roesset, 1975) or approximate methods (Wolf and Preisig, 2003; Baidya et al., 2006; Nogami and Mahbub, 2005) and semi-analytical (Gazetas, 1980) or analytical methods (Ardeshir-Behrestaghi et al., 2013; Eskandari-Ghadi et al., 2014)

The thin-layer method is semi-analytical widely used approach in frequency-domain formed on an axisymmetric Finite Element formulation. The Finite Element discretization matches the direction of layering.

Another semi-analytical approach, the Scaled Boundary Finite Element Method (SBFEM) (Wolf, 2003), is particularly capable of dynamic analysis in unbounded domains. The SBFEM combines some important benefits of the Finite Element method and the boundary element method.

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Not only does it reduce the spatial dimension of the problem but it also doesn't require any fundamental solution. The material anisotropy can be implemented straightforwardly by modification of the constitutive matrix. A combined SBFE-FEM formulation could be used for dynamic SSI problems (Genes, 2012; Yaseri et al., 2012; Syed and Maheshwari, 2015; Rahnema et al., 2016).

SBFEM is originally uses a scaling center from which the whole boundary shall be observable. By the geometry transformation, it leads to an analytical formulation in the radial direction which could be solved numerically in the circumferential direction.

Some modified SBFE formulations for parallel boundaries two-dimensional domains (Li et al., 2005), three-dimensional prismatic domains (Krome et al., 2017) and axisymmetric domain (Doherty and Deeks, 2003) have been presented. However, none of these formulations can be utilized to model a truly three-dimensional layered media. In order to overcome this shortcoming, the scaling center is replaced by a scaling line in the modified formulation of (Birk and Behnke, 2012). It fully couples the 3D FEM with 2D SBFEM. In order to reduce the computational cost and accuracy of the solution, the formulation of (Birk and Behnke, 2012) has been modified to an axisymmetric SBFEM which only a line discretization is needed to find the solution.

## AXI-SBFEM

Consider a scaling line, identical to the axis of symmetry (the z-axis) in AXI-SBFEM as shown in Figure 1. The radial direction ξ is perpendicular to the scaling line. ξ is 0 at the z-axis and 1 at the line S.

In this paper, the line S is revolved around the axis of symmetry in order to define the domain boundary, as shown in Figure1. The corresponding transformation is formulated as:

$$r = \xi . r_s\ (s) \tag{1a}$$

$$z = z_s(s) \tag{1b}$$

$$\theta = \theta \tag{1c}$$

The displacement components of an axisymmetric solid in the cylindrical coordinate system (r, z, $\theta$) can be written as:

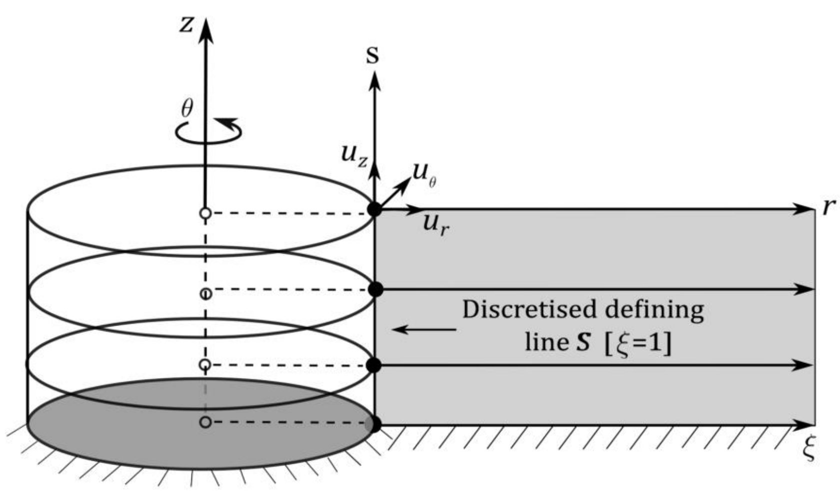



**Fig. 1.** Axisymmetric domain modeled by SBFEM

$$u_r(r,z,\theta) = \sum_{n=0}^{\infty} u_r(r,z,\theta,n) = \sum_{n=0}^{\infty} (u_r^s(r,z,n) cosn\theta + u_r^a(r,z,n) \sin n\theta) \tag{2a}$$

$$u_z(r,z,\theta) = \sum_{n=0}^{\infty} u_z(r,z,\theta,n) = \sum_{n=0}^{\infty} (u_z^s(r,z,n) cosn\theta + u_z^a(r,z,n) \sin n\theta) \tag{2b}$$

$$u_\theta(r,z,\theta) = \sum_{n=0}^{\infty} u_\theta(r,z,\theta,n) = \sum_{n=0}^{\infty} (-u_\theta^s(r,z,n) sin\, n\theta + u_\theta^a(r,z,n) \cos n\theta) \tag{2c}$$

The governing equation for the linear elstodynamics state

$$\bar{L}^T \sigma + \omega^2 \rho u = 0 \tag{3}$$

where $\sigma$, u and $\bar{L}$: are stresses, displacements and the equilibrium operator in cylindrical coordinate. The strains follow from the displacements as

$$\epsilon = Lu \tag{4}$$

which $L$: is a differential operator in cylindrical coordinate and can be found in Aslmand et al. (2018). The approximate solution of Eq. (3) is obtained using a Fourier series in the direction of $\theta$ and linear shape functions along the S which leads to a series of ODEs in terms of $\xi$. Hence, a solution is written in the form

$$\begin{Bmatrix} u_r(\xi,s,\theta) \\ u_z(\xi,s,\theta) \\ u_\theta(\xi,s,\theta) \end{Bmatrix} = \sum_{n=0}^{\infty} \{[F_u^s(\theta,n)][N(s)]\{u^s(\xi,n)\} + [F_u^a(\theta,n)][N(s)]\{u^a(\xi,n)\}\} \tag{5}$$

where

$$[F_u^s(\theta,n)] = \begin{bmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & -\sin n\theta \end{bmatrix}, [F_u^s(\theta,n)] = \begin{bmatrix} \sin n\,\theta & 0 & 0 \\ 0 & \sin n\,\theta & 0 \\ 0 & 0 & \cos n\theta \end{bmatrix} \tag{6}$$

[N(s)]: is the shape functions matrix, and $\{u^s(\xi,n)\}$ and $\{u^a(\xi,n)\}$: stand for the variation of the nodal displacement in the $\xi$ direction for the symmetric and anti-symmetric Fourier terms, respectively.

Mapping to the coordinate system of scaled boundary, the linear operator $L$ is broken in parts such that:

$$[L] = [L^1]\frac{\partial}{\partial r} + [L^2]\frac{\partial}{\partial z} + [L^3]\frac{1}{r} + [L^4]\frac{1}{r}\frac{\partial}{\partial\theta} + [L^5]\frac{1}{r}\frac{\partial}{\partial\theta} \tag{7}$$

The operators $\delta/\delta r$ and $\delta/\delta z$: are mapped to the scale boundary coordinate and the operator is then expressed as:

$$[L] = [b^1(s)]\frac{\partial}{\partial r} + [b^2(s)]\frac{\partial}{\partial z} + [b^3(s)]\frac{1}{r} + [b^4(s)]\frac{1}{r}\frac{\partial}{\partial\theta} + [b^5(s)]\frac{1}{r}\frac{\partial}{\partial\theta} \tag{8}$$

where $[b^1]$ to $[b^5]$ can be found in Aslmand et al. (2018). Multiplying Eq. (8) by Eq. (5) and substituting into strain-stress relationship, terms can be gathered resulting the following approximate stresses:

$$\{\sigma(\xi,s,\theta)\} = \begin{Bmatrix} \sigma_r(\xi,s,\theta) \\ \sigma_z(\xi,s,\theta) \\ \sigma_\theta(\xi,s,\theta) \\ \tau_{rz}(\xi,s,\theta) \\ \tau_{z\theta}(\xi,s,\theta) \\ \tau_{r\theta}(\xi,s,\theta) \end{Bmatrix} = \sum_{n=0}^{\infty}\{[D(s)][F_\varepsilon^s(n,\theta)]([B^1\,(s)]\,\{u^s(\xi,n)\}_{,\xi}\left(\frac{1}{\xi}[B^2(s,n)] + [B^3(s)]\right)\{u^s(\xi,n)\}) + [D(s)][\,F_\varepsilon^a(n,\theta)]([B^1(s)]_{,\xi}\left(\frac{1}{\xi}[B^2(s,n)] + [B^3(s)]\right)\{u^a(\xi,n)\})\} \tag{9}$$

where the terms

$$[B^1\,(s)] = [b^1(s)][N(s)] \tag{10}$$

$$[B^2(s,n)] = \Big[\,[b^3(s)] + \,n\,[b^4(s)] - \,n\,[b^5(s)]\Big]\,[N\,(s)] \tag{11}$$

$$[B^3(s)] = [b^2(s)][N(s)]_{,s} \tag{12}$$

are introduced for convenience. $[\,F_\varepsilon^s(n,\theta)]$ and $[\,F_\varepsilon^a(n,\theta)]$: stand for the variation in the circumferential direction.

$$[\,F_\varepsilon^s(n,\theta)] = \begin{bmatrix} \cos n\,\theta & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos n\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos n\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos n\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sin n\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sin n\theta \end{bmatrix} \tag{13}$$

and

$$[\,F_\varepsilon^s(n,\theta)] = \begin{bmatrix} \sin n\,\theta & 0 & 0 & 0 & 0 & 0 \\ 0 & \sin n\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & \sin n\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin n\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos n\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos n\theta \end{bmatrix} \tag{14}$$

The virtual work's principle for the dynamic case states:

$$\delta U + \,\delta K - \,\delta W \,= 0 \tag{15}$$

where $U$, $K$ and $W$: are internal strain energy, structure's kinetic energy and external work caused by boundary traction $\{t(s,\theta)\}$, respectively (It is supposed that all of the surface tractions are in the near field). The contribution of the three terms to the virtual work will be derived in the following. To derive these three terms, the following form of virtual displacement.

$$\{\delta u\,(\xi,s,\theta)\} = \sum_{n=0}^{\infty}\{[F_u^s(n,\theta)]\,[N(s)]\{\delta u^s(\xi,n)\} + [F_u^a(n,\theta)]\,[N(s)]\{\delta u^a(\xi,n)\}] \tag{16}$$

is applied to the structure, where $\{\delta u^s(\xi,n)\}$ and $\{\delta u^a(\xi,n)\}$ includes symmetric and anti-symmetric virtual nodal displacements for the $n^{th}$ term of the Fourier series. The corresponding virtual strain field proceed from Eq. (9) as:

$$\{\delta\varepsilon(\xi,s,\theta)\} = \sum_{m=0}^{\infty}\Big\{[F_\varepsilon^s(m,\theta)]\Big([B^1(s)]\,\{\delta u^s(\xi,m)\},_\xi(\frac{1}{\xi}[B^2(s,m)] + [B^3(s)]\Big)\{\delta u^s(\xi,m)\}[F_\varepsilon^a(m,\theta)]([B^1(s)\{\delta u^a(\xi,m)\},_\xi(\frac{1}{\xi}[B^2(s,m)] + [B^3(s)])\{\delta u^a(\xi,m)\})\Big\} \tag{17}$$

The virtual work equation (Eq. (15)) can be written as:

$$\int_V \{\delta\varepsilon\,(\xi,s,\theta)\}^T\{\sigma\,(\xi,s,\theta)\}dV + \int_V \{\delta u(\xi,s,\theta)\}\,\rho\{\ddot{u}(\xi,s,\theta)\}dV - \int_s\int_0^{2\pi}\{\delta u\ \ (\delta u\,(s,\theta)\}^T\{t(s,\theta)\}|J(s)|d\theta ds = 0 \tag{18}$$

where

$$dV = |J(s)|\xi r_s(s)d\theta\, d\xi ds \tag{19}$$

$|J(s)|$: is the discretized boundary's Jacobian. Expansion of the internal virtual work could be done by substituting Eqs. (9), (17) and (19) into the first term of Eq. (18). It should be noted that for all $m$ and $n$:

$$\int_0^{2\pi}[F_\varepsilon^s\,(n,\theta)]\,[F_\varepsilon^a\,(m,\theta)]d\theta = \int_0^{2\pi}[F_\varepsilon^a\,(n,\theta)][F_\varepsilon^s\,(m,\theta)]d\theta = [0] \tag{20}$$

and for all $m$ and $n$ with $m \neq n$ :

$$\int_0^{2\pi}[F_\varepsilon^s\,(n,\theta)]\,[F_\varepsilon^s\,(m,\theta)]d\theta = \int_0^{2\pi}[F_\varepsilon^a\,(n,\theta)][F_\varepsilon^a\,(m,\theta)]d\theta = [0] \tag{21}$$

The following terms for convenience are introduced:

$$[D^a(n,s)] = [D(s)]\int_0^{2\pi}[F_\varepsilon^s\,(n,\theta)][F_\varepsilon^s\,(n,\theta)]d\theta \tag{22}$$

$$[D^a(n,s)] = [D(s)]\int_0^{2\pi}[F_\varepsilon^a\,(n,\theta)][F_\varepsilon^a\,(n,\theta)]d\theta \tag{23}$$

and the series become:

$$\begin{aligned}\int_V \{\delta\varepsilon(\xi,s,\theta)\}^T\{\sigma(\xi,s,\theta)\}dV \\ = \sum_{n=0}^{\infty}\int_1^{\infty}\int_S (\{[B^1(s)]\{\delta u^s(\xi,n)\}_{,\xi} \\ + \left(\frac{1}{\xi}[B^2(s,n)]+[B^3(s)]\right)\{\delta u^s(\xi,n)\}\}^T.[D^a(n,s)].\{[B^1(s)]\Big\{\{u^s(\xi,n)\}_{,\xi} \\ + \left(\frac{1}{\xi}[B^2(s,n)]+[B^3(s)]\right)\{u^s(\xi,n)\}\Big\} \\ + \{[B^1(s)]\Big\{\{\delta u^a(\xi,n)\}_{,\xi} \\ + \Big(\frac{1}{\xi}[B^2(s,n)] \\ + [B^3(s)]\Big)\{\delta u^a(\xi,n)\}\Big\}^T.[D^a(n,s)].\Big\{[B^1(s)]\{u^a(\xi,n)\}_{\xi}\Big(\frac{1}{\xi}[B^2(s,n)] \\ + [B^3(s,n)]\Big)\{u^a(\xi,n)\}\Big\})\xi|J(s)|r_s(s)dsd\xi\end{aligned} \tag{24}$$

Using Green's theorem, the area integrals involving $\{\delta u(\xi,n)\}_{\xi}$ are integrated with respect to $\xi$ in line around S and leads to:

$$\begin{aligned}\delta U = \sum_{n=0}^{\infty}(\{\delta u^s\,(n)\}^T\{[E^{0s}(n)]\{u^s(n)\}_{\xi} + \{[E^{1s}(n)^T + [E^{3s}(n)]\}\{u^s(n)\} \\ - \int_1^{\infty}\{\,\delta u^s(\xi,n)\}^T\{[E^{0s}(n)]\xi\{\delta u^s(\xi,n)\}_{\xi\xi} + \{[E^{0s}(n) - [E^{1s}(n)]) \\ + [E^{1s}(n)\}^T + \ \xi[E^{3s}(n)] - \xi[E^{3s}(n)]^T)\{u^s(\xi,n)\}_{\xi} \\ + \Big(-\frac{1}{\xi}[E^{2s}(n)] + [E^{3s}(n)] - [E^{4s}(n)] - [E^{4s}(n)]^T \\ - \xi[E^{5s}(n)]\Big)\{u^s(\xi,n)\}\,d\xi + \ \{\delta u^a(n)\}^T\{[E^{0a}(n)]\{u^a(n)\}_{\xi} \\ + ([E^{1a}(n)]^T + [E^{3a}(n)^T + [E^{3a}(n)])\,\{u^a(n)\}\} \\ - \int_1^{\infty}\{\,\delta u^a(\xi,n)\}^T\{[E^{0s}(n)]\xi\{\delta u^a(\xi,n)\}_{\xi\xi} + \{[E^{0a}(n) - [E^{1a}(n)]) \\ + [E^{1a}(n)] + E^{1a}(n)\}^T + \ \xi[E^{3a}(n)] - \xi[E^{3a}(n)]^T\}\{u^a(\xi,n)\}_{\xi} \\ + \Big(-\frac{1}{\xi}[E^{2a}(n)] + [E^{3a}(n)] - [E^{4a}(n)] - [E^{4a}(n)]^T \\ - \xi[E^{5a}(n)]\Big)\{u^a(\xi,n)\}\}\,d\xi)\end{aligned} \tag{25}$$

where the components are:

$$[E^{0s/a}(n)] = \int_s [B^1\,(s)]^T\big[D^{s/a}(n,s)\big][B^1(s)]|J(s)|r_s(s)ds \tag{26a}$$

$$[E^{1s/a}(n)] = \int_S [B^2\,(s,n)]^T \left[D^{s/a}(n,s)\right][B^1(s)]|J(s)|r_s(s)ds \tag{26b}$$

$$[E^{2s/a}(n)] = \int_S [B^2\,(s,n)]^T \left[D^{s/a}(n,s)\right][B^2(s,n)]|J(s)|r_s(s)ds \tag{26c}$$

$$[E^{3s/a}(n)] = \int_S [B^1\,(s)]^T \left[D^{s/a}(n,s)\right][B^3(s)]|J(s)|r_s(s)ds \tag{26d}$$

$$[E^{4s/a}(n)] = \int_S [B^2\,(s,n)]^T \left[D^{s/a}(n,s)\right][B^3(s)]|J(s)|r_s(s)ds \tag{26e}$$

$$[E^{5s/a}(n)] = \int_S [B^3\,(s)]^T \left[D^{s/a}(n,s)\right][B^1(s)]|J(s)|r_s(s)ds \tag{26f}$$

The discretized boundary $S$ can be split into linear 2-node elements. The global coefficient matrices are the assembly of the computed coefficient matrices for each element over the discretized boundary.

The virtual kinetic energy can be written as:

$$\begin{aligned}\delta k = \int_V \{\delta u(\xi,s,\theta)\}\,\rho\{\ddot{u}(\xi,s,\theta)\}dV \\ = \sum_{n=0}^{\infty}\int_1^{\infty}\int_S \;(\{[F_u^s\,(n,\theta)][N(s)]\{\,\delta u^s(\xi,n)] \\ + [F_u^a\,(n,\theta)][N(s)]\{\delta u^a(\xi,n)\}\}^T.\rho.[F_u^s\,(n,\theta)][N(s)]\{\ddot{u}^s(\xi,n)\} \\ + [F_u^a\,(n,\theta)][N(s)]\{\ddot{u}^s(\xi,n)\}\})\xi|J(s)|r_s(s)dsd\xi\end{aligned} \tag{27}$$

with the mass density $\rho$. The mass matrices are found as:

$$\left[M^{0s/a}\right](n) = \int_S [N(s)]^T\,[T^{s/a}(n)\ddot{\rho}[N(s)]r_s(s)|J(s)|ds \tag{28}$$

where

$$[T^s(n)] = \int_0^{2\pi} [F_u^s\,(n,\theta)]\,[F_u^s\,(n,\theta)]d\theta = \int_0^{2\pi}\begin{bmatrix} cos^2\,n\theta & 0 & 0 \\ 0 & cos^2\,n\theta & 0 \\ 0 & 0 & sin^2\,n\theta \end{bmatrix} \tag{29a}$$

$$[T^a(n)] = \int_0^{2\pi} [F_u^a\,(n,\theta)]\,[F_u^a\,(n,\theta)]d\theta = \int_0^{2\pi}\begin{bmatrix} sin^2\,n\theta & 0 & 0 \\ 0 & sin^2\,n\theta & 0 \\ 0 & 0 & cos^2\,n\theta \end{bmatrix} \tag{29b}$$

Eq. (27) leads to:

$$\delta K = \sum_{n=0}^{\infty}\int_1^{\infty}\{\delta u^s(\xi,n)\}^T\,[M_0^s\,(n)]\{\ddot{u}^s(\xi,n)\}\xi d\xi + \{\delta u^a(\xi,n)]^T[M_0^a(n)]\{\ddot{u}^a(\xi,n)\}\xi d\xi \tag{30}$$

Formulation of the boundary tractions follows:

$$\{t(s,\theta)\} = \sum_{n=0}^{\infty}\{ [F_u^s\ (n,\theta)]\{t^s(s,n)\} + [F_u^a\ (n,\theta)]\ \{t^a(s,n)\}\} \tag{31}$$

where $\{t^s(s,n)\}$ and $\{t^a(s,n)\}$: are amplitudes of tractions. The external virtual work of surface tractions (Eq. (18)) then becomes:

$$\begin{aligned}\int_1 \int_0^{2\pi} \{\delta u(s,\theta)\}^T\{t(s,\theta)\}|J(s)|d\theta ds = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\int_S\int_0^{2\pi}\{\{\delta u^s(m)\}^T[N(s)]^T[F_u^s\ (m,\theta)] \\ + \{\delta u^s(m)\}^T[N(s)]^T[F_u^a\ (m,\theta)]\}\{[F_u^a\ (n,\theta)]\{t^s(s,n)\} \\ + [F_u^a\ (n,\theta)]\{t^a\ (s,n)\}\}|J(s)|d\theta ds\end{aligned} \tag{32}$$

Decoupling the symmetric and anti-symmetric terms with respect to $\theta$ leads to:

$$\delta W = \sum_{n=0}^{\infty}\{\{\delta u^s(n)\}^T\{P^s(n)\} + \{\delta u^a(n)\}^T\ \{P^a(n)\}\} \tag{33}$$

where $\{P^s(n)\}$ and $\{P^a(n)\}$: are equivalent nodal forces obtained from load decomposition into Fourier series, and expressed as:

$$\left[P^{s/a}(n)\right] = \int_S [N(s)]^T\left[T^{s/a}(n)\right]\{t^s(s,n)\}|J(s)|ds \tag{34}$$

Eq (15) then becomes:

$$\begin{aligned}&\textstyle\sum_{n=0}^{\infty}\{\delta u^s(n)\}^T\{[E^{0s}(n)]\left\{u^s(n)\}_\xi + ([E^{1s}(n)]^T + [E^{3s}(n)])\{u^s(n)\}\right\} - \\ &\textstyle\int_1^{\infty}\{\delta u^s(\xi,n)\}^T\{[E^{0s}(n)]\xi\{\delta u^s(\xi,n)\}_{\xi\xi} + \{[E^{0s}(n) - [E^{1s}(n)]) + [E^{1s}(n)]^T + \xi[E^{3s}(n)] - \\ &\xi[E^{3s}(n)]^T)\{u^s(\xi,n)\}_\xi + \left(-\frac{1}{\xi}[E^{2s}(n)] + [E^{3s}(n)] - [E^{4s}(n)] - [E^{4s}(n)]^T - \right. \\ &\left.\xi[E^{5s}(n)]\right)\{u^s(\xi,n)\}\}d\xi + \textstyle\int_1^{\infty}\{\delta u^s(\xi,n)\}^T[M_0^s(n)]\,\{\ddot{u}^s(\xi,n)\}\xi d\xi - \sum_{n=0}^{\infty}\{\delta u^s(n)\}^T\{P^s(n)\} + \\ &\left\{\{\delta u^a(n)\}^T\{\left[E^{0a}(n)[u^a(n)]\right\}_\xi + ([E^{1a}(n)]^T[E^{3a}(n)])\{u^a(n)\}\right\} - \\ &\textstyle\int_1^{\infty}\{\delta u^a(\xi,n)\}^T\{[E^{0a}(n)]\xi\{\delta u^a(\xi,n)\}_{\xi\xi} + \{[E^{0a}(n) - [E^{1a}(n)]^T) + \xi[E^{3a}(n)] - \\ &\xi[E^{3a}(n)]^T)[u^a(\xi,n)\}_\xi + \left(-\frac{1}{\xi}[E^{2a}(n)] + [E^{3a}(n)] - [E^{4a}(n)] - [E^{4a}(n)]^T - \right. \\ &\left.\xi[E^{5a}(n)]\right)\{u^a(\xi,n)\}\,d\xi\} + \textstyle\int_1^{\infty}\{\delta u^s(\xi,n)\}^T[M_0^s(n)]\,\{\ddot{u}^a(\xi,n)\}\xi d\xi - \sum_{n=0}^{\infty}\{\delta u^a(n)\}^T\ \{P^a(n)\}) = 0\end{aligned} \tag{35}$$

in order for Eq. (35) to be satisfied for all $\delta u^s(\xi,n)$ and $\delta u^a(\xi,n)$, the following requirements must be met for each of the symmetric/anti-symmetric Fourier series term:

$$\{P^s(n)\} = [E^{0s}(n)u^s(n)_\xi + ([E^{1s}(n)]^T + [E^{3s}(n)])\{u^s(n)\} \tag{36}$$

$$\begin{aligned}[E^{0s}(n)]\xi\,\{u^s(\xi,n)]_{\xi\xi} \\ + ([E^{0s}(n)] - [E^{1s}(n)] + [E^{1s}(n)]^T + \xi[E^{3s}(n)] - \xi[E^{3s}(n)]^T)\{u^s(\xi,n)\}_\xi \\ + \omega^2[M_0^s(n)]\xi\{u^s(\xi,n)\} = 0\end{aligned} \tag{37}$$

$$\{P^a(n)\} = [E^{0a}(n)u^a(n)_\xi + ([E^{1a}(n)]^T + [E^{3a}(n)])\{u^a(n)\} \tag{38}$$

$$\begin{aligned}[E^{0a}(n)]\xi\,\{u^a(\xi,n)]_{\xi\xi} \\ + ([E^{0a}(n)] - [E^{1a}(n)] + [E^{1a}(n)]^T + \xi[E^{3a}(n)] - \xi[E^{3a}(n)]^T)\{u^a(\xi,n)\}_\xi \\ + \omega^2[M_0^a(n)]\xi\{u^a(\xi,n)\} = 0\end{aligned} \tag{39}$$

Eqs. (36-39) and (38-39) are pairs of equations. Based on the number of required terms to represent the load or displacement, a number of equations could be solved separately, and the total stresses and displacements are obtained by superposition. Eqs. (37) and (39) are the displacement SBFE equations for each term of Fourier series in an unbounded layered system.

In order to derive an equation in dynamic stiffness which is the fraction of the coupling forces to the coupling displacement, a transformation of equations is made in the following.

## AXI-SBFE Equation in Dynamic Stiffness

The internal nodal forces $\{Q^s(\xi,n)\}$ and $\{Q^a(\xi,n\}$ could be expressed applying the virtual work's principle.

$$\begin{aligned}\{Q^s(\xi,n)\} \\ = [E^{0s}(n)]\xi\{u(\xi,n)\}_{,\xi} \\ + ([E^{1s}(n)]^T + \xi\,[E^{3s}(n)])\{u(\xi,n)\}\end{aligned} \tag{40}$$

$$\begin{aligned}\{Q^a(\xi,n)\} \\ = [E^{0a}(n)]\xi\{u(\xi,n)\}_{,\xi} \\ + ([E^{1a}(n)]^T + \xi\,[E^{3a}(n)])\{u(\xi,n)\}\end{aligned} \tag{41}$$

In the case of an unbounded domain, the external nodal loads ($\{R^s(\xi,n)\}$ and $\{R^a(\xi,n)\}$) are related to the internal nodal forces ($\{Q^s(\xi,n)\}$ and $\{Q^a(\xi,n)\}$) as follows:

$$\{R^s(\xi,n)\} = -\{Q^s(\xi,n)\} \tag{42a}$$

$$\{R^a(\xi,n)\} = -\{Q^a(\xi,n)\} \tag{42b}$$

The dynamic stiffness $S^s(\omega,\xi,n)$and $S^s(\omega,\xi,n)$ are defined as:

$$\begin{aligned}\{R^s(\xi,n)\} = \{S^s(\omega,\xi,n)]\{u^s(\xi,n)\} \\ - \{R^{Fs}(\xi,n)\}\end{aligned} \tag{43a}$$

$$\begin{aligned}\{R^a(\xi,n)\} = \{S^a(\omega,\xi,n)]\{u^a(\xi,n)\} \\ - \{R^{Fa}(\xi,n)\}\end{aligned} \tag{43b}$$

The terms $\{R^{Fs}(\xi,n\})$ and $\{R^{Fa}(\xi,n)\}$ in Eqs. (43a-43b) correspond to the symmetric and anti-symmetric nodal loads arising from volume forces. Substituting Eqs. (40-41) and (43a-43b) in Eqs. (42a-42b) yields:

$$\begin{aligned}-\{S^s(\omega,\xi,n)]\{u^s(\xi,n)\} \\ + \{R^{Fs}(\xi,n)\} \\ = [E^{0s}(n)]\xi\{u^s(\xi,n)\}_{,\xi} \\ + ([E^{1s}(n)]^T \\ + \xi\,[E^{3s}(n)])\{u^s(\xi,n)\}\end{aligned} \tag{44a}$$

$$\begin{aligned}-\{S^s(\omega,\xi,n)]\{u^s(\xi,n)\} \\ + \{R^{Fs}(\xi,n)\} \\ = [E^{0s}(n)]\xi\{u^s(\xi,n)\}_{,\xi} \\ + ([E^{1s}(n)]^T \\ + \xi\,[E^{3s}(n)])\{u^s(\xi,n)\}\end{aligned} \tag{44b}$$

Differentiating Eqs. (44a-44b) with respect to $\xi$ yields:

$$
\begin{aligned}
&-\{S^a(\omega,\xi,n)],_\xi \{u^s(\xi,n)\} \\
&- \{S^s(\omega,\xi,n)\}\{u^s(\xi,n)\},_\xi \\
&+ \{R^{FS}(\xi,n)\},_\xi \\
&- [E^{0s}(n)]\xi\{u^s(\xi,n)\},_{\xi\xi} \\
&- ([E^{0s}(n)] + [E^{1s}(n)]^T \\
&+ \xi[E^{3s}(n)])\{u^s(\xi,n)\},_\xi \\
&- [3^{3s}(n)]\{u^s(\xi,n)\} = 0
\end{aligned} \tag{45a}
$$

$$
\begin{aligned}
&-\{S^a(\omega,\xi,n)],_\xi \{u^s(\xi,n)\} \\
&- \{S^a(\omega,\xi,n)\}\{u^a(\xi,n)\},_\xi \\
&+ \{R^{Fa}(\xi,n)\},_\xi \\
&- [E^{0s}(n)]\xi\{u^a(\xi,n)\},_{\xi\xi} \\
&- ([E^{0a}(n)] + [E^{1a}(n)]^T \\
&+ \xi[E^{3a}(n)])\{u^a(\xi,n)\},_\xi \\
&- [E^{3a}(n)]\{u^a(\xi,n)\} = 0
\end{aligned} \tag{45b}
$$

Adding Eqs. (45a-45b) and the SBFE Eqs. (37-39) and multiplying by $\xi$ yields:

$$
\begin{aligned}
&\xi\,(-[S^s(\omega,\xi,n)] - [E^{1s}(n)] \\
&- \xi[E^{3s}(n)]^T)\{u^s(\xi,n)\},_\xi \\
&+ \xi(-[S^s(\omega,\xi,n)],_\xi - \frac{1}{\xi}[E^{2s}(n)] \\
&- [E^{4s}(n)] - [E^{4s}(n)]^T \\
&- \xi[E^{5s}(n)])\,\{u^s(\xi,n)\} \\
&+ \omega^2[M^{0s}(n)]\xi^2\{u^s(\xi,n)\} \\
&+ \xi\{R^{Fs}(\xi,n)\},_\xi = 0
\end{aligned} \tag{46a}
$$

$$
\begin{aligned}
&\xi\,(-[S^a(\omega,\xi,n)] - [E^{1s}(n)] \\
&- \xi[E^{3a}(n)]^T)\{u^a(\xi,n)\},_\xi \\
&+ \xi(-[S^a(\omega,\xi,n)],_\xi - \frac{1}{\xi}[E^{2a}(n)] \\
&- [E^{4a}(n)] - [E^{4a}(n)]^T \\
&- \xi[E^{5a}(n)])\,\{u^a(\xi,n)\} \\
&+ \omega^2[M^{0a}(n)]\xi^2\{u^a(\xi,n)\} \\
&+ \xi\{R^{Fa}(\xi,n)\},_\xi = 0
\end{aligned} \tag{46b}
$$

Eqs. (44a-44b) are solved for $\xi\, u_{,\xi,}$.

$$
\begin{aligned}
&[E^{0s}(n)]^{-1}((-[S^s(\omega,\xi,n)] \\
&- [E^{1s}(n)]^T - \xi\,[E^{3s}(n)])\{u^s(\xi,n)\} \\
&+ \{R^{FS}(\xi,n)\}) = \xi\{u^s(\xi,n)\},_\xi
\end{aligned} \tag{47a}
$$

$$
\begin{aligned}
&[E^{0a}(n)]^{-1}((-[S^a(\omega,\xi,n)] \\
&- [E^{1a}(n)]^T - \xi\,[E^{3a}(n)])\{u^a(\xi,n)\} \\
&+ \{R^{Fa}(\xi,n)\}) = \xi\{u^a(\xi,n)\},_\xi
\end{aligned} \tag{47b}
$$

Eqs. (46a-46b) are substituted in Eqs. (47a-47b).

$$
\begin{aligned}
&(-[S^s(\omega,\xi,n)] - [E^{1s}(n) \\
&- \xi[E^{3s}(n)^T)[E^{0s}(n)]^{-1}(-[S^s(\omega,\xi,n)] \\
&- [E^{1s}(n)]^T - \xi[E^{3s}(n)])\{u^s(\xi,n)\} \\
&+ \xi(-[S^s(\omega,\xi,n)],_\xi - \frac{1}{\xi}[E^{2s}(n) \\
&- [E^{4s}(n)]^T - \xi[E^{5s}(n)])\{u^s(\xi,n)\} \\
&+ \omega^2[M^0(n)]\xi^2\{u^s(\xi,n)\} \\
&- \xi\{R^{FS}(n)(\xi)\},_\xi - (-[S^s(\omega,\xi)] \\
&- [E^{1s}(n)] \\
&- [E^{3s}(n)]^T)[E^{0s}(n)]^{-1}\{R^{Fs}(\xi,n)\} = 0
\end{aligned} \tag{48a}
$$

$$
\begin{aligned}
&(-[S^a(\omega,\xi,n)] - [E^{1s}(n) \\
&- \xi[E^{3a}(n)^T)[E^{0a}(n)]^{-1}(-[S^a(\omega,\xi,n)] \\
&- [E^{1a}(n)]^T - \xi[E^{3a}(n)])\{u^a(\xi,n)\} \\
&+ \xi(-[S^a(\omega,\xi,n)],_\xi - \frac{1}{\xi}[E^{2a}(n) \\
&- [E^{4a}(n)]^T - \xi[E^{5a}(n)])\{u^a(\xi,n)\} \\
&+ \omega^2[M^0(n)]\xi^2\{u^s(\xi,n)\} \\
&- \xi\{R^{Fa}(n)(\xi)\},_\xi - (-[S^a(\omega,\xi)] \\
&- [E^{1a}(n)] \\
&- [E^{3a}(n)]^T)[E^{0a}(n)]^{-1}\{R^{Fa}(\xi,n)\} = 0
\end{aligned} \tag{48b}
$$

For any displacements, the terms related with $\{u^s(\xi,n)\}$ and $\{u^a(\xi,n)\}$ should disappear. Thus, Equations for the dynamic stiffness matrix is acquired.

$$
\begin{aligned}
&([S^s(\omega,\xi,n)] + [E^{1s}(n)] \\
&+ \xi[E^{3a}(n)]^T[E^{0s}(n)]^{-1}[[S^s(\omega,\xi,n)] \\
&+ (E^{1s}(n)]^T + \xi[E^{3s}(n)]) \\
&- \xi[S^s(\omega,\xi,n)],_\xi - \xi\,([E^{4s}(n)] \\
&+ [E^{4s}(n)]^T) - \xi^2[E^{5s}(n)] \\
&+ \omega^2\xi^2[M^{0s}(n)] = 0
\end{aligned} \tag{49a}
$$

$$
\begin{aligned}
&([S^a(\omega,\xi,n)] + [E^{1a}(n)] \\
&+ \xi[E^{3a}(n)]^T[E^{0a}(n)]^{-1}[[S^a(\omega,\xi,n)] \\
&+ (E^{1a}(n)]^T + \xi[E^{3a}(n)]) \\
&- \xi[S^a(\omega,\xi,n)],_\xi - ([E^{2a}(n)] \\
&- \xi([E^{4a}(n)]^T) - [E^{4a}(n)]^T) \\
&- \xi^2[E^{5a}(n)] + \omega^2\xi^2\,[M^{0a}(n)] = 0
\end{aligned} \tag{49b}
$$

Eqs. (49a-49b) are the AXI-SBFE equation in dynamic stiffness for an infinite layered medium.

The nonlinear first-order differential equation of variable $\xi$ is solved for a particular frequency $\omega^*$, employing a Runge-Kutta numerical integration method. Utilizing an asymptotic expansion of the dynamic stiffness, the required initial value for the numerical integration is calculated in the following.

**Expression of Dynamic Stiffness as Power Series in $\xi$**

The unknown matrix of dynamic stiffness $[S^{\infty}(\xi, n)]$ for both symmetric and anti-symmetric is expanded as a decreasing exponent power series in $(\xi)$. In the following the superscript s and a and the series number *n* has been omitted for convenience.

$$[S^{\infty}(\xi)] \approx (\xi)^1[C_\infty] + (\xi)^0[K_\infty] + \sum_{j=1}^{m} \frac{1}{(\xi)^j}[A_j] \tag{50}$$

The eigenvalue problem

$$\begin{aligned} &[M^0][\Phi] = [E^0][\Phi][m^0], \\ &[\Phi]^T[E^0][\Phi] = [I], \\ &[\Phi]^T[M^0][\Phi] = [m^0]. \end{aligned} \tag{51}$$

is used to transform Eq. (49a-49b) into:

$$\begin{aligned} ([s^{\infty}(\xi)] + [e^1] + \xi[e^3]^T)\,([s^{\infty}(\xi)] + [e^1]^T + \xi[e^3]\ ) \\ - \xi\,[s^{\infty}(\xi)]_{,\xi} \\ - [e^2] - \xi([e^4] + [e^4]^T) - \xi^2[e^5] \\ + \omega^2\xi^2[m^0] = 0, \end{aligned} \tag{52}$$

with

$$[s^{\infty}(\xi)] = [\Phi]^T[S^{\infty}(\xi)]\,[\Phi] \tag{53}$$

$$[e^j] = [\Phi]^T[E^j][\Phi], \quad j = 1,2, \ldots ,5. \tag{54}$$

Eq. (50) could be written as:

$$[s^{\infty}(\xi)] \approx (\xi)^1[c_\infty] + (\xi)^0[k_\infty] + \sum_{j=1}^{m} \frac{1}{(\xi)^j}[a_j]. \tag{55}$$

where

$$[c_\infty] = [\Phi]^T[C_\infty][\Phi], \qquad [k_\infty] = [\Phi]^T\,[k_\infty] = [\Phi], \quad [a_j] = [\Phi]^T[A_j][\Phi]. \tag{56}$$

The derivative with respect to $[s^{\infty}(\xi)]_{,\xi}$ is expressed as:

$$[s^{\infty}(\xi)]_{,\xi} \approx [c_\infty] - \sum_{j=1}^{m} \frac{j}{(\xi)^j}[a_j], \tag{57}$$

The power series Eq. (55) and its derivative Eq. (57) are substituted in Eq. (52). Sorting by decreasing powers of $\xi$ and equating the associated terms with zero, the coefficients $[c_\infty]$, $[k_\infty]$ and $[a_j]$ could be found. The quadratic term yields:

$$[c_\infty][c_\infty] + [e^3]^T[c_\infty][e^3] + [e^3]^T[e^3] - [e^5] + \omega^2[m^0] = 0 \tag{58}$$

Eq. (58) is an algebraic Riccati equation for the coefficient $[C_\infty]$ which could be solved by the Schur decomposition of related Hamiltonian matrix. The linear term yields:

$$([c_\infty] + [e^3]^T)[k_\infty] + [k_\infty]\,[c_\infty] + [\,e^3] = -[(c_\infty] + [e^3]^T)[e^1]^T - [e^1]([c_\infty] + [e^3]) + [c_\infty] + ([e^4] + [e^4]^T). \tag{59}$$

Eq. (59) is a Lyapunov equation for the coefficient $[k_\infty]$ which its solution involves solving a Sylvester equation (Bartels and Stewart, 1972) followed by a Schur decomposition. The constant term yields:

$$([c_\infty] + [e^3]^T)[a_1] + [a_1]([c_\infty] + [\,e^3] = -([k_\infty] + [e^1])([k_\infty] + [e^1]^T) + [e^2]) \tag{60}$$

Similar to Eq. (59), Eq. (60) is also a Lyapunov equation for the coefficient $[a_1]$.

For higher order terms $[a_j]$, $j > 1$, equations can be obtained in a similar approach. Construction of the initial value of the dynamic stiffness is straightforward by assessing Eq. (65) for a large finite value $\xi_{h,}$.

$$[S^{\infty}(\omega^*, \xi_h)] \approx ([\Phi]^{-1})^T \{\xi_h [C_\infty] + [k_\infty] + \sum_{j=1}^{m} \frac{j}{(\xi_h)^j} [a_j]\} [\Phi]^{-1} \quad (61)$$

In order to consider material damping in the formulation given above, Linear hysteretic material damping has been applied by using a complex shear modulus $G^*$ instead of the real one $G$.

$$G^* = (1 + i.2D)G \quad (62)$$

where the damping ratio and the imaginary unit has been indicated by symbols $D$ and $i$, respectively. Using Eq. (66), the complex coefficient matrices $[E^{0*}]$ – $[E^{5*}]$ follow from those given in Eqs. (26a-26f)

$$[E^{i*}] = (1 + i.2D)[E^i], \ \ i = 0,1, \dots ,5. \quad (63)$$

The mass matrix $[M^0]$ stays unchanged due to the material damping.

## NUMERICAL EXAMPLES

In this section, dynamic response of an infinite soil layer has been calculated based on the derivation presented before. An axisymmetric SBFEM code has been developed in the MATLAB programming language. Despite the general formulation of SBFEM, just a straight line is discretized. The obtained numerical results using the proposed method is compared to the well known thin layer method (Lysmer and Waas, 1972; Kausel and Roesset, 1975) for different terms of Fourier series.

### Uniformly Distributed Pressure along the Depth in Homogenous Material

Consider a hollow unbounded cylinder of radius 1, thickness 1 and Poisson's ratio υ = 1/3. The discretization used with one element of order 10 (Figure 2). Using the Gaussian–Lobatto–Legendre quadrature for the numerical integrations alongside the element, the nodes and integration points match together. Two cases with free or fixed bottom have been calculated in the analysis. An axisymmetric harmonic uniform pressure of amplitude 1 and in a frequency range of 0 to 10 with 100 steps has been applied in horizontal direction along the depth and the displacement of top node due to this loading has been calculated and compared with the TLM solution in Figures 3 and 4. Due to the axisymmetric loading, the computation was done for the first term of Fourier series (n = 0). The results of the current approach are in excellent agreement with the TLM.

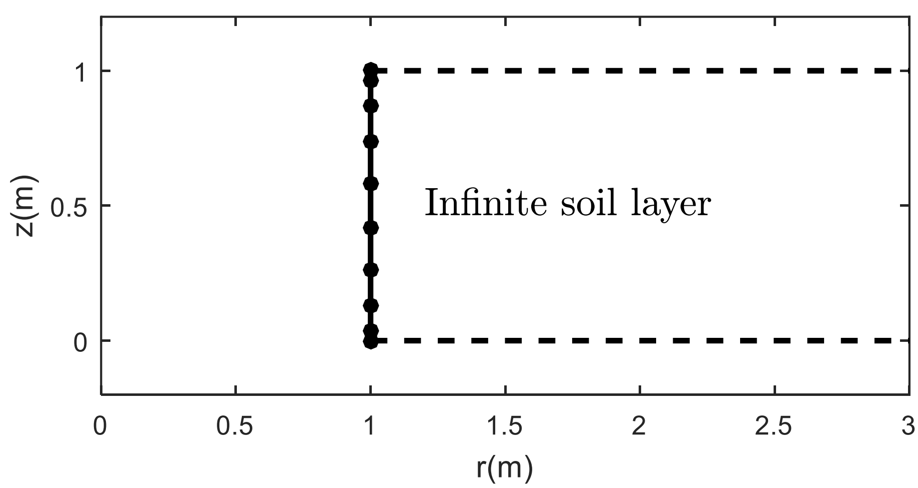
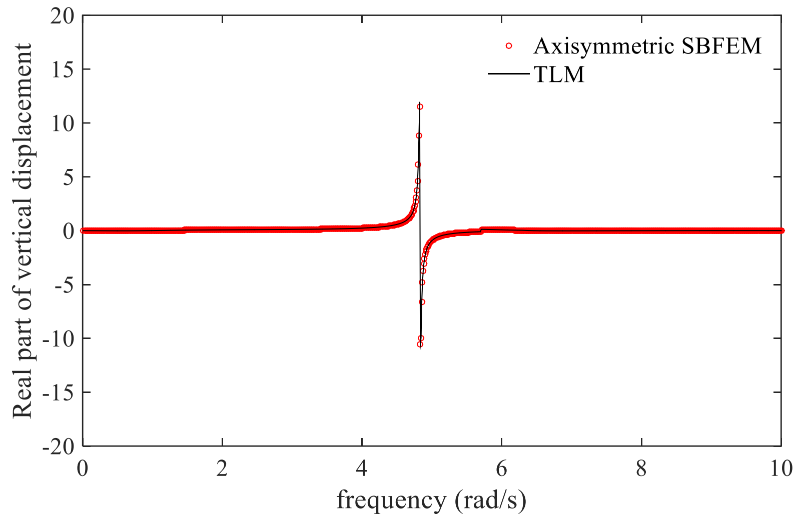
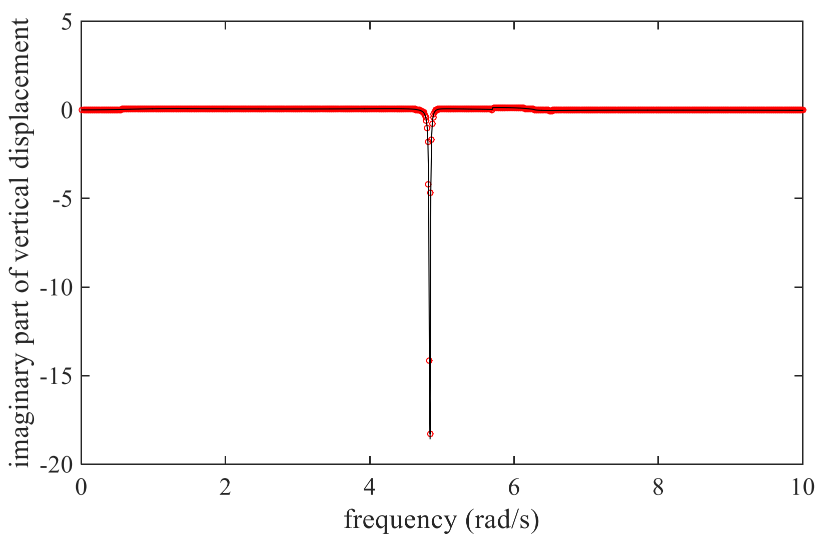
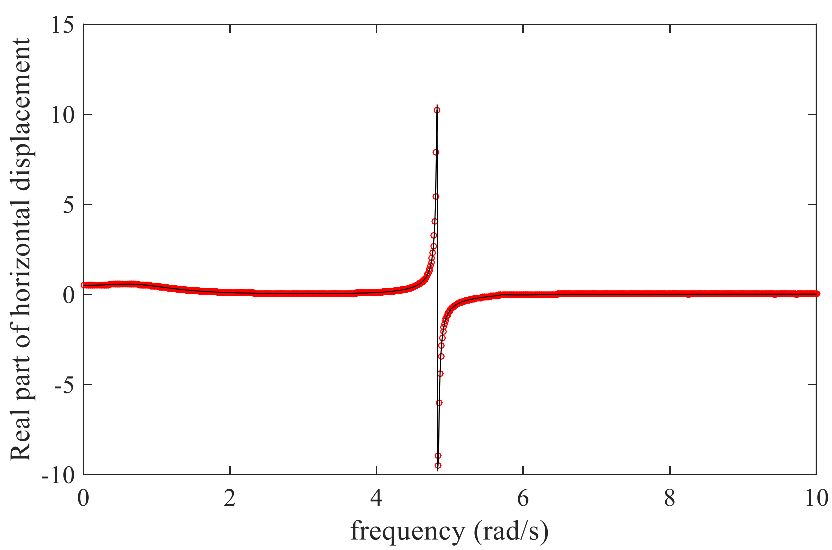



**Fig. 2.** Discretization of an unbounded cylinder


Axisymmetric SBFEM
TLM
Real part of vertical displacement
frequency (rad/s)


imaginary part of vertical displacement
frequency (rad/s)


Real part of horizontal displacement
frequency (rad/s)

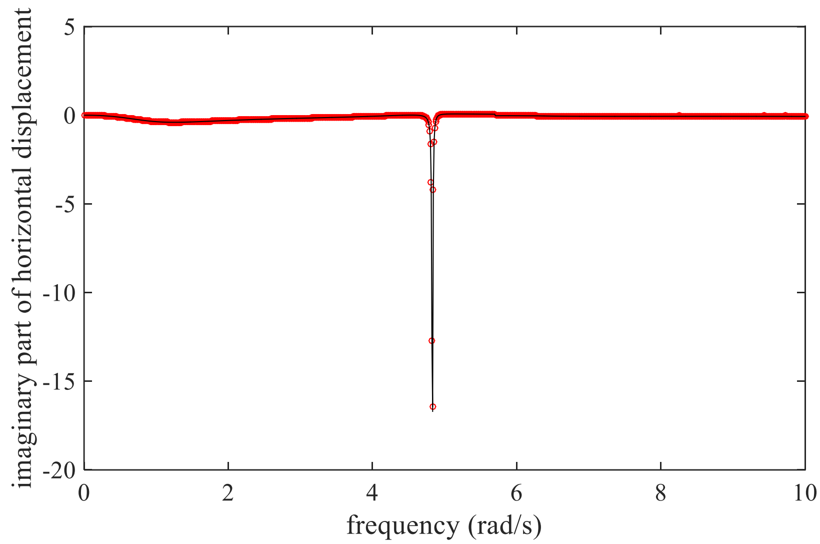
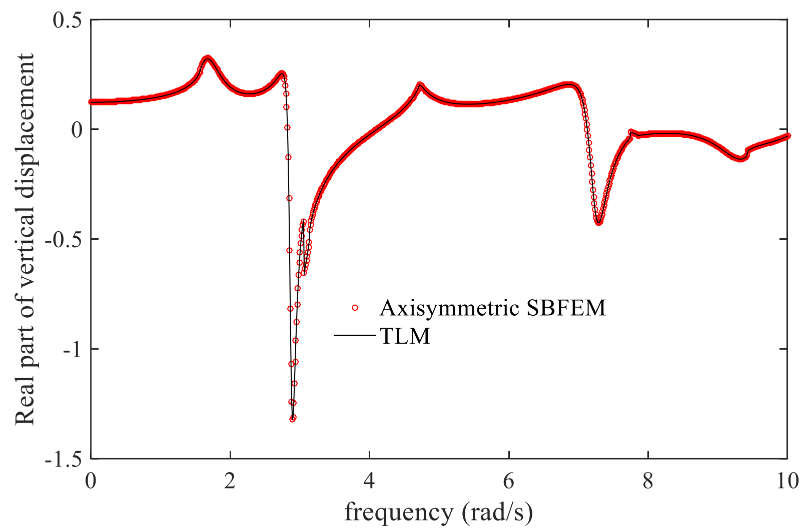
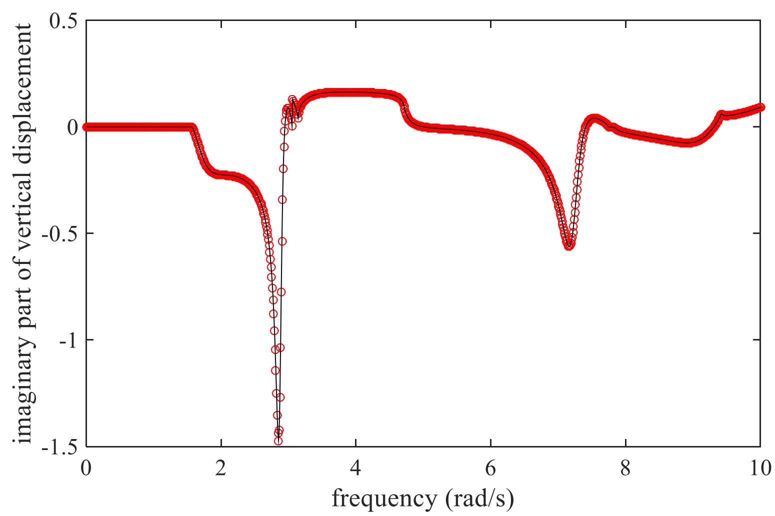



**Fig. 3.** Real and imaginary displacement of top node due to applied time harmonic pressure (Free bottom)



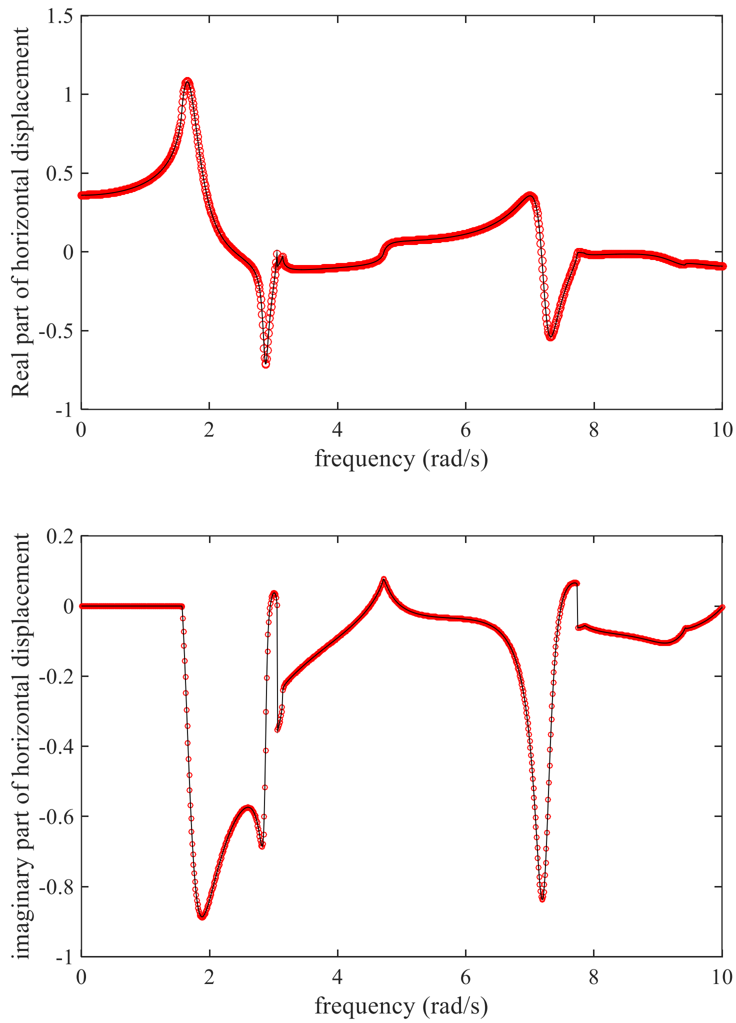
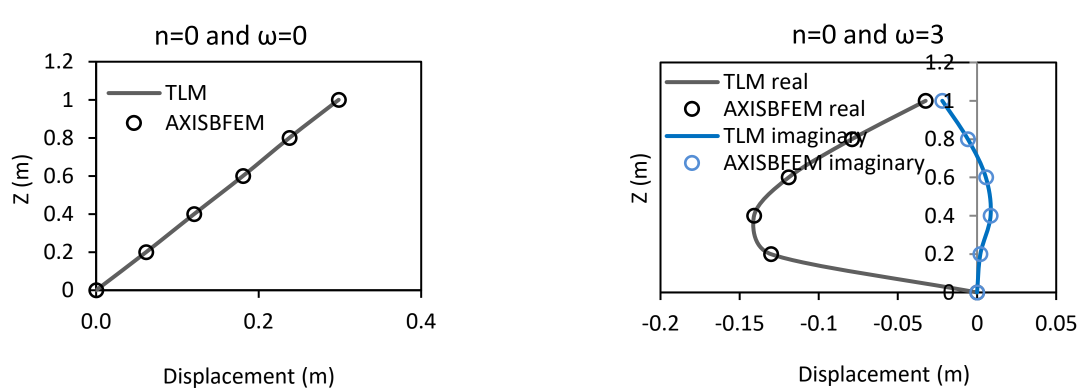



**Fig. 4.** Real and imaginary displacement of top node due to applied time harmonic pressure (fixed bottom)

**Linearly Varying Pressure along the Depth in a Homogenous Material**

In this example, a linearly antisymmetric varying pressure of amplitude 1 at the top and 0 at the bottom has been applied in horizontal direction and the displacement has been calculated and compared with the TLM solution in Figure 5. The comparison has been done for different terms of Fourier series (n = 0,1,2) along the depth and for 5 layers discretization (Figure 5). Again the results match very well with the TLM solutions.



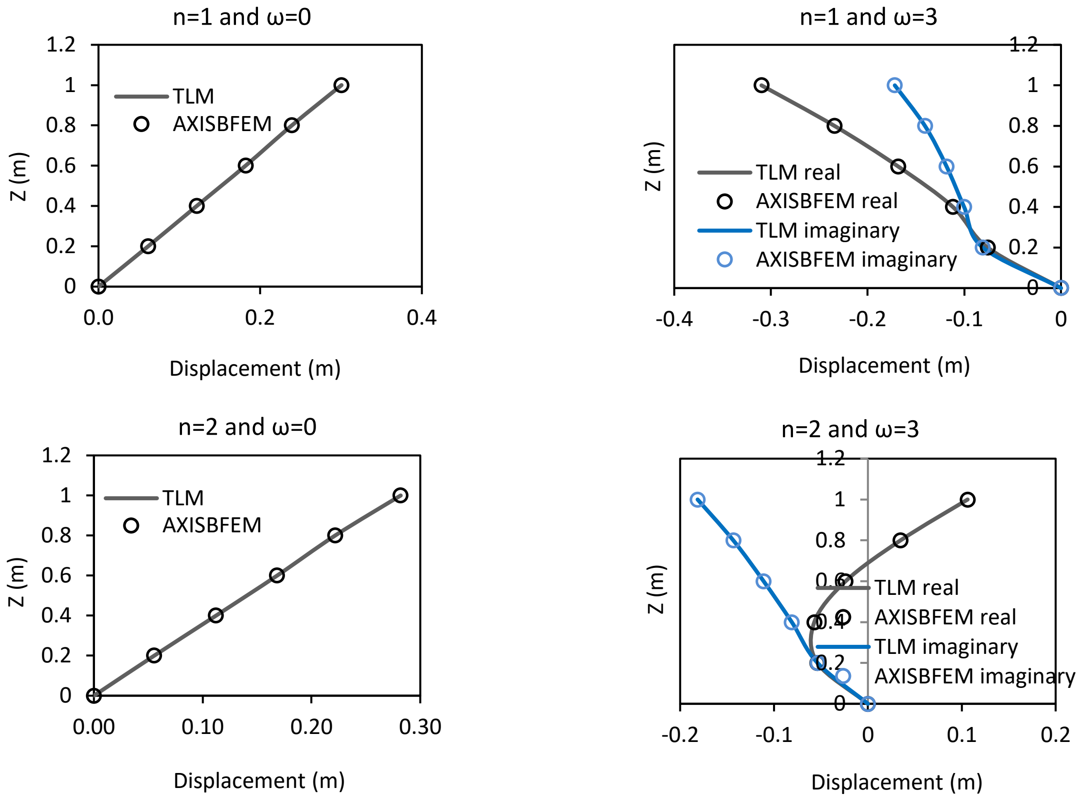



**Fig. 5.** The displacement of discretized line for different terms of series (n = 0,1,2) and for different circular frequencies (ω = 0,3)

## CONCLUSIONS

In this paper, an axisymmetric scaled boundary Finite Element formulation for elasto-dynamic analysis of three-dimensional unbounded layered media is derived. The formulation was based on different terms of Fourier series and a numerical integration is required to solve for the dynamic stiffness. Although the examples presented for the isotropic case, the current approach is not restricted to isotropic material. Generally, anisotropic material can be modeled simply by modifying the elasticity matrix in program.

The novel axisymmetric SBFEM formulation has been validated with the well-known thin layer method formulation for different example.

Summarizing, it can be said that the proposed formulation is very well suited for the wave propagation of unbounded layered systems and it could be coupled seamlessly to a formulation of near field to study the problem of dynamic soil structure interaction.

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