Economical Design of Double Variables Acceptance Sampling With Inspection Errors

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Abstract

The paper presents an economical model for double variable acceptance sampling with inspection errors. Taguchi cost function is used as acceptance cost while quality specification functions are normal with known variance. An optimization model is developed for double variables acceptance sampling scheme at the presence of inspection errors with either constant or monotone value functions. The monotone value functions could be descending or ascending exponentially. In the case that inspection errors have exponentially functions, we can find the best value for inspection errors regarding to the sample number and other economical parameters. Finally sensitivity analysis has done on model parameters and some numerical examples are given to demonstrate how the developed model is applied.

Keywords: Acceptance Sampling- Double Variable - Inspection Errors - Optimization- Taguchi Cost function

Introduction

The literature in variable acceptance sampling to control the ratio of non-conformity is very little, that Jackson [1] remarked this area undeveloped. The variable acceptance sampling international standard (ISO 3951:1989), in paragraph 9g of section 1.2.b. notes that in the case of more than one variable acceptance sampling, sampling method must be applied for all factors, and the lot will be accepted if and only if all factors are accepted. It is clear that in this case OC curve is different from single variable type and consumer's risk is smaller than the one in single variable type and producer risk is greater than the one in single variable type, in the case that factors increase in multi variable acceptance sampling , the efficiency of this method will decrease.

Other authors like Montgomery [2], Ryan [3], proposed methods for multi variable acceptance models using m factors with single variable methods. Shakun [4] presented a model for alternative variable sampling when the covariance matrix is known and the specification limits are approximately elliptical. Dantziger and Papp [5] developed a single variable method for alternative variable where the specifications are independent. Wesolowsky [6] developed the graph for double variable acceptance sampling, in his method he set the control limits with paying attention to economical specifications, assuming the variance and the covariance are known.

In recent decade applying acceptance sampling methods brought many questions in quality control and now the main target was production specification and reducing manufacturing tolerances, but in many cases because of human and manufacturing system errors, acceptance sampling is a desired method.

Vanderman [7] and Schilling [8] kept working on how much accuracy of the acceptance sampling is used in qualified environments. Hamilton and Lesperance [9] developed a method for single and multi variable acceptance sampling assuming that the process quality can be found out with estimation from lot defects while the variance and mean are known. Tagares [10] proposed an economical model for single variable acceptance sampling plan by using Taguchi cost model when inspections are free of errors. Arshadi [11] presents a new model for single variable acceptance sampling plan considering inspection errors.

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In this paper we use Taguchi cost function by considering inspection errors for economical design of double variable acceptance sampling problems.The main reason for using Taguchi cost function is its view on cost of deviation as below:
When there is a deviation from target value , in traditional method the cost of this deviation was a constant value regardless the measure of this deviation ,but in Taguchi model the cost of this deviation is related to the square distance from the target(x^2).it seems that this view is more effective to decrease the deviations.

Notations and assumptions
Notations

y_1 =Measured variable for specification type1
y_2 =Measured variable for specification type2
μ_{01}= Target value for lot in specification type1
μ_{02}= Target value for lot in specification type2
μ_1= Deviation of the mean of the quality characteristic type 1 in a given inspection lot from target value (μ_{01}).
μ_2 = Deviation of the mean of the quality characteristic type 2 in a given inspection lot from target value (μ_{02}).
$\sigma_1^{\ 2}$ =Variance of y_1 in a given inspection lot
$\sigma_2^{\ 2}$ =Variance of y_2 in a given inspection lot
N= Lot size
n_1= Sample size for type 1
n_2= Sample size for type 2
$x_1 = y_1 - \mu_{01}$= Deviation from target value in each inspection for type 1
$x_2 = y_2 - \mu_{02}$= Deviation from target value in each inspection for type 2
L_1=Lower acceptance limit for type 1
L_2=Lower acceptance limit for type 2
U_1= Upper acceptance limit for type1
U_2= Upper acceptance limit for type 2
ci = Variable sampling and inspection cost per unit
cr =Rejected cost per unit
k= Constant of the quality cost kx^2
α =Type 1 inspection error
β= Type 2 inspection errors

Assumptions

1) The variance of x_i, σ_i^2 is known and constant
2) The variance of $\mu_i, \sigma^2_{\mu i}$ is known and constant, $\sigma^2_{\mu i} = \sigma_i^{\ 2} / D_i$, D_i is positive constant, expected to be larger than 1.
3) Measurement are not free of errors
4) The distribution of $x, f(x_i / \mu_i)$ is normal with mean μ_2 .
5) The distribution of $\mu_i, h(\mu_i)$ is normal with mean 0
6) L_i+U_i= $2\mu_{i0}$ and $L_i = \mu_{i0} - z_i, U_i = \mu_{i0} + z_i$
7) Inspections are destructive
8) Variables are independent

Description of cost model

Two samples are taken randomly with sizes n_1, n_2, after measuring y_1, y_2 then x_1,x_2, $\bar{x}_1, \bar{x}_2$ will be calculated and if $\bar{y}_1$ lies between L $_1$, U_1 or $\bar{x}_1$ lies between z_1,-z_1 and $\bar{y}_2$ lies between L_2 , U_2 or $\bar{x}_2$ lies between z_2,-z_2 the lot will be accepted otherwise the lot is rejected and rejected lots will be returned to the suppliers with cr cost.
In this model screening of rejected lots is not considered because in some cases screening is not a practically feasible solution.
Regarding to the above notations and assumptions, the following three types of cost are recognized:

1) Inspection cost (CI)
2) Acceptance cost (CA)
3) Rejection cost (CR)

In this model these three costs are compared with each other and the solution will be gained through minimizing the expected total cost:
a) Expected total cost per inspection (ETCI)
b) Expected total cost without sampling/accept the lot (ETCA)
c) Expected total cost in rejection (ETCR)
And ETC=Expected totals cost of model=min (ETCI, ETCA, ETCR)
When there is no inspection error as mentioned by Tagares [10], single variable model, Pa (μ), the probability of acceptance of a lot with given μ is:

$$\text{Pa}(\mu) = \int_{-z}^{z} g(\bar{x} \mid \mu) d\bar{x}$$

But when we have inspection error this probability will be written as (single variable):

Pa (μ) =P (accept the lot | lot is ok)× P (lot is ok) +P (accept the lot | lot is not ok)× P (lot is not ok)

So

$$Pae(\mu)=(1-\alpha)\times \int_{-z}^{z} g(\bar{x}\mid\mu)d\bar{x} + \beta\times(1-\int_{-z}^{z} g(\bar{x}\mid\mu)d\bar{x}) \quad (1)$$

So we have:

$$Pae=\int_{\mu}\int_{\bar{x}}(1-\alpha-\beta)g(\bar{x}\mid\mu)h(\mu)d\bar{x}d\mu+\int_{\mu}\beta h(\mu)d\mu \quad (2)$$

When inspection errors are predetermined and fixed, we have:

$$Pae=(1-\alpha-\beta)\times\int_{\mu}\int_{\bar{x}} g(\bar{x}\mid\mu)h(\mu)d\bar{x}d\mu+\beta \quad (3)$$

When we have two independent variables, we can write:

Probability of lot acceptance = (probability of lot acceptance by first criteria)×(probability of lot acceptance by second criteria)

$$Pae(\mu_1)=(1-\alpha)\times\int_{-z_1}^{z_1} g(\bar{x}_1\mid\mu_1)d\bar{x}_1+\beta\times(1-\int_{-z_1}^{z_1} g(\bar{x}_1\mid\mu_1)d\bar{x}_1) \quad (4)$$

$$Pae(\mu_2)=(1-\alpha)\times\int_{-z_2}^{z_2} g(\bar{x}_2\mid\mu_2)d\bar{x}_2+\beta\times(1-\int_{-z_2}^{z_2} g(\bar{x}_2\mid\mu_2)d\bar{x}_2) \quad (5)$$

then

$$Pae1=(1-\alpha-\beta)\times\int_{\mu_1}\int_{\bar{x}_1} g(\bar{x}_1\mid\mu_1)h(\mu_1)d\bar{x}_1d\mu_1+\beta \quad (6)$$

and

$$Pae2=(1-\alpha-\beta)\times\int_{\mu_{21}}\int_{\bar{x}_2} g(\bar{x}_2\mid\mu_2)h(\mu_2)d\bar{x}_2d\mu_2+\beta \quad (7)$$

$$Pae=Pae1\times Pae2=(1-\alpha-\beta)^2\times\int_{\mu_1}\int_{\bar{x}_1} g(\bar{x}_1\mid\mu_1)h(\mu_1)d\bar{x}_1d\mu_1\times\int_{\mu_2}\int_{\bar{x}_2} g(\bar{x}_2\mid\mu_2)h(\mu_2)d\bar{x}_2d\mu_2+\beta(1-\alpha-\beta)\times\int_{\mu_1}\int_{\bar{x}_1} g(\bar{x}_1\mid\mu_1)h(\mu_1)d\bar{x}_1d\mu_1+\beta(1-\alpha-\beta)\times\int_{\mu_2}\int_{\bar{x}_2} g(\bar{x}_2\mid\mu_2)h(\mu_2)d\bar{x}_2d\mu_2+\beta^2 \quad (8)$$

for double independent variable we have :

$$q(x_1,x_2)=k_1x_1^2+k_2x_2^2 \quad (9)$$

in the case that we have double variable model:

$$CA=\int CA(\mu_1,\mu_2)d(\mu_1,\mu_2),$$

$$CA(\mu_1,\mu_2)=(N-n_1-n_2)\times\int_{\mu} q(x_1,x_2)f(x_1,x_2\mid\mu_1,\mu_2)d(x_1,x_2)\times Pae(\mu_1,\mu_2)$$

, $Pae(\mu_1,\mu_2)=$

$$(1-\alpha)\times\int_{-z}^{z} g(\bar{x}_1,\bar{x}_2\mid\mu_1,\mu_2)d(\bar{x}_1,\bar{x}_2)+\beta\times(1-\int_{-z}^{z} g(\bar{x}_1,\bar{x}_2\mid\mu_1,\mu_2)d(\bar{x}_1,\bar{x}_2)) \quad (10)$$

, CR=(N-n_1-n_2)cr× (1-Pae)

and

$$CI=cs+(n_1+n_2)ci \quad (11)$$

here we have ETCI= CA+CR+CI

ETCR=N×cr (12)

$$ETCA=N\int_{\mu_2}\int_{\mu_1}\int_{x_2}\int_{x_1} q(x_1,x_2)f(x_1,x_2\mid\mu_1,\mu_2)\times h(\mu_1)h(\mu_2)d\mu_1d\mu_2d\bar{x}_1d\bar{x}_2$$

$$=N(k_1(\sigma_1^2+\sigma_{\mu_1}^2)+k_2(\sigma_2^2+\sigma_{\mu_2}^2)) \quad (13)$$

(for modeling problem see appendix)

ETC=min(ETCA,ETCI,ETCR) (14)

From Tagares [10] we have :

$$\int_{\mu}[g(z\mid\mu)+g(-z\mid\mu)]h(\mu)d\mu=2\psi(z), z\sim N(0,\sigma^2(n+D)/nD) \quad (15)$$

and

$$\int_{\mu}\mu^2[g(z\mid\mu)+g(-z\mid\mu)]h(\mu)d\mu=2\psi(z)\times\{n^2z^2/(n+D)^2+\sigma^2/(n+D)\} \quad (16)$$

by using above relations for μ_1,μ_2 and relations in Appendix to find the optimal solution for this problem we must have the first order condition for z_i ($\partial ETCI/\partial z_i$) as follows:

$\{Q1=n_1^2 z_1^2/(n_1+D_1)^2+\sigma_1^2/(n_1+D_1)$

&

$Q2=n_2^2 z_2^2/(n_2+D_2)^2+\sigma_2^2/(n_2+D_2)\}$

$$\partial ETCI/\partial z_1 =$$
$$(1-\alpha-\beta)^2\{k_1\times 2\psi(z_1)\times Q1\times\int\psi(z_2)$$
$$+k_2\times 2\psi(z_1)\times\int\psi(z_2)\times Q2\}+(1-\alpha-\beta)^2\times$$
$$(k_1\sigma_1^2+k_2\sigma_2^2-cr)\times 2\psi(z_1)\int\psi(z_2)+$$
$$\beta(1-\beta-\alpha)(k_1\sigma_1^2+k_2\sigma_2^2)\times 2\psi(z_1)+$$
$$\beta(1-\alpha-\beta)\{k_1\times 2\psi(z_1)\times Q1+k_1\times$$
$$(\sigma_1^2/D_1)\int\psi(z_2)+k_2\times 2\psi(z_2)\times Q2+$$
$$k_2(\sigma_2^2/D_2)\int\psi(z_1)=0 \qquad (17)$$

by similar way we have the same equation for z_2 .

In above equations we must have $k_1\sigma_1^2+k_2\sigma_2^2-cr<0$, because all of above statements are positive and if these equations turn to be zero then we have:

$$k_1\sigma_1^2+k_2\sigma_2^2-cr<0 \qquad (18)$$

If we want to have the absolute minimum of this model, the second –order condition must be calculated and the Hessian Matrix must be absolutely positive:

by replacing $\alpha=\beta=0$ and calculating the second order conditions we have :

$$\partial^2 ETCI/\partial^2 z_1 = 2\psi'(z_1)\{k_1Q_1\int\psi(z_2)+$$
$$k_2\int(\psi(z_2)Q_2)+(k_1\sigma_1^2+k_2\sigma_2^2-cr)\int\psi(z_2)\}$$

and

$$\partial^2 ETCI/\partial^2 z_2 = 2\psi'(z_2)\{k_2Q_2\int\psi(z_1)+$$
$$k_1\int(\psi(z_1)Q_1)+(k_1\sigma_1^2+k_2\sigma_2^2-cr)\int\psi(z_1)\}$$

and also:

$$\partial^2 ETCI/\partial z_2\partial z_1=\partial^2 ETCI/\partial z_1\partial z_2=k_1Q_1 2\psi(z_1)\psi(z_2)$$
$$+k_2Q_2 2\psi(z_1)\psi(z_2)+(k_1\sigma_1^2+k_2\sigma_2^2-cr)2\psi(z_1)\psi(z_2)$$

by considering above statement, determinant of Hessian Matrix is:

$$\partial^2 ETCI/\partial^2 z_1\times\partial^2 ETCI/\partial^2 z_2 -$$
$$\partial^2 ETCI/\partial z_2\partial z_1\times\partial^2 ETCI/\partial z_1\partial z_2$$

(Note: $\psi(z_1)$, $\psi(z_2)$ are normal distributions so $\psi'(z_1),\psi'(z_2)$) are negative ,then for determining the sign of second order statement we must have both of $\partial^2 ETCI/\partial^2 z_1$ and $\partial^2 ETCI/\partial^2 z_2$ by the same sign to have always the absolute minimum value for this model because of:{(-)(-)(+)(+)-(+)>0 or (-)(-)(-)(-)-(+)>0}

In this section we consider that both of them ($\partial^2 ETCI/\partial^2 z_1$ and $\partial^2 ETCI/\partial^2 z_2$) are positive:

In this case

$$\{k_1Q_1\int\psi(z_2)+k_2\times\int(\psi(z_2)Q_2)+(k_1\sigma_1^2+k_2\sigma_2^2-cr)$$
$$\int\psi(z_2)\}>0$$

and

$$\{k_2Q_2\int\psi(z_1)+k_1\times\int(\psi(z_1)Q_1)+(k_1\sigma_1^2+k_2\sigma_2^2-cr)$$
$$\int\psi(z_1)\}>0$$

and for evaluation the above statement we should have the values of (n_1,z_1,n_2,z_2) and these values must be found after the problem solution .

we should compare ETCI with ETCA and ETCR as follows:

ETCA=

$$N\iiiint q(x_1,x_2)f(x_1,x_2\mid\mu_1,\mu_2)h(\mu_1)$$
$$h(\mu_2)d\mu_1 d\mu_2 dx_1 dx_2 =$$
$$N\iiiint(k_1x_1^2+k_2x_2^2)f(x_1\mid\mu_1)f(x_2\mid\mu_2)$$

and

$$h(\mu_1)h(\mu_2)d\mu_1 d\mu_2 dx_1 dx_2$$
$$=N\{k_1(\sigma_1^2+\sigma_{\mu 1}^2)+(k_2(\sigma_2^2+\sigma_{\mu 2}^2)\}=$$
$$N\{k_1(\sigma_1^2+\sigma_1^2/d_1)+(k_2(\sigma_2^2+\sigma_2^2/d_2)\}$$

ETCR=N×Cr

By solving equation (14) we will found the optimal method for this problem. In this problem we can not say the model has the absolute minimum answer for ETCI cost model (not for model),but by modeling this problem by Maple software and solving the model for problem , and checking the feasibility condition of the problem , we can have the optimum answer for the model .The following example could explain it clearly.

Example

Let:

σ_1=0.8, σ_2=0.65, N=100,000, d_1=5, d_2=5

Cs=10, Ci=5, Cr=2.5, k_1=2, k_2=2.2, α=β=0

Then by modeling and programming with Maple 9.5 and comparing this problem with the case that we have only one variable ((K=1 ,σ=.8,),(K=2.2,σ=.65)) we have the following table(Table 1 in index):

For solving this problem (double variable model) we must check $k_1\sigma_1^{\ 2}+k_2\sigma_2^{\ 2}-cr<0$:
$2\times.8$^2+2.2$\times$.65^2-2.5<0

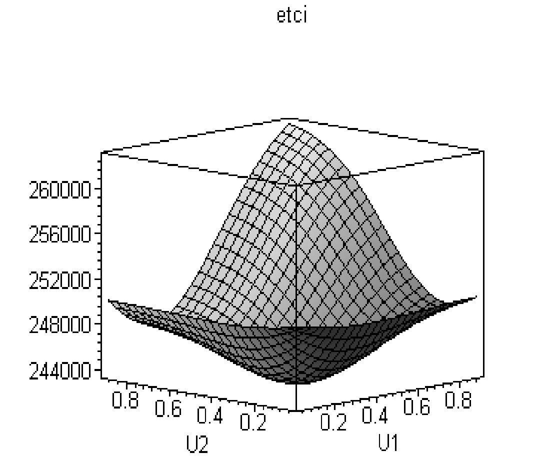

Figure 1: Double variables model for α=β=0.

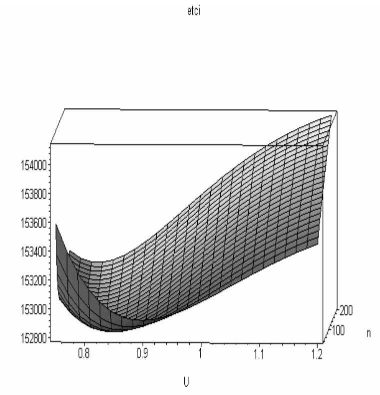

Figure 2: Single variable for α=β=0,k=2,s=0.8.

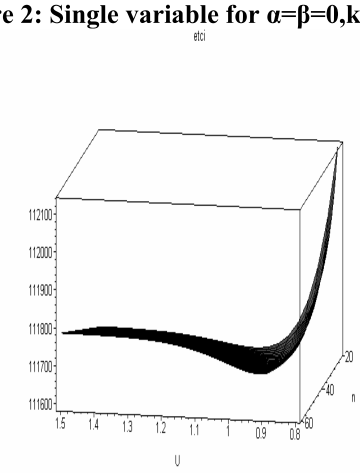

Figure 3: Single variable for S=.65,K=2.2,α=β=0.

Let have inspection errors with deterministic values: in above example we only change inspection error values to α=5%, β=10%.

Using above equations and programming the model by maple software, we have:
z_1=0.33, z_2=0.3, n_1=109, n_2=109, Pae=43.69%,ETCI=245689.3, ETCA=265140, ETCR=250000 So the optimal decision is acceptance with inspection (Figure 4).

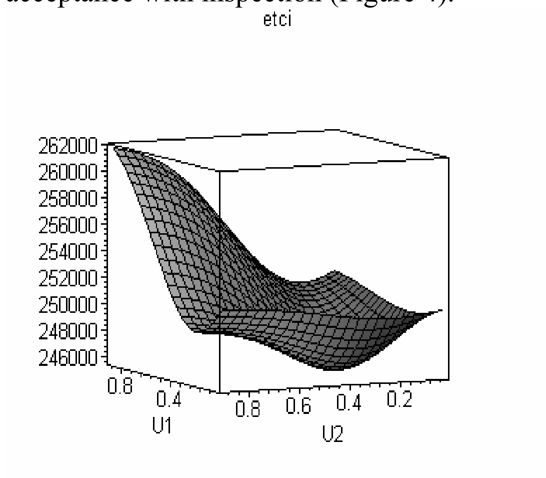

Figure 4: Double variable for k_1=2, σ_1=0.8, k_2=2.2, σ_2=0.65,α=5%,β=10%.

Let
k_1=1.2, k_2=0.8, σ_1=1, σ_2=1,α=5%,β=10%,cr= 2.5, $d_1=d_2$=5,N=100000
First of all, we should check the condition: $k_1\sigma_1^{\ 2}+k_2\sigma_2^{\ 2}-cr<0$, this condition holds true, so modeling the problem by above equations and programming by Maple 9.5, we have :
n_1=79, n_2=109, z_1=0.71, z_2=0.59,
ETCI=232762, ETCA=2400000,
ETCR=250000
so the best decision is acceptance with inspection corresponding to these inspection error values.

Note: If we want to have two discrete sampling plan by variables No.1 and No.2
We must have:
n_1=50, z_1=1.1, ETCI=148980 ETCA=144000, ETCR=250000 and n_2=50, z_2=1.7, ETCI=103895.8, ETCA=960000, ETCR=250000
in these two problems, the best solution is lot acceptance without sampling with the total cost 240000(96000+144000), but when we have a double variable acceptance sampling problem with two independent variable, in the same time the best solution is acceptance

sampling by inspection with lower cost(232762<240000).

Exponentially inspection errors

In this section we propose two types of increasing and decreasing exponential functions for error types:

a) Increasing type

In this model inspection error will be increased by the number of sample size. We consider inspection error as below:

$e(n)=e(-(n_1+n_2)/1000)-1$ and $\alpha=e(n)/5$, $\beta=4e(n)/5$

by replacing above statement in double variable inspection model and modeling with Maple we have following results:

n1= 70,n2=100,Z1=.73,Z2=.6,
Pae=230203,Pae=71.53%,
α=0.34%,β=1.37%

Note

In this model the best value for inspection errors will be calculated by model and by knowing this information about the best value for inspection error parameter we can have a good sight to calibrate inspection instrument, considering cost values.

b) Decreasing type

In this model inspection error will be decreased by the number of sample size:

We consider inspection error as follows:

$e(n)=e(-(n_1+n_2)/7000)-0.36$ and $\alpha=e(n)/5$, $\beta=4e(n)/5$

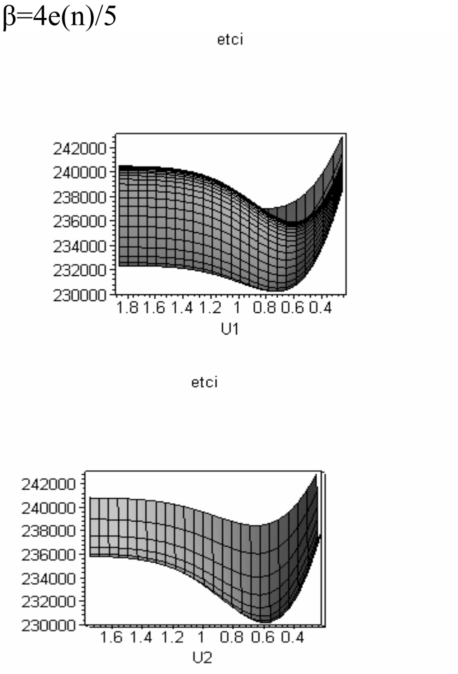

Figure 5: double variable with exponentially increasing errors.

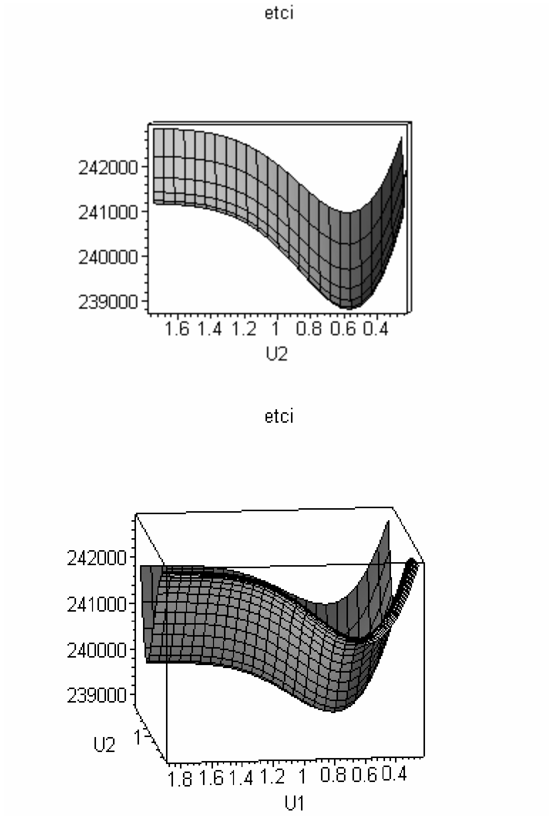

Figure 6: double variable with exponentially decreasing errors.

By replacing above statement in double variable inspection model and modeling with Maple we have following results:

n1=70,Z1=.74,n2=100,Z2=.58,Pae=66.6%,ETCA=240000,ETCI=230216.7,ETCR=250000

so the best decision is acceptance sampling with best values for inspection errors as:

α=12.3%, β=49.2%

Concluding remarks

An new economical model for the selection of cost minimizing acceptance sampling plans for double variable model with two independent variables has been developed when inspection errors are present. A cost model is proposed for situations of fixed and variable inspection errors and also using quadratic cost in Taguchi method.

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Table 1: Results for single and double variables modeling.

α (%)	B (%)	K	σ	n_1	z_1	n_2	z_2	ETCI	Pae(%)	ETCA	ETCR	Decision
0	0	2	0.8	70	.83	----	----	152797.3	97.49	153200	250000	ETCI
0	0	2.2	0.6 5	---	---	50	.92	111668.3	99.74	111540	250000	ETCA
0	0	----	---	139	.34	109	.33	243607.07	47.61	265140	250000	ETCI
5	10	----	----	109	.33	109	.3	245689.3	43.69	265140	250000	ETCI

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Appendix

a) $$\partial/\partial z_1\{\int_{\mu_2}\int_{\mu_1}\int_{\overline{x_2}}\int_{\overline{x_1}}\{g(\overline{x}_1\mid\mu_1)\times g(\overline{x}_2\mid\mu_2)\}h(\mu_1)h(\mu_2)d\overline{x}_1 d\overline{x}_2 d\mu_1 d\mu_2\} =$$

$$\int_{\mu_1}\int_{\overline{x_{x1}}}\int_{\mu_1}\{g(-z_1\mid\mu_1)+g(z_1\mid\mu_1)\}h(\mu_1)g(\overline{x}_2\mid\mu_2)h(\mu_2)d\overline{x_2}d\mu_2 d\mu_1 \quad =$$

(x_1,x_2 are independent) so:

$$\int_{\mu_2}\int_{\overline{x_2}}2\psi(z_1)g(\overline{x_2}\mid\mu_2)h(\mu_2)d\overline{x}_2 d\mu_2 = 2\psi(z_1)\times\int_{-z_2}^{z_2}\psi(z_2)d\mu_2$$

b) $$\partial/\partial z_2\{\int_{\mu_2}\int_{\mu_1}\int_{\overline{x_2}}\int_{\overline{x_1}}\{g(\overline{x}_1\mid\mu_1)\times g(\overline{x}_2\mid\mu_2)\}h(\mu_1)h(\mu_2)d\overline{x}_1 d\overline{x}_2 d\mu_1 d\mu_2\} =$$

$$\int_{\mu_1}\int_{\overline{x_{x1}}}\int_{\mu_1}\{g(-z_2\mid\mu_2)+g(z_2\mid\mu_2)\}h(\mu_2)g(\overline{x}_1\mid\mu_1)h(\mu_1)d\overline{x_1}d\mu_1 d\mu_2 \quad =$$

(x_1,x_2 are independent) so:

$$\int_{\mu_1}\int_{\overline{x_1}}2\psi(z_2)g(\overline{x_1}\mid\mu_1)h(\mu_1)d\overline{x}_1 d\mu_1 = 2\psi(z_2)\times\int_{-z_1}^{z_1}\psi(z_1)d\mu_1$$

c) $$\partial/\partial z_1\{\int_{\mu_2}\int_{\mu_1}\int_{\overline{x_2}}\int_{\overline{x_1}}\{k1\mu_1^2 g(\overline{x}_1\mid\mu_1)\times g(\overline{x}_2\mid\mu_2)\}h(\mu_1)h(\mu_2)d\overline{x}_1 d\overline{x}_2 d\mu_1 d\mu_2\}$$

$$=k1\int_{\mu_1}\int_{\mu_2}\int_{\overline{x_2}}\mu_1^2\{g(z_1\mid\mu_1)+g(-z_1\mid\mu_1)\}h(\mu_1)d\mu_1 g(\overline{x_2}\mid\mu_2)d\overline{x_2}d\mu_2$$

(x_1,x_2 are independent) so:

$$k1\int_{\mu_x}\int_{\overline{x_x}}2\psi(z_1)\{n_1^2 z_1^2/(n_1+D_1)^2+\sigma_1^2/(n_1+D_1)\}g(\overline{x_2}\mid\mu_2)d\overline{x_2}d\mu_2$$

d) $$\partial/\partial z_2\{\int_{\mu_2}\int_{\mu_1}\int_{\overline{x_2}}\int_{\overline{x_1}}\{k2\mu_2^2 g(\overline{x}_2\mid\mu_2)\times g(\overline{x}_1\mid\mu_1)\}h(\mu_2)h(\mu_1)d\overline{x}_1 d\overline{x}_2 d\mu_1 d\mu_2\}$$

$$=k2\int_{\mu_1}\int_{\mu_2}\int_{\overline{x_1}}\mu_2^2\{g(z_2\mid\mu_2)+g(-z_2\mid\mu_2)\}h(\mu_2)d\mu_2 g(\overline{x_1}\mid\mu_1)d\overline{x_1}d\mu_1$$

(x_1,x_2 are independent) so:

$$k2\int_{\mu_x}\int_{\overline{x_1}}2\psi(z_2)\{n_2^2 z_2^2/(n_2+D_2)^2+\sigma_2^2/(n_2+D_2)\}g(\overline{x_1}\mid\mu_1)d\overline{x_1}d\mu_1$$

e) $$\partial/\partial z_1\{\int_{\mu_2}\int_{\mu_1}\int_{\overline{x_1}}g(\overline{x_1}\mid\mu_1)h(\mu_1)h(\mu_2)d\mu_1 d\mu_2 d\overline{x_1}\} =$$

$$\int_{\mu_2}\int_{\mu_1}\{g(-z_1\mid\mu_1)+g(z_1\mid\mu_1)\}h(\mu_1)h(\mu_2)d\mu_1 d\mu_2 = 2\psi(z_1)$$

Similarly:

f) $\partial/\partial z_2\{\int_{\mu_2}\int_{\mu_1}\int_{\overline{x_2}} g(\overline{x_2}\mid\mu_2)h(\mu_2)h(\mu_1)d\mu_1 d\mu_2 d\overline{x_2}\} = 2\psi(z_2)$

g) $\partial/\partial z_1\{\int_{\mu_2}\int_{\mu_1}\int_{\overline{x_1}} \mu_1^{\,2} g(\overline{x_1}\mid\mu_1)h(\mu_1)h(\mu_2)d\overline{x_1}d\mu_1 d\mu_2\} = 2\psi(z_1)\times$

$\{n_1^{\,2}z_1^{\,2}/(n_1+D_1)^2+\sigma_1^{\,2}/(n_1+D_1)\}$

h) $\partial/\partial z_2\{\int_{\mu_2}\int_{\mu_1}\int_{\overline{x_2}} \mu_2^{\,2} g(\overline{x_2}\mid\mu_2)h(\mu_1)h(\mu_2)d\overline{x_2}d\mu_1 d\mu_2\} = 2\psi(z_2)\times$

$\{n_2^{\,2}z_2^{\,2}/(n_2+D_2)^2+\sigma_2^{\,2}/(n_2+D_2)\}$

i) $\partial/\partial z_1$

$$\int\int\int \mu_1^{\,2} g(\overline{x_1}\mid\mu_1)h(\mu_1)h(\mu_2)d\overline{x}_1 d\mu_1 d\mu_2 = 2\psi(z_1)\{n_1^{\,2}z_1^{\,2}/(n_1+D_1)^2+\sigma_1^{\,2}/(n_1+D_1)\}$$

j) $\partial/\partial z_2$

$$\int\int\int \mu_2^{\,2} g(\overline{x_2}\mid\mu_2)h(\mu_1)h(\mu_2)d\overline{x}_2 d\mu_1 d\mu_2 = 2\psi(z_2)\{n_2^{\,2}z_2^{\,2}/(n_2+D_2)^2+\sigma_2^{\,2}/(n_2+D_2)\}$$

k)
$\partial/\partial z_1$

$$\int\int\int\int \mu_2^{\,2} g(\overline{x}_1\mid\mu_1)g(\overline{x}_2\mid\mu_2)h(\mu_2)h(\mu_1)d\overline{x}_1 d\overline{x_2}d\mu_1 d\mu_2 =$$

$$\int\int \mu_2^{\,2} g(\overline{x}_2\mid\mu_2)h(\mu_2)d\overline{x_2}d\mu_2\times\partial/\partial z_1\int\int g(\overline{x}_1\mid\mu_1)h(\mu_1)d\overline{x}_1 d\mu_1 =$$

$$2\psi(z_1)\int\int \mu_2^{\,2} g(\overline{x}_2\mid\mu_2)h(\mu_2)d\overline{x_2}d\mu_2 = 2\psi(z_1)\times\int\psi(z_2)n_2^{\,2}z_2^{\,2}/(n_2+D_2)^2+\sigma_2^{\,2}/(n_2+D_2)d\mu_2$$

l)

$$\int\int\int\int \mu_1^{\,2} g(\overline{x}_1\mid\mu_1)g(\overline{x}_2\mid\mu_2)h(\mu_2)h(\mu_1)d\overline{x}_1 d\overline{x_2}d\mu_1 d\mu_2 =$$

$$\partial/\partial z_2\int\int \mu_1^{\,2} g(\overline{x}_1\mid\mu_1)h(\mu_1)d\overline{x_1}d\mu_1\times\partial/\partial z_2\int\int g(\overline{x}_2\mid\mu_2)h(\mu_2)d\overline{x}_2 d\mu_2 =$$

$$2\psi(z_2)\int\int \mu_1^{\,2} g(\overline{x}_1\mid\mu_1)h(\mu_1)d\overline{x_1}d\mu_1 = 2\psi(z_2)\times\int\psi(z_1)n_1^{\,2}z_1^{\,2}/(n_1+D_1)^2+\sigma_1^{\,2}/(n_1+D_1)d\mu_1$$

$$\partial/\partial z_2\int\int\int \mu_2^{\,2} g(\overline{x}_1\mid\mu_1)h(\mu_1)h(\mu_2)d\overline{x}_1 d\mu_1 d\mu_2 = \int\int(\int \mu_2^{\,2}h(\mu_2)d\mu_2)g(\overline{x}_1\mid\mu_1)h(\mu_1)d\overline{x}_1 d\mu_1 =$$

$$(\sigma_2^{\,2}/D_2)\times\int\int g(\overline{x_1}\mid\mu_1)h(\mu_1)d\overline{x_1}d\mu_1 = (\sigma_2^{\,2}/D_2)\int\psi(z_1)$$