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Analytical and Numerical Investigation of Second Grade Magnetohydrodynamics Flow over a Permeable Stretching Sheet

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ABSTRACT

In this paper, the steady laminar boundary layer flow of non-Newtonian second grade conducting fluid past a permeable stretching sheet, under the influence of a uniform magnetic field is studied. Three different methods are applied for solving the problem; numerical Finite Element Method (FEM), analytical Collocation Method (CM) and 4th order Runge-Kutta numerical method. The FlexPDE software package is used for modeling and solving the problem by FEM. In most new analytical methods used for solving nonlinear equations, it is impossible to solve problems with infinity boundary conditions. In this article by using a special technique, the infinity boundary condition transformed to a finite one, then the governing equation solved analytically. In the physical aspect, the effects of the non-Newtonian, magnetic and permeability parameters on the velocity distribution have been investigated. As a result, the present suggested technique can be used for the analytical solution of many such problems with infinite boundary conditions. Moreover, the comparison between the results obtained from our modified analytical method and numerical solutions shows an excellent agreement.

1. Introduction

Nowadays the behavior of the boundary layer on a moving sheet is one of the most important types of flow which is seen in many industrial and engineering processes. In many cases the used fluids are non-Newtonian and researchers interested in study non-Newtonian fluids. One of these fluids is viscoelastic fluid which has numerous applications in various industries

Flow and heat transfer in a second grade fluid over a stretching sheet is examined numerically by Vajravelu and Roper [1]. The surface temperature was constant and the effects of frictional heat, heat source or sink within the fluid and work due to the elastic deformation was determined. It was shown that by increasing the elastic parameter and the source rate, the temperature increases and with increasing Prandtl number, the temperature decreases.

The boundary layer flow of a second grade non-Newtonian fluid with variable heat flux on the wall was investigated by Massoudi [2]. The central difference method with non-uniform grid was used to investigate the effect of the non-Newtonian parameter and the heat flux variation on the thermal boundary layer.

Vajravelu and Rollins [3] examined hydrodynamic flow of a second grade fluid over a stretching sheet numerically. They concluded that by increasing the permeability and the magnetic parameters, the dimensionless velocity decreases and by increasing the viscoelastic parameter, the dimensionless velocity increases. Moreover, they showed that by increasing the viscoelastic parameter and reducing the magnetic and the permeability parameters, the transverse velocity increases.

Flow and heat transfer of an electrically conducting second grade fluid over a stretching sheet with suction which subjected to the transverse magnetic field was studied by Cortell [4]. The Runge-Kutta method was used to solve the problem. It was concluded that by increasing of the viscoelastic parameter, Prandtl number, the permeability parameter and the wall temperature gradient increases, thus more heat transferred from sheet and reducing the thickness of the thermal boundary layer. However, Eckert number and the magnetic parameter have opposite effect on the wall temperature gradient and the thermal boundary layer.

The first study on the fourth grade viscoelastic fluid was performed by Sajid et al. [5]. The steady flow of a fourth grade flow on a porous sheet was studied. The problem was solved by analytical homotopy method. Two years later, the same problem was solved by modified homotopy method [6]. It was showed that like second and third grade fluid, by increasing the

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parameters related to the second, third and fourth grade fluid that are available in governing equations of the flow, fluid velocity decreases.

Sajid et al. [7] numerically investigated the third grade fluid flow on a horizontal sheet with slip condition. The results showed that the suction velocity and the slip parameter have an important role in control of the boundary layer thickness.

The Second grade viscoelastic fluid flow over a stretching sheet subjected to the transverse magnetic field with heat and mass Transfer was investigated by Aiboud and Saouli [8]. The viscoelastic and magnetic parameters effects on the velocity and the length of the sheet, also the effects of the magnetic parameter, source and Prandtl number on the temperature has been studied. It was shown that by increasing the viscoelastic and magnetic parameters, the velocity and the length decreases. Moreover, by increasing the magnetic parameter and source, the temperature increases and by increasing of the Prandtl number, the temperature decreases.

Sahoo [9] investigated the effects of slip condition on the sheet-driven flow and heat transfer of a non-Newtonian fluid past a stretching sheet. The results showed that the momentum boundary layer thickness reduces and the thermal boundary layer thickness increases with the slip condition. However, third grade fluid parameter has an opposite effect on the velocity and temperature boundary layers.

Convection heat and mass transfer in a hydromagnetic flow of a second grade fluid in the presence of thermal radiation and thermal diffusion was studied by Olajuwon [10]. The results show that with increasing the second grade parameter, the rate of the heat and mass transfer and fluid flow increases. By increasing the heat radiation parameter, mass transfer rate decreases while heat transfer rate increases. By increasing Schmidt number, the mass transfer rate increases but the heat transfer rate reduces.

Islam et al. [11] investigated the optimal solution for the flow of a fourth-grade fluid with partial slip. The modified homotopy solution was used. The results showed that with increasing of the slip parameter, the velocity decreases and by decreasing of the non-Newtonian parameter and axial pressure gradient, the velocity reduces.

Flow and heat transfer of a third grade fluid past an exponentially stretching sheet with partial slip boundary condition was studied by Sahoo and Poncet [12]. It was concluded that the third grade fluid parameter increases the momentum boundary layer thickness and decreases the thermal boundary layer thickness. The study showed that the dimensionless slip parameter has a significant effect on the thickness of the boundary layers.

MHD flow of a third grade fluid in a porous half space with plate suction or injection with an analytical approach was studied by Aziz and Aziz [13]. The effects of the several parameters on the velocity distribution were discussed.

Ganji et al [14] studied a Non-Newtonian fluid flow in an axisymmetric channel with porous wall. The analytic method (OHAM) was used to solve the problem. The results compared with the numerical results and there was good agreement between them.

Analysis of a thin film of a second grade fluid flow over a vertical oscillating belt with was investigated by Islam et al [15]. Governing equation for velocity field with suitable boundary condition was solved by analytical method (ADM). For comparison the OHAM was used.

The flow of a second grade fluid over a stretching sheet with the variable thermal conductivity and viscosity in the presence of a heat source / sink was investigated by Akinbobola and Okoya [16]. The effect of the heat source/sink, heat radiation and viscous dissipation was considered. It was assumed that the viscosity was a nonlinear function of the temperature and the thermal conductivity coefficients are linear functions of temperature. The results showed that by increasing the viscoelastic parameters, the horizontal and the vertical components of the velocity increase while the temperature decreases.

Mustafa [17] studied a viscoelastic fluid flow and heat transfer over a Non-Linearly stretching sheet. The OHAM was used for solving governing equations. The results showed that by increasing of the nonlinear index (n) and the second grade fluid parameter, the horizontal velocity component increases. Also by increasing of (n), Prandtl number and the second grade fluid parameter, the dimensionless temperature decreases.

The main purpose of this study is to solve the momentum equation of a second order non-Newtonian fluid flow over a permeable stretching sheet under the influence of a magnetic field by three different methods; Finite Element Method (FEM), the Collocation Method (CM) and 4th order Range-Kutta (RK4). Also, effect of the magnetic, permeability and non-Newtonian parameters on the velocity components were investigated.

2. The Problem Statement

Let's consider the steady laminar boundary layer of a conducting second grade non-Newtonian fluid over a permeable stretching sheet under the influence of a uniform magnetic field, as shown in Fig. 1. The x and y axes are considered along and perpendicular to the sheet, respectively. Two equal opposite forces in direction of x axis stretch the sheet from its center. Due to these forces, the sheet stretches with velocity, $u_1=cx$, and the fluid flow. It is assumed that the magnetic Reynolds number is small so the induced magnetic field can be neglected.

The governing equations of the described problem are written as [4]:

$$\nabla \cdot V = 0, \tag{1}$$

$$\rho \frac{DV}{Dt} = -\nabla p + divT + j \times B, \tag{2}$$

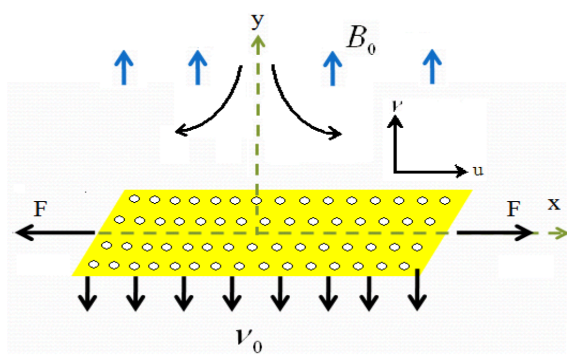


Figure 1. Problem schematic

where V is the velocity vector, ρ is the constant density, D/Dt is the substantial derivative, T is the stress tensor, p is the

pressure, j is the electric current density and $B=B_0+b$ is the total magnetic field (where B_0 is the external magnetic field and b is the induced magnetic field). In the present study, the magnetic Reynolds number is assumed small therefore the induced magnetic field can be neglected.
For defined problem, the B can be written as [18, 19]:

$$B = B_0 \vec{j}, \tag{3}$$

And in absence of the electric field, j is obtained by Ohm's law as [18, 19]:

$$j = \sigma_0 (V \times \mathrm{B}), \tag{4}$$

Where σ_0 is fluid electrical conductivity.
The stress tensor for second order fluid is defined as [20]:

$$T = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2, \tag{5}$$

Where μ is the viscosity and α_1, α_2 are the material coefficients that usually referred to the coefficients of the normal stress and A_1, A_2 are defined as follow:

$$A_1 = (gradV) + (gradV)^T, \tag{6}$$

$$A_2 = \frac{D}{Dt} A_1 + A_1 \cdot (gradV) + (gradV)^T \cdot A_1. \tag{7}$$

In this model (Eq. 5) due to the thermodynamics fundamentals, the following relationships must be established [21]:

$$\mu \geq 0, \;\; \alpha_1 \geq 0, \;\; \alpha_1 + \alpha_2 = 0. \tag{8}$$

By applying Eqs. (2-8), the momentum equations in x and y direction is obtained as:

$$\begin{aligned}
& u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{1}{\rho}\sigma_0 B_0^2 u + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \\
& 5\frac{\alpha_1}{\rho}\left(\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial y \partial x}\right) + 4\frac{\alpha_1}{\rho}\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y \partial x}\right) \\
& +3\frac{\alpha_1}{\rho}\left(\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial y^2}\right) + 10\frac{\alpha_1}{\rho}\left(\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\right) + 3\frac{\alpha_1}{\rho}\left(\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}\right) + \\
& 2\frac{\alpha_1}{\rho}\left(\frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial y^2}\right) + \frac{\alpha_1}{\rho}\left(\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2}\right) \\
& +3\frac{\alpha_1}{\rho}\left(\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x^2}\right) + 4\frac{\alpha_1}{\rho}\left(\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial x^2}\right) + \frac{\alpha_1}{\rho}\left(\frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial y \partial x}\right) + \\
& 4\frac{\alpha_1}{\rho}\left(\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y \partial x}\right) + 2\frac{\alpha_1}{\rho}\left(u\frac{\partial^3 u}{\partial x^3}\right) \\
& +2\frac{\alpha_1}{\rho}\left(v\frac{\partial^3 u}{\partial x^2 \partial y}\right) + \frac{\alpha_1}{\rho}\left(u\frac{\partial^3 v}{\partial x^2 \partial y}\right) + \frac{\alpha_1}{\rho}\left(v\frac{\partial^3 v}{\partial y^2 \partial x}\right) + \\
& \frac{\alpha_1}{\rho}\left(u\frac{\partial^3 u}{\partial y^2 \partial x}\right) + \frac{\alpha_1}{\rho}\left(v\frac{\partial^3 u}{\partial y^3}\right),
\end{aligned} \tag{9a}$$

$$\begin{aligned}
& u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + 3\frac{\alpha_1}{\rho}\left(\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial y^2}\right) + \\
& 3\frac{\alpha_1}{\rho}\left(\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y \partial x}\right) + 2\frac{\alpha_1}{\rho}\left(\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y \partial x}\right) \\
& +10\frac{\alpha_1}{\rho}\left(\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y^2}\right) + 4\frac{\alpha_1}{\rho}\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\right) + \frac{\alpha_1}{\rho}\left(\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y \partial x}\right) + \\
& 3\frac{\alpha_1}{\rho}\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x^2}\right) + 3\frac{\alpha_1}{\rho}\left(\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y \partial x}\right) \\
& +\frac{\alpha_1}{\rho}\left(\frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial y \partial x}\right) + \frac{\alpha_1}{\rho}\left(\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial x^2}\right) + 2\frac{\alpha_1}{\rho}\left(\frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial x^2}\right) + \\
& 3\frac{\alpha_1}{\rho}\left(\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial x^2}\right) + 4\frac{\alpha_1}{\rho}\left(\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial y \partial x}\right) \\
& +2\frac{\alpha_1}{\rho}\left(u\frac{\partial^3 v}{\partial y^2 \partial x}\right) + \frac{\alpha_1}{\rho}\left(v\frac{\partial^3 v}{\partial y^3}\right) + \frac{\alpha_1}{\rho}\left(u\frac{\partial^3 u}{\partial x^2 \partial y}\right) + \\
& \frac{\alpha_1}{\rho}\left(v\frac{\partial^3 u}{\partial y^2 \partial x}\right) + \frac{\alpha_1}{\rho}\left(u\frac{\partial^3 v}{\partial x^3}\right) \\
& +\frac{\alpha_1}{\rho}\left(v\frac{\partial^3 v}{\partial x^2 \partial y}\right) + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right).
\end{aligned} \tag{9b}$$

Using the Prandtl's boundary layer assumptions and the order of magnitude technique, Eqs. (9a) and (9b) can be simplified as:

$$\begin{aligned}
& u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \upsilon\frac{\partial^2 u}{\partial y^2} - \frac{\sigma_0 B_0^2}{\rho}u + \\
& \frac{\alpha_1}{\rho}\left[\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2} + v\frac{\partial^3 u}{\partial y^3}\right. \\
& \left. + \frac{\partial}{\partial x}\left(u\frac{\partial^2 u}{\partial y^2}\right) + 2\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)^2\right],
\end{aligned} \tag{10a}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} + 2\frac{\alpha_1}{\rho}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^2 = 0, \tag{10b}$$

where ν is kinematic viscosity.

New parameter $\hat{p}$ is defined as follow:

$$\hat{p} = p - 2\alpha_1\left(\frac{\partial u}{\partial y}\right)^2. \tag{11}$$

By using Eq. (11), Eqs. (10a) and (10b) can be rewritten as:

$$\begin{aligned}
& u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial \hat{p}}{\partial x} + \upsilon\frac{\partial^2 u}{\partial y^2} - \frac{\sigma_0 B_0^2}{\rho}u + \\
& \frac{\alpha_1}{\rho}\left[\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2} + v\frac{\partial^3 u}{\partial y^3} + \frac{\partial}{\partial x}\left(u\frac{\partial^2 u}{\partial y^2}\right)\right],
\end{aligned} \tag{12a}$$

$$-\frac{1}{\rho}\frac{\partial \hat{p}}{\partial y} = 0. \tag{12b}$$

It can be deduced from Eq. (12b) that $\hat{p}$ is only the function of x, $\hat{p} = \hat{p}(x)$.

It is known from the Prandtl approximation, the flow outside of the boundary layer can be considered potential and thus the viscous term can be neglected. Besides, the velocity outside of the boundary layer (in the potential region) equals to upstream flow velocity. By applying the above conditions on Eq. (12a), we have:

$$u_\infty \frac{\partial u_\infty}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{\sigma_0 B_0^2}{\rho} u_\infty, \tag{13}$$

Where u_∞ is flow velocity outside of boundary layer. Outside of the boundary layer the fluid is still therefore, we have

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = 0. \tag{14}$$

According to Eq. (11), outside of the boundary layer the viscous forces are zero and we have:

$$\hat{p} = p. \tag{15}$$

Therefore,

$$\frac{\partial \hat{p}}{\partial x} = 0. \tag{16}$$

Thus, the governing equations for problem are simplified as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{17}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \upsilon \frac{\partial^2 u}{\partial y^2} - \frac{\sigma_0 B_0^2}{\rho} u + \frac{\alpha_1}{\rho}\left[\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2} + v\frac{\partial^3 u}{\partial y^3} + \frac{\partial}{\partial x}\left(u\frac{\partial^2 u}{\partial y^2}\right)\right]. \tag{18}$$

As problem definition, the boundary conditions are considered as follow:

$$u = u_1 = cx, \quad v = -v_0 \quad at\ y = 0, \tag{19}$$

$$u \to 0, \quad \frac{\partial u}{\partial y} \to 0 \quad as\ y \to \infty. \tag{20}$$

In Eq. (19), c is the sheet stretching rate of and has positive value.

The similarity parameter η and $f'(\eta)$ are defined as

$$\eta = \left(\frac{c}{\upsilon}\right)^{1/2} y, \qquad f'(\eta) = \frac{u}{u_1}, \tag{21}$$

Combining eq. (17) and (21) gives:

$$u = cxf'(\eta), \quad v = -(c\upsilon)^{1/2} f(\eta), \tag{22}$$

Applying similarity parameters (Eq. 21) in the governing equations (Eqs. 17-18) and the boundary conditions (Eqs. 19-20) yields:

$$(f')^2 - ff'' + Mf' = f''' + \lambda_1\left[-(f'')^2 - ff^{iv} + 2ff'''\right]. \tag{23}$$

$$f = R, \quad f' = 1 \quad at \quad \eta = 0, \tag{24}$$

$$f' \to 0, \quad f'' \to 0 \quad as \quad \eta \to \infty, \tag{25}$$

In which

$$M = \frac{\sigma_0 B_0^2}{\rho c}, \quad \lambda_1 = \frac{\alpha_1 c}{\rho \upsilon}, \quad R = \frac{v_0}{(c\upsilon)^{1/2}}. \tag{26}$$

Where M, λ_1, R are the magnetic parameter, the non-Newtonian parameter and the permeability parameter respectively.

3. Solution of the problem

In this work, three different methods are used for solving the problem. The first method is numerical finite element method (FEM) and the second one is the analytical collocation method. For checking the accuracy of the solutions, the 4th order Rung – Kutta numerical method is used. More details of these solutions are described as follow.

3.1 Finite element method (FEM)

The geometry of the problem was modeled in FlexPDE. The physics of the problem, boundary conditions, governing equations, variables and the precision of the solution is coding and accordingly the solution is performed. Using the default meshing sequence of software, the finite element method is applied for solving.

The basic idea in the finite-element method is to find the solution of a complicated problem by replacing it by a simpler one. Since the actual problem is replaced by a simpler one in finding the solution, we will be able to find only an approximate solution rather than the exact solution. The existing mathematical tools will not be sufficient to find the exact solution (and sometimes, even an approximate solution) of most of the practical problems. Thus, in the absence of any other convenient method to find even the approximate solution of a given problem, we have to prefer the finite element method. Moreover, in the finite element method, it will often be possible to improve or refine the approximate solution by spending more computational effort. In the finite element method, the solution region is considered as built up of many small, interconnected sub-regions called finite elements [18, 22].

3.2 Collocation Method

The collocation method is used to solve obtained equations [23, 24]. As defined earlier, this problem has some infinite boundary conditions (eq.25). By using a suitable transformation, the infinite boundary condition can be transformed into a finite boundary condition. For this purpose some new parameters are defined as:

$$z = \frac{\eta}{k}, \quad g(z) = \frac{f(\eta)}{k}. \tag{27}$$

If k is selected large enough, by using Eq. (27), the Eqs. (23-25) are transformed as below:

$$2\lambda_1 g'g'' - \lambda_1 (g'')^2 - \lambda_1 gg^{iv} + k^2 gg'' - k^2 (g')^2 + g''' - k^2 Mg' = 0, \tag{28}$$

$$g' = 1, \quad g = \frac{R}{k} \quad at \quad z = 0, \tag{29}$$

$$g' \to 0, \quad g'' \to 0 \quad as \quad z \to 1. \tag{30}$$

The prim represents the derivative with respect to z and $z \in [0,1]$. To obtain an approximate solution for the problem (Eq. 28), *g(z)* a trial polynomial function in terms of *z*, has been defined such that satisfies boundary conditions (29-30) as follow:

$$g(z)=\frac{R}{k}+z-z^2+\frac{z^3}{3}+c_0\left(2z^2-\frac{8}{3}z^3+z^4\right)+ c_1\left(5z^2-5z^3+z^5\right)+c_2\left(9z^2-8z^3+z^6\right) +c_3\left(14z^2-\frac{35}{3}z^3+z^7\right)+c_4\left(20z^2-16z^3+z^8\right) +c_5\left(27z^2-21z^3+z^9\right)+c_6\left(35z^2-\frac{80}{3}z^3+z^{10}\right) +c_7\left(44z^2-33z^3+z^{11}\right). \quad (31)$$

By substituting Eq. (31) in Eq. (28), residual function *R(z)* obtained as

$$\begin{aligned} R(z) &= 2-8c_0-(70/3)c_1-(392/9)c_2-(1820/27)c_3 \\ &-(7664/81)c_4-(10150/81)c_5-(38800/243)c_6- \\ &(144232/729)c_7+...-(1420/729)k^2c_0c_1 \\ &-(8000/2187)k^2c_0c_2+...-(24496/2187)\lambda_1c_0^2- \\ &(49706020/531441)\lambda_1c_0^2-(1596146980/4782969)\lambda_1c_0^2+...+ \\ &-24\lambda_1Rc_0/k-(160/3)\lambda_1Rc_1/k-(640/9)\lambda_1Rc_2/k- \\ &(17920/243)\lambda_1Rc_3/k+...-(32/729)k^2Mc_0-(1040/6561)k^2Mc_1 \\ &+...-(11811160064/3486784401)k^2Mc_7=0 \end{aligned} \quad (32)$$

By try and error the k value is chosen as 6.0 which is large enough to satisfy the infinite boundary conditions.

To find constants c_0 to c_7, eight points are selected in the interval of $0<z<1$. The residual function must approach to zero at these points. Then we have:

$$R\left(\frac{1}{9}\right)=0,\quad R\left(\frac{2}{9}\right)=0,\quad R\left(\frac{3}{9}\right)=0,\quad R\left(\frac{4}{9}\right)=0,\quad R\left(\frac{5}{9}\right)=0,\quad R\left(\frac{6}{9}\right)=0,\quad R\left(\frac{7}{9}\right)=0,\quad R\left(\frac{8}{9}\right)=0. \quad (33)$$

Thus, eight equations with eight unknowns are formed. By using Maple 18.0 to solve these equations, constant values $c_0,c_1,...,c_7$ are obtained for different values of M,λ_1,R are obtained. By inserting constants $c_0,c_1,...,c_7$ in Eq. (31), *g(z)* can be calculated. Then by using transformation (27), $f(\eta)$ and $f'(\eta)$ which are approximation solutions of the momentum equation, can be achieved. For example, for $R=1.5,\ M=5,\ \lambda_1=2$ the functions *g(z)* and $f(\eta)$ are obtained as below:

$$\begin{aligned} g(z) &= 0.2636203866+z-3.133611297z^2 \\ &+6.543115066z^3-10.21969661z^4 \\ &+12.64625077z^5-12.70069256z^6 \\ &+10.28143387z^7-6.460892259z^8+2.931285417z^9 \\ &-0.8426870665z^{10}+0.1138152987z^{11}, \end{aligned} \quad (34)$$

$$\begin{aligned} f(\eta) &= 1.500000000+0.9999999998\eta- \\ &0.5507225477\eta^2+0.2020970735\eta^3- \\ &0.05547549725\eta^4+0.01206459434e\eta^5 \\ &-0.002129443252\eta^6+0.0003029563694\eta^7 \\ &3.345851284\times10^{-5}\eta^8+2.667841202\times10^{-6}\eta^9 \\ &-1.347894590\times10^{-7}\eta^{10}+3.199469524\times10^{-9}\eta^{11}. \end{aligned} \quad (35)$$

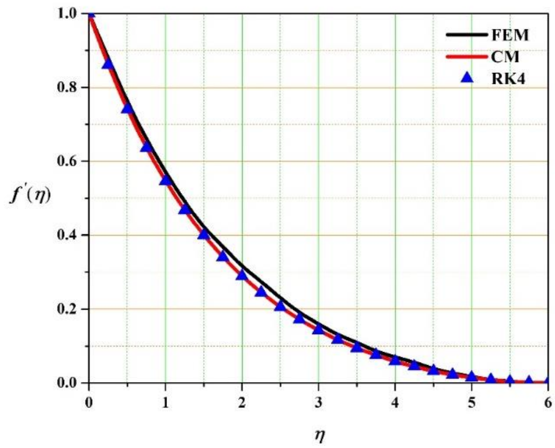


Figure 2. Comparison among results obtained from three different methods ($R=5,\ M=10,\ \lambda_1=10$)

4. Results and discussion

In the present study, the velocity profiles are provided by using three different methods. Fig. 2 presents a comparison between the results of three methods for a special case. Also, Table 1 is prepared to show the results of these different solutions. It can be observed that the suggested methods which used to solve the problem have adequate accuracy.

Table 1. The results of three different methods for $f'(\eta)$ ($R=5,\ M=10,\ \lambda_1=10$)

η	FEM	CM	RK4
0.0	1.00000	1.00000	1.00000
0.5	0.76354	0.74045	0.74045
1.0	0.57093	0.54586	0.54586
1.5	0.42028	0.39955	0.39955
2.0	0.31502	0.28917	0.28917
2.5	0.23208	0.20562	0.20562
3.0	0.16902	0.14224	0.14224
3.5	0.11324	0.09425	0.09425
4.0	0.07088	0.05827	0.05827
4.5	0.03809	0.03202	0.03202
5.0	0.01797	0.01403	0.01403
5.5	0.00434	0.00349	0.00349
6.0	0.00000	0.00000	0.00000

The vectors of $f'(\eta)$ over the permeable stretching sheet are presented in Fig. 3.

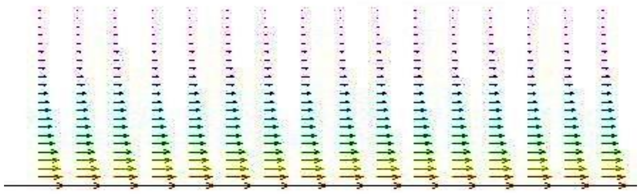
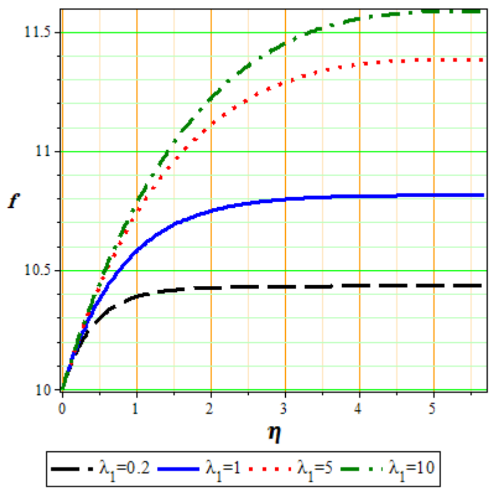


Figure 3. The vectors of $f'(\eta)$ over the sheet

Figs. 4-6 show the effect of the physical parameters on the fluid behavior. Figs. 4 (a and b) demonstrate the effect of non-Newtonian parameter on $f(\eta)$ and $f'(\eta)$. It is shown that for a special η, by increasing of λ_1, $f(\eta)$ and $f'(\eta)$ increases. Therefore, according to Eq. (20) the value of the velocity components increase. It means that, the surface movement has more effect on the fluid if the material coefficient of normal stress (non-Newtonian property) increases.

(a)



(b)

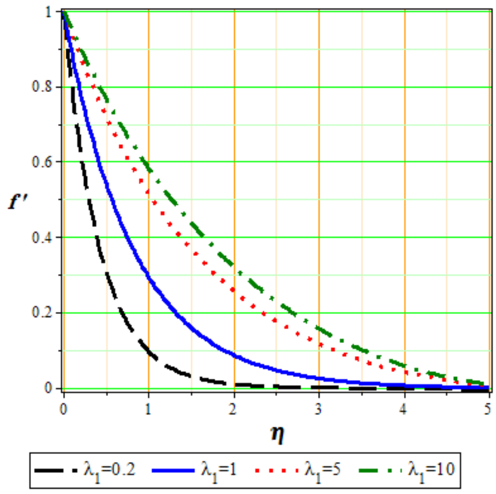
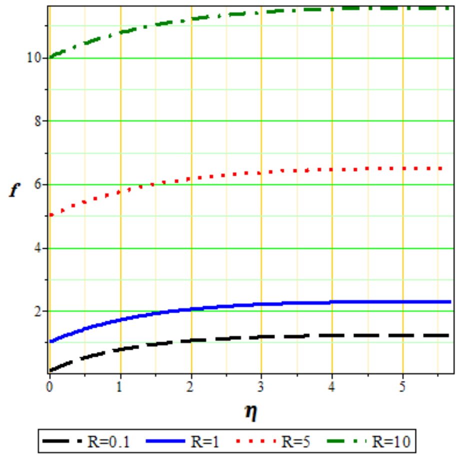


Figure 4. The effect of the non-Newtonian parameter ($R=10,\ M=8$)

According to Figs. 5 (a and b), by increasing of the permeability parameter, $f(\eta)$ and $f'(\eta)$ increase. As expected, the more suction velocity results the more influence of the permeability on the velocity in fluid region.

(a)



(b)

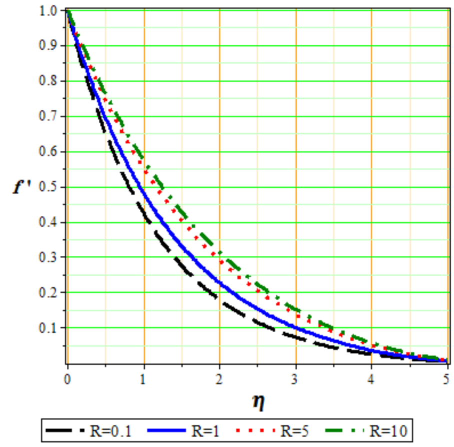
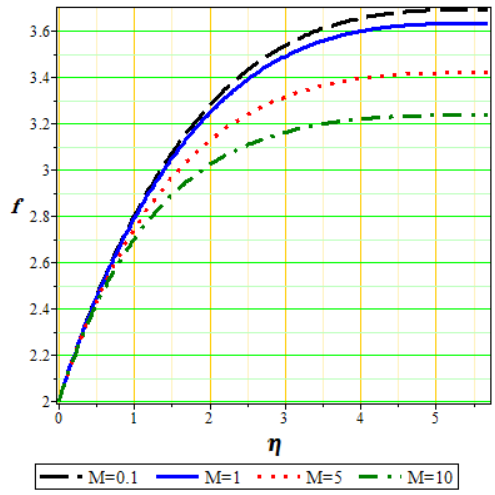


Figure 5. The effect of the permeability parameter ($M=6,\ \lambda_1=8$)

(a)



(b)

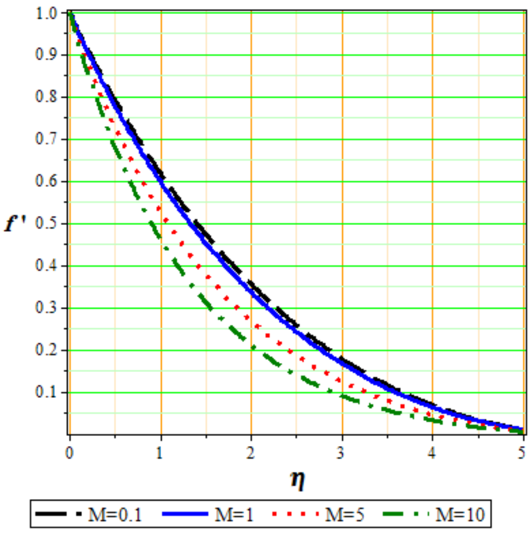


Figure 6. The effect of the magnetic parameter ($R=2,\ \lambda_1=8$)

The effect of the magnetic parameter on flow field has been shown in Figs. 6 (a and b). It is shown that by increasing of *M*, $f(\eta)$ and $f'(\eta)$ decreases. In physical aspect by increasing magnetic field the resistance Lorentz force increases, then *u* and *v* decreases.

4. Conclusion

In this paper, the second grade non-Newtonian fluid flow over a permeable stretching sheet under the influence of the magnetic field has been studied. Three different methods were applied for solution of the problem; FEM, CM and RK4. The FEM was applied by using the FlexPDE software package. In the analytical method, by using a special technique, the infinity boundary condition transformed to a finite one and the governing equation was solved analytically. These solution methods were compared for a special case. It seems that the technique suggested for analytical method can be used for a wide range of nonlinear problems with infinite boundary conditions. Furthermore, the effect of the physical parameters affecting the flow field was studied.

5. References

[1] K. Vajravelu, T. Roper, Flow and heat transfer in a second grade fluid over a stretching sheet, *International Journal of Non-Linear Mechanics*, Vol. 34, No. 6, pp. 1031-1036, 1999/11/01, 1999.

[2] M. Massoudi, Boundary layer flow of a second grade fluid with variable heat flux at the wall, *Appl. Math. Comput.*, Vol. 143, No. 2-3, pp. 201-212, 2003.

[3] K. Vajravelu, D. Rollins, Hydromagnetic flow of a second grade fluid over a stretching sheet, *Applied Mathematics and Computation*, Vol. 148, No. 3, pp. 783-791, 1/30/, 2004.

[4] R. Cortell, Flow and heat transfer of an electrically conducting fluid of second grade over a stretching sheet subject to suction and to a transverse magnetic field, *International Journal of Heat and Mass Transfer*, Vol. 49, No. 11–12, pp. 1851-1856, 6//, 2006.

[5] M. Sajid, T. Hayat, S. Asghar, On the analytic solution of the steady flow of a fourth grade fluid, *Physics Letters A*, Vol. 355, No. 1, pp. 18-26, 6/19/, 2006.

[6] V. Marinca, N. Herişanu, C. Bota, B. Marinca, An optimal homotopy asymptotic method applied to the steady flow of a fourth-grade fluid past a porous plate, *Applied Mathematics Letters*, Vol. 22, No. 2, pp. 245-251, 2//, 2009.

[7] M. Sajid, R. Mahmood, T. Hayat, Finite element solution for flow of a third grade fluid past a horizontal porous plate with partial slip, *Computers & Mathematics with Applications*, Vol. 56, No. 5, pp. 1236-1244, 9//, 2008.

[8] S. Aïboud, S. Saouli, Second Law Analysis of Viscoelastic Fluid over a Stretching Sheet Subject to a Transverse Magnetic Field with Heat and Mass Transfer, *Entropy*, Vol. 12, No. 8, pp. 1867, 2010.

[9] B. Sahoo, Effects of slip on sheet-driven flow and heat transfer of a non-Newtonian fluid past a stretching sheet, *Computers & Mathematics with Applications*, Vol. 61, No. 5, pp. 1442-1456, 3//, 2011.

[10] B. I. Olajuwon, Convection heat and mass transfer in a hydromagnetic flow of a second grade fluid in the presence of thermal radiation and thermal diffusion, *International Communications in Heat and Mass Transfer*, Vol. 38, No. 3, pp. 377-382, 3//, 2011.

[11] S. Islam, Z. Bano, I. Siddique, A. M. Siddiqui, The optimal solution for the flow of a fourth-grade fluid with partial slip, *Computers & Mathematics with Applications*, Vol. 61, No. 6, pp. 1507-1516, 3//, 2011.

[12] B. Sahoo, S. Poncet, Flow and heat transfer of a third grade fluid past an exponentially stretching sheet with partial slip boundary condition, *International Journal of Heat and Mass Transfer*, Vol. 54, No. 23–24, pp. 5010-5019, 11//, 2011.

[13] A. Aziz, T. Aziz, MHD flow of a third grade fluid in a porous half space with plate suction or injection: An analytical approach, *Applied Mathematics and Computation*, Vol. 218, No. 21, pp. 10443-10453, 7/1/, 2012.

[14] M. Hosseini, Z. Sheikholeslami, D. D. Ganji, Non-Newtonian fluid flow in an axisymmetric channel with porous wall, *Propulsion and Power Research*, Vol. 2, No. 4, pp. 254-262, 12//, 2013.

[15] T. Gul, S. Islam, R. A. Shah, I. Khan, S. Shafie, M. A. Khan, Analysis of thin film flow over a vertical oscillating belt with a second grade fluid, *Engineering Science and Technology, an International Journal*, Vol. 18, No. 2, pp. 207-217, 6//, 2015.

[16] T. E. Akinbobola, S. S. Okoya, The flow of second grade fluid over a stretching sheet with variable thermal conductivity and viscosity in the presence of heat source/sink, *Journal of the Nigerian Mathematical Society*, Vol. 34, No. 3, pp. 331-342, 12//, 2015.

[17] M. Mustafa, Viscoelastic Flow and Heat Transfer over a Non-Linearly Stretching Sheet: OHAM Solution, *Journal of Applied Fluid Mechanics*, Vol. 9, No. 3, pp. 1321-1328, 2016.

[18] S. M. Ebrahimi, M. Javanmard, M. H. Taheri, M. Barimani, Heat transfer of fourth-grade fluid flow in the plane duct under an externally applied magnetic field with convection on walls, *International Journal of Mechanical Sciences*, Vol. 128–129, pp. 564-571, 8//, 2017.

[19] P. G. Moakher, M. Abbasi, M. Khaki, Fully developed flow of fourth grade fluid through the channel with slip condition in the presence of a magnetic field, *Journal of Applied Fluid Mechanics*, Vol. 9, No. 7, pp. 2239-2245, 2016.

[20] R. S. Rivlin, J. L. Ericksen, *Stress-Deformation Relations for Isotropic Materials*, in: G. I. Barenblatt, D. D. Joseph, *Collected Papers of R.S. Rivlin: Volume I and II*, Eds., pp. 911-1013, New York, NY: Springer New York, 1997.

[21] J. E. Dunn, R. L. Fosdick, Thermodynamics, stability, and boundedness of fluids of complexity 2 and fluids of second grade, *Archive for Rational Mechanics and Analysis,* Vol. 56, No. 3, pp. 191-252, 1974.

[22] S. A. R. Sahebi, H. Pourziaei, A. R. Feizi, M. H. Taheri, Y. Rostamiyan, D. D. Ganji, Numerical analysis of natural convection for non-Newtonian fluid conveying nanoparticles between two vertical parallel plates, *The European Physical Journal Plus,* Vol. 130, No. 12, pp. 238, 2015.

[23] P. G. Moakhar, M. Abbasi, M. Khaki, Fully Developed Flow of Fourth Grade Fluid through the Channel with Slip Condition in the Presence of a Magnetic Field, *Journal of Applied Fluid Mechanics,* Vol. 9, No. 5, pp. 2239-2245, 2016.

[24] S. M. Ebrahimi, M. Abbasi, M. Khaki, Fully Developed Flow of Third-Grade Fluid in the Plane Duct with Convection on the Walls, *Iranian Journal of Science and Technology, Transactions of Mechanical Engineering,* Vol. 40, No. 4, pp. 315-324, 2016.