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Modelling Decision Problems Via Birkhoff Polyhedra

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ABSTRACT

A compact formulation of the set of tours neither in a graph nor its complement is presented and illustrates a general methodology proposed for constructing polyhedral models of decision problems based upon permutations, projection and lifting techniques. Directed Hamilton tours on n vertex graphs are interpreted as (n-1)-permutations. Sets of extrema of Birkhoff polyhedra are mapped to tours neither in a graph nor its complement and these sets are embedded into disjoint orthogonal spaces as the solution set of a compact formulation. An orthogonal projection of its solution set into the subspace spanned by the Birkhoff polytope is the convex hull of all tours neither in a graph nor its complement. It's suggested that these techniques might be adaptable for application to linear programming models of network and path problems.

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1 Introduction

A compact formulation of the set of tours neither in a graph nor its complement is presented and illustrates a general methodology for constructing polyhedral models of decision problems based upon permutations, projection and lifting techniques. Directed Hamilton tours on n vertex graphs are interpreted as (n-1)-permutations, and sets of extrema of the Birkhoff polytope are mapped to tours neither in a graph nor its complement. A polynomial number of Birkhoff polyhedra of varied dimension are embedded into disjoint orthogonal spaces so that an orthogonal projection of its solution set into the subspace spanned by the Birkhoff polytope is the convex hull of all tours neither in a graph nor its complement. These modelling techniques can be adapted to model paths in a network. It's suggested that this be investigated for possible use in linear programming models of network and path problems.
Motivation for this paper came about through curiosity, in association with Ted Swart [1]. Ted claimed in 1988 that **P**=**NP** having created a compact formulation closely related to the TSP polytope [2]. But Yannakakis [3] proved shortly thereafter that no *symmetric* compact formulation of the TSP polytope exists, refuting Ted's claim[2]. At that time, Ted and I worked together as 'professor and student' and we decided to study the tractable Birkhoff polytope in search of understanding how and why symmetry appeared to go 'hand in hand' with tractability. That is, we wondered about the existence of non-symmetric compact formulations of the TSP polytope i.e. it was suspected and now known [4], that symmetry impacts upon the size of an extended formulation. We became friends and colleagues, and our continued collaborations led us to study extended formulations using sets of Birkhoff polyhedra. Hence this paper.
The ideas presented here were probably first formalized by Balas
Let graph G=$G(V, E)$ be simple, connected and directed, $|V| = n > 4$, $E \neq \emptyset$ neither empty nor complete. A Hamilton tour or *tour* in G is a circuit containing each vertex in G, and G is said to be *Hamiltonian* if and only if there exists a tour in G. Otherwise G is *non-Hamiltonian*.

Definition 1.1 *A p-set polytope*[3]*, is the convex hull of a set of permutation matrices. Members are written as $\mathcal{O}(n^2)$ vectors, whose components are variables $p_{ui} \in \{0, 1\}^4$.*

Definition 1.2 *An extended formulation is a system of rational linear equations and inequalities whose finite number of extreme point solutions under an orthogonal projection are the set of solutions of a discrete problem.*

Definition 1.3 *A compact formulation is a polynomial sized extended formulation [11].*

It should be noted that the **coNP**-complete decision problem *Is a graph non-Hamiltonian?* is expressed as deciding equivalence of a p-set polytope and the Birkhoff polytope, also **coNP**-

[1] *Dedicated to my mother Shirley Frances Gismondi 1931-2012.*
[2] *It's now known that the TSP polytope has no compact formulation [12]*

complete. Modelling **NP**-complete (**coNP**-complete) problems using ideas in this paper therefore does not increase their computational complexity.

1.1 Terminology and Definitions

Define a tour as a permutation of vertex labels of a graph, where every tour starts and ends at vertex n, and, assign p_{ui}=1 (of an (n-1)-permutation matrix) if and only if the i^{th} arc of a tour enters vertex u. Let P be the set of n-1! (n-1)-permutations corresponding to all tours in a complete graph. Denote the Birkhoff polytope, the convex hull of all permutations as $conv(P)$. Given an instance of G, let P_G be the set of permutations corresponding to the set of tours in G. Let P_{G^c} be the set of permutations corresponding to the set of tours in G^c, where G^c is the arc complement of G. Let P_S be the set of permutations corresponding to the set of tours neither in G nor in G^c. These sets are mutually disjoint, related by Equation 1.

$$P = P_G \cup P_{G^c} \cup P_S \tag{1}$$

Definition 1.4 *An extraneous tour of G (G^c) is a tour not in G (G^c). The corresponding permutation is called an extraneous permutation of G (G^c).*

Definition 1.5 *A straddling tour is an extraneous tour of both G and G^c. The corresponding permutation is called a straddling permutation.*

The sets of extraneous permutations of G and G^c are respectively

$$\overline{P_G} = P_{G^c} \cup P_S \tag{2}$$

$$\overline{P_{G^c}} = P_G \cup P_S \tag{3}$$

It follows that

$$P_S = \overline{P_G} \cap \overline{P_{G^c}} \tag{4}$$

Remark 1.1 Tours are commonly modelled as extrema of the asymmetric traveling salesman problem (ATSP) polytope[5], the convex hull of all n-1! n-cycles [19, 20, 21, 13, 14, 22, 23, 24, 25, 3], a *subset* of the set of all n-permutations. Variables represent arcs of an instance of G, and arc (i, j) participates in a tour if and only if $x_{i,j} = 1$. Unlike arc variables of a graph, where variable $x_{i,j}$ represents the weight assigned to arc (i, j) in G, p_{ui} variables represent *sequence positions*[6] i.e. arc i of a tour enters vertex u [10].

[3] *The term p-set polytope is used rather than the more natural term permutation polytope since a permutation polytope is already defined as the convex hull of the members of a subgroup of permutations, common in literature. It is noted that faces of the Birkhoff polytope are permutation polyhedra, from which the model presented in this paper is constructed. To learn more about permutation polyhedra, see [13, 14, 15, 16, 17, 18].*

[4] *Note that pairs of subscripts of p variables are not separated by a comma. e.g. p_{14}.*

[5] *An external representation of the ATSP is unknown in general.*

[6] *Ted Swart originally proposed this idea in 1990 [1].*

2 Methodology

An overview is now presented that describes the most important details of coding extraneous tours and straddling tours of G as permutations, fundamental to an understanding the models presented in Sections 2.2 and 3.

2.1 Extraneous Permutations

Imagine the task of coding a set of tours that use a particular arc not in a graph, for example, say for some six vertex graph G_6. Thus choose any arc in G_6^c, say arc $(6,3)$, assign it to be the first arc of all tours in the set, and then list all tours by permuting the remaining set of four vertex labels e.g. $6-3-2-4-5-1-6$ is one such tour. A convenient coding for this set of tours must somehow indicate and assign arc $(6,3)$ as a first arc, while accounting for the list of tours whose permuted vertex labellings might instead be coded as permutations. A set of equations that code double stochastity can be constructed, whose solution set is a p-set polytope, i.e. the convex hull of all such permutations (mapped from tours) that correspond to permutations of the vertex labelling, but assigning component $p_{31}=1$ indicating that arc one enters vertex three. Of course the implementation presented below fixes vertex n as the start and end vertex of every tour, and arc (6,3) is allowed only to play the role of a first arc. But in general, if an arc is allowed to play the role of a second arc in a tour not in G, a third arc in a tour not in G and so on, then many extraneous tours are created. A *family* of extraneous tours is associated with an arc not in G, and is the set of all extraneous tours that use this arc. The family of extraneous tours corresponds with a family of permutations allowing for the external representation of each p-set polytope to be the solution set of the polynomial sized assignment constraints, some variables set at unit level.
Extend this idea by choosing all arcs in G^c, allowing each arc to play the role of a first, second,...,n^{th} arc. Lifting and embedding these polyhedra into disjoint orthogonal subspaces permits the construction of a compact formulation whose, 1) image is a p-set polytope, and 2) set of extrema is the complete set of permutations corresponding to the complete set of tours that use an arc in G^c. That is, a polynomial sized linear system can now be constructed by creating its solution set to be the convex combination of all solution sets of all corresponding sets of assignment constraints that define all p-set polyhedra associated with all arcs in G^c. Under an orthogonal projection into a subspace, the image of every extraneous permutation is extreme, and every extreme point in the image space is the image of at least one extraneous permutation.

2.2 Straddling Permutations

Recall that an extraneous permutation of G corresponds to an extraneous tour of G that uses at least one arc in G^c, and possibly all arcs in G^c, i.e. tours in G^c. Consider now the subset of extraneous permutations of G that also use at least one arc in G, i.e. tours neither in G

nor G^c, straddling tours. By choosing a particular arc in G and a particular arc in G^c, called an *arc pairing*, a family of straddling permutations can be coded by allowing these arcs to be sequenced as first, second,...,n^{th} arcs from which to enumerate corresponding straddling permutations. By creating the set of all arc pairings and generating all subsequent families of straddling permutations, the complete set of straddling permutations can be coded.
Imagine first the task of creating an arc pairing composed of arc (a, b) in G and arc (c, d) in G^c, where each arc is allowed to be sequenced as a first, second,...,n^{th} arc of a tour, in order to create a corresponding family of straddling permutations. Some arc pairings are not allowed to be sequenced in any straddling permutation, for example, in the case where arcs enter or exit the same vertex. Likewise some arc sequence combinations are also not allowed to be assigned to any straddling permutation, say in the case where two consecutively sequenced arcs imply that the same sequenced arc enter different vertices. These restrictions are naturally enforced by use of permutations, illustrated in Section 3 during the process of creating arc pairings. Thus the term *permissible* is now introduced and refers to the idea that a particular arc pairing together with at least one arc sequence combination can generate at least one straddling permutation. Note that arcs that exit and enter vertex n are defined a priori as first and n^{th} arcs of a tour and therefore cannot be sequenced as second, third,...,n-1^{th} arcs of a tour. These restrictions are accommodated by seven special cases presented in Section 3.2.
Thus choose all $O(n^4)$ arc pairings, coding all $O(n^2)$ arc sequencings resulting in an model of the complete set of all families of straddling permutations.

Remark 2.1 Straddling tours can be further classified into n-1 disjoint sets corresponding to extraneous tours that use: one arc in G and therefore n-1 arcs in G^c, two arcs in G and therefore n-2 arcs in G^c and so on. Modelling techniques used in this paper can be generalized to model any of these sets. Implementation details become intricate.

Remark 2.2 Tours not in G are modelled as extrema of a p-set polytope in [10], and the complement set of extrema with respect to $conv(P)$ is the set of permutations that correspond to tours in G. In this way the set of tours in G are conveniently represented by what they are not, namely extrema of the p-set polytope. This leads to the idea of investigating the Hamilton tour decision problem via classification of graphs by the sets of tours that they do not instantiate. Thus for each n, perform a one-time assignment of vertex labels, and partition the complete set of graphs into equivalence classes. Two graphs belong to the same equivalence class if an only if they instantiate identical sets of tours in G and hence identical sets of tours not in G. While each equivalence class can be represented by any one of its members, there now exists precisely one representative p-set polytope. Further, two equivalence classes are defined to be isomorphic if and only if there exists a permutation of graph vertex labels in one class such that representative p-set polyhedra of each class become equal. Interestingly however, there is no issue of isomorphism in the case of non-hamiltonian graphs. They all belong to the same equivalence class represented by $conv(P)$, and deciding Hamiltonicity (non-Hamiltonicity) of a graph is the problem of deciding if its p-set polytope is (is not) $conv(P)$. This problem can be

$$\underset{G_9}{\begin{bmatrix} - & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & - & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & - & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & - & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & - & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & - & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & - & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & - & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & - \end{bmatrix}} \underset{G_9^c}{\begin{bmatrix} - & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & - & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & - & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & - & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & - & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & - & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & - & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & - & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & - \end{bmatrix}}$$

Figure 1: Adjacency Matrix Representation of G_9 and G_9^c

expressed as that of deciding strict inclusion (equivalence) of a pair of p-set polyhedra. While viewing tours as permutations may be mere novelty, Ted [1] and I like to think of the $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ problem as that of determining the complexity of deciding whether or not a particular p-set polytope is $conv(P)$.

3 A Model of All Straddling Permutations and Main Theorem

Consider an example. Given an instance of nine vertex graph, say G_9, imagine the existence of a set (part of a family) of straddling tours whose second arc enters a vertex along an arc in G_9 and whose fourth arc enters a vertex along an arc in G_9^c. See Figure 1 below, adjacency matrices corresponding to G_9 and G_9^c. Suppose also that this set of straddling tours incorporates arc $(1, 2)$ in G_9 and arc $(3, 5)$ in G_9^c as second and fourth arcs. This set of straddling tours is now coded below, illustrating a natural link to straddling permutations.

3.1 Signal Matrices and Doublets

Construct signal matrix S, used to define sets of permutations that correspond to arc sequencings. To encode the idea that the second arc of a tour enters vertex two along arc $(1, 2)$ in G_9, set $s_{22} = 1$, and then force the first arc of a tour to enter vertex one, i.e. set $s_{11} = 1$. Encode the fourth arc of a tour to enter vertex five along arc $(3, 5)$ in G_9^c, set $s_{54} = 1$, and then force the third arc of a tour to enter vertex three, i.e. set $s_{33} = 1$. See Figure 2.

Every permutation permitted by S is a straddling permutation for which the second arc of a tour enters vertex two along arc $(1, 2)$ in G_9 and whose fourth arc enters vertex five along arc $(3, 5)$ in G_9^c. Thus if assignment constraints were imposed upon S, using unassigned variables s_{ij}, then the set of extrema of the solution set of these constraints is the complete set of these straddling permutations. For ease of book keeping, associate this signal matrix with its set of straddling tours via the solution set of these constraints, interchangeably using the notation

$\{p_{11}, p_{22}, p_{33}, p_{54}\}$ or $p_{11} = p_{22} = p_{33} = p_{54} = 1$. Note that $\{p_{11}, p_{22}, p_{33}, p_{54}\}$ is referred to as a 4-doublet. See [26].

There commonly exist 2-doublets, 3-doublets and 4-doublets, conveniently written as 2,3,4-doublets. Observe that when arc $(1, 2)$ in G_9 and arc $(3, 5)$ in G_9^c are respective second and fifth arcs, second and sixth arcs in proposed extraneous tours of G_9, 4-doublets $\{p_{11}, p_{22}, p_{34}, p_{55}\}$ and $\{p_{11}, p_{22}, p_{35}, p_{56}\}$ are referenced, and so on. Imagine then creating a list of all permissible 4-doublets allowing arcs $(1, 2)$ and $(3, 5)$ to play the roles of non-consecutive second, third,...,eighth sequenced arcs in a straddling tour, i.e. keep only those 4-doublets whose associated signal matrix admits at least one permutation solution. In this way, create a polynomial number of sets of assignment constraints, whose solution sets are p-set polyhedra, whose convex combination is the convex hull of a set of straddling permutations of G_9, whose extrema are a family of straddling permutations specific to an arc pairing.

As noted earlier, some arc sequence combinations are not allowed to be assigned to any straddling permutation. For example, referring to arcs $(1, 2)$ and $(3, 5)$, these arcs cannot be sequenced consecutively since there is no common vertex. This is naturally enforced by use of permutations since the corresponding signal matrix admits no permutation solution. That is, sequencing these arcs as k-1^{th} and k^{th} arcs in a proposed straddling tour causes the following assignments to signal matrix S: to sequence the k^{th} arc to enter vertex five, assign $s_{5k} = 1$ and $s_{3k-1} = 1$ to sequence the k-1^{th} arc to enter vertex two, assign $s_{2k-1} = 1$ and s_{1k-2}. But this is not permissible since the k-1^{th} arc enters two distinct vertices (violating a column sum), vertices two and three.

Consecutively sequenced arcs define a common vertex and other than issues associated with vertex n, generally result in assignment of three components to S. Consecutively sequenced arcs that define a subtour, e.g. (1,2) and (2,1) are naturally disallowed, since this results in two distinct arcs entering the same vertex (violating a row sum). Likewise some arc pairings are not allowed to be sequenced in any straddling permutation, say in the case of arcs $(1, 2)$ in G_9 and $(1, 3)$ in G_9^c. These restrictions are likewise enforced by use of permutations, i.e. a corresponding signal matrix admits no permutation solution. That is, if the k^{th} arc of a tour

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{45} & s_{46} & s_{47} & s_{48} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{65} & s_{66} & s_{67} & s_{68} \\ 0 & 0 & 0 & 0 & s_{75} & s_{76} & s_{77} & s_{78} \\ 0 & 0 & 0 & 0 & s_{85} & s_{86} & s_{87} & s_{88} \end{bmatrix}$$

Figure 2: Sample Signal Matrix

enters vertex three, then the k-1^{th} arc enters vertex one. Then if the j^{th} arc enters vertex two, then the j-1^{th} arc must also enter vertex one, which is again not permissible, since two distinct arcs j and k both enter vertex one. Similarly for arc pairings of the form (a, b) and (c, b). Sequencing of arcs in proposed straddling permutations for which arcs enter or exit the same vertex is naturally disallowed.

3.2 Characterization of All Straddling Permutations

Generalizing, create all arc pairings (a, b) in G and (c, d) in G^c. For each arc pairing, allow each arc to play all permissible roles of first, second,...,n^{th} arcs in straddling tours of G. Characterizing all straddling tours, it is now convenient to view a straddling tour as composed of three pieces, namely 1) a first arc, 2) a second, third,...,n-1^{th} arc, and, 3) an n^{th} arc. Seven characterizations result, presented below.

Characterization 3.1 ***A first arc in G and a second,third,..., n-1^{th} arc in G^c.***

Observe arcs of G that exit vertex n, first arcs of potential straddling tours. Referring to G?s adjacency matrix, create permissible signal matrices as follows. If $g_{nu} = 1, u = 1, 2, ..., n$-1, assign $s_{u1} = 1$ in order to force the first arc to enter vertex u. Unlike the example presented above where two s_{ij} must be specified in order to force tours to use specified arcs, every first arc is defined as ?leaving the origin? and it is sufficient to assign only $s_{u1} = 1$. For each arc pairing $(n, u), u = 1, 2, ..., n$-1 in G and (c, d) in G^c, $c, d = 1, 2, ..., n$-1, $c \neq d$, create each permissible signal matrix allowing arc (c, d) to play the role of a second, third,...,n-1^{th} arc of a straddling tour, e.g. by assigning $s_{d2} = s_{c1} = 1$, $s_{d3} = s_{c2} = 1, ..., s_{dn-1} = s_{cn-2} = 1$. Note that in the case of a common vertex e.g. $c = u$, arc (c, d) can only play the role of a second arc of a straddling tour and signal matrices have two unit entries. In the case of no common vertex, arc (c, d) can only play the role of a third, fourth,...,n-1^{th} arc of a straddling tour and signal matrices have three unit entries. The complete family of straddling permutations for each arc pairing is therefore the set of extrema of each corresponding solution set of assignment constraints, associated with a set of 2-doublets or 3-doublets. The convex hull of the union of all solution sets associated with all 2,3-doublets, is the convex hull of all straddling tours whose first arc is in G and whose second, third,...,n-1^{th} arc is in G^c, and a convenient representation of these extrema is simply the set of all 2,3-doublets.

Characterization 3.2 ***A first arc in G and an n^{th} arc in G^c.***

Observe arcs of G that exit vertex n, first arcs of potential straddling tours. Referring to adjacency matrices of G and G^c, create permissible signal matrices as follows. For each arc pairing of the form $g_{nv} = 1, v = 1, 2, ..., n$-1 and $g^c_{un} = 1, u = 1, 2, ..., n$-1, assign $s_{v1} = 1$ in order to force the first arc of a straddling tour to enter vertex v along an arc in G, and assign $s_{un-1} = 1$ in order to force the n-1^{th} arc of a straddling tour to enter vertex u so that the n^{th} arc perforce enters vertex n along arc (u, n) in G^c. Each signal matrix now has two unit entries. The complete family of straddling permutations for each arc pairing is therefore the set of extrema associated with each corresponding set of 2-doublets, and the union of all 2-doublets associated with all families of solutions is likewise a convenient representation of the set of all straddling tours whose first arc is in G and whose n^{th} arc is in G^c.

Characterization 3.3 ***A second, third,...,n-1th arc in G and a first arc in G^c.***

Repeat Characterization 3.1. Swap roles of G and G^c.

Characterization 3.4 ***A second, third,...,n-1th arc in G and a second, third,...,n-1th arc (different from that of G) in G^c.***

Observe arcs of G and G^c, other that those that enter and / or exit vertex n. Referring to G?s and G^{c}'s adjacency matrices, create permissible signal matrices as follows. For arc (a, b) in G, $a, b = 1, 2, ..., n$-1, $a{\neq}b$ and arc (c, d) in G^c, $c, d = 1, 2, ..., n$-1, $c{\neq}d$, create each permissible signal matrix allowing arcs (a, b) and (c, d) to play the roles of a second, third,...,n-1th arc of a straddling tour, e.g. by assigning $s_{b2} = s_{a1} = 1, s_{b3} = s_{a2} = 1, ..., s_{bn-1} = s_{an-2} = 1$, and $s_{d2} = s_{c1} = 1, s_{d3} = s_{c2} = 1, ..., s_{dn-1} = s_{cn-2} = 1$. Note that in the case of a common vertex, e.g. $b = c$, or $a = d$ (since (c, d) may be sequenced before (a, b)) paired arcs play the roles of second and third arcs, third and fourth arcs,...,n-2th and n-1th arcs, and signal matrices have three unit entries. In the case of no common vertex, signal matrices have four unit entries. The complete family of straddling permutations for each arc pairing is therefore the set of extrema of each solution set associated with each set of 3,4-doublets and the union of all 3,4-doublets associated with all families of solutions represents the set of all straddling tours whose 1) second, third,...,n-1th arc is in G, and, 2) second, third,...,n-1th arc (different from that of G) is in G^c.

Characterization 3.5 ***A second, third,...,n-1th arc in G and an n^{th} arc in G^c.***

Observe arcs of G^c that enter vertex n, n^{th} arcs of potential straddling tours. Referring to G^c?s adjacency matrix, create permissible signal matrices as follows. If $g^c_{un} = 1, u = 1, 2, ..., n$-1, assign $s_{un-1} = 1$ in order to force the n-1th arc to enter vertex u. It is sufficient to assign only $s_{un-1} = 1$ since this defines the n^{th} arc of a tour as originating from vertex u. For each arc pairing $(u, n), u = 1, 2, ..., n$-1 in G^c and (c,d) in G, $c, d = 1, 2, ..., n$-1, $c{\neq}d$, create each permissible signal matrix allowing arc (c, d) to play the role of a second, third,...,n-1th arc of a straddling tour. Note that in the case of a common vertex e.g. $d = u$, arc (c, d) can only play the role of an n-1th arc of a straddling tour and signal matrices have two unit entries. In the case of no common vertex, arc (c, d) can only play the role of a second, third,...,n-2th arc of a straddling tour and signal matrices have three unit entries. The complete family of straddling permutations for each arc pairing is therefore the set of extrema of each solution set, associated with each set of 2,3-doublets. The union of all 2,3-doublets associated with all families of solutions represents the set of all straddling tours whose n^{th} arc is in G^c and whose second, third,...,n-1th arc is in G.

Characterization 3.6 ***An n^{th} arc in G and a first arc in G^c.***

Repeat Characterization 3.2. Swap roles of G & G^c.

Characterization 3.7 ***An n^{th} arc in G and a second, third, ..., n-1th arc in G^c.***

Repeat Characterization 3.5. Swap roles of G and G^c.

3.3 Main Theorem

P_S is the set of straddling permutations of G.

Theorem 3.1 *Conv(P_S) has a compact formulation.*

Proof. By construction. Generate all 2,3,4-doublets using the polynomial time Characterizations 3.1 through 3.7 above. For each 2,3,4-doublet, create the corresponding set of assignment constraints in an extended formulation. This embeds their solution sets into mutually disjoint orthogonal spaces. Create a convex combination of these solution sets noting that its projection into the subspace spanned by the p_{ij} variables is $conv(P_S)$.

Lemma 3.1 *Given G, at least one straddling tour exists.*

Proof. Let any instance of G be given. Recall that G is neither complete nor empty. Thus there exists directed arc (x, y) in G. If the out-degree of vertex y is n-1, re-label vertex y as vertex x, and choose any other vertex y with out-degree less than or equal to n-2. Such a vertex y must exist, otherwise all vertices in G have out-degree n-1 and G is complete, a contradiction. CASE I. There exists vertex z such that arc $(y, z \neq x)$ is in G^c. In this way, path $x - y - z$ uses an arc in G and an arc in G^c, and, can be completed as a straddling tour by adding arcs either from G or G^c, i.e. simply choose any permutation of the remaining vertices to complete the tour. CASE II. There does not exist vertex z such that arc $(y, z \neq x)$ is in G^c. See Figure 3. Thus every arc leaving vertex y enters a vertex different from x, exhausting all n-2 $z \neq x$ vertices in G, *and*, arc (y, x) is in G^c. Consider any two such distinct vertices $z_1 \neq x$, $z_2 \neq x, z_1 \neq z_2$. If arc (z_1, y) is in G, then path $z_1 - y - x$ uses an arc in G and an arc in G^c. Otherwise arc (z_1, y) is in G^c, and path $z_1 - y - z_2$ uses an arc in G^c and an arc in G. Use this arc pairing to complete a straddling tour, i.e. choose any permutation of the remaining vertices to complete the tour.

Corollary 3.1 $Conv(P_S) \neq \emptyset$.

Corollary 3.2 $Conv(P_S) = conv(P) \Leftrightarrow G$ *and* G^c *are non-Hamiltonian.*

Section 4 presents implementation details of constructing $conv(P_S)$, using a sample six vertex graph.

4 Mini-Tutorial

Consider G_6 and G_6^c shown graphically in Figure 4, and in adjacency matrix form in Figure 5. A sample set of arc pairings is first introduced leading to the creation of a set of 3-doublets for which an extended formulation is then presented. The complete set of all 2,3,4-doublets are then created using a particular arc in G_6, including reference to each of the Characterizations 3.1 through 3.7, sufficient to complete the example.

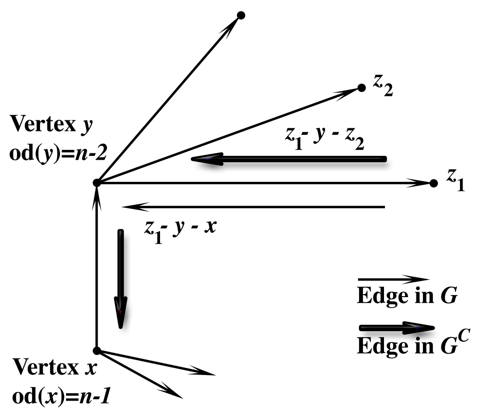

Figure 3: CASE II of Lemma 3.1

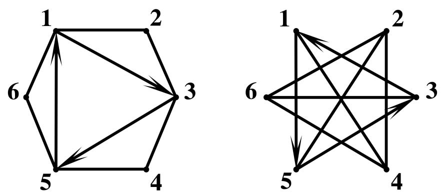

Figure 4: Graphical Representation of G_6 and G_6^c

$$\underset{G_6}{\begin{bmatrix} - & 1 & 1 & 0 & 0 & 1 \\ 1 & - & 1 & 0 & 0 & 0 \\ 0 & 1 & - & 1 & 1 & 0 \\ 0 & 0 & 1 & - & 1 & 0 \\ 1 & 0 & 0 & 1 & - & 1 \\ 1 & 0 & 0 & 0 & 1 & - \end{bmatrix}} \quad \underset{G_6^c}{\begin{bmatrix} - & 0 & 0 & 1 & 1 & 0 \\ 0 & - & 0 & 1 & 1 & 1 \\ 1 & 0 & - & 0 & 0 & 1 \\ 1 & 1 & 0 & - & 0 & 1 \\ 0 & 1 & 1 & 0 & - & 0 \\ 0 & 1 & 1 & 1 & 0 & - \end{bmatrix}}$$

Figure 5: Adjacency Matrix Representation of G_6 and G_6^c

Table 1: Family of 3-doublets

Arc Pairing		
$(1,2)$	$(2,4)$	Permissible 3-doublets
2^{nd}	3^{rd}	$p_{11} = p_{22} = 1$ & $p_{22} = p_{43} = 1 \Rightarrow \{p_{11}, p_{22}, p_{43}\}$
3^{rd}	4^{th}	$p_{12} = p_{23} = 1$ & $p_{23} = p_{44} = 1 \Rightarrow \{p_{12}, p_{23}, p_{44}\}$
4^{th}	5^{th}	$p_{13} = p_{24} = 1$ & $p_{24} = p_{45} = 1 \Rightarrow \{p_{13}, p_{24}, p_{45}\}$

4.1 How to Model Straddling Permutations for an Arc Pairing

Consider arc $(1,2)$ in G_6, and an initial choice of another arc from G_6^c. Noting that arcs from the same rows are not permissible (i.e. an arc pairing that exits the same vertex is not permissible since this leads to sequencing of the same arc as entering two distinct vertices), begin the coding of straddling permutations corresponding to straddling tours that use arc pairing $(1,2)$ in G_6 and $(2,4)$ in G_6^c. These permutations are Characterization 3.4, in the case of a common vertex. Since vertex two is common, these arcs must be sequenced consecutively. In the first case, where arc $(1,2)$ is a second arc of a tour, this implies that the first arc enters vertex one and the second arc enters vertex two e.g. $p_{11} = p_{22} = 1$, and, where arc $(2,4)$ is simultaneously a third arc of a tour, this implies (again) that the second arc enters vertex two and the third arc enters vertex four e.g. $p_{22} = p_{43} = 1$. Accounting for these arcs to be consecutively sequenced as third and fourth arcs, then as fourth and fifth arcs, a family of 3-doublets is generated, shown in Table 1.

Corresponding signal matrices and assignment constraints (re: construction of an extended formulation) are scaled by non-negative variable α shown in Figure 6, in order to prepare for the construction of a convex combination of their solution sets.

For example, for $\alpha_{\{p_{11},p_{22},p_{43}\}} = 1$ or $\alpha_{\{p_{12},p_{23},p_{44}\}} = 1$ or $\alpha_{\{p_{13},p_{24},p_{45}\}} = 1$, the complete family of straddling tours that use arc $(1,2)$ in G_6 and arc $(2,4)$ in G_6^c, sequenced in every permissible way as second, third,...,fifth arcs in a straddling tour is a member in the union of the sets of extrema of these three solution sets. Alternatively, these straddling permutations comprise the union of the set of extrema of the convex hull of these three solutions sets, modelled in Figure 7 below, whose image is a p-set polytope.

The matrix S^1 referenced in the upper left block of the coefficient matrix in Figure 7 is structured as a general coefficient matrix re: the assignment constraints, but with respect to the first signal matrix associated with $\alpha_{\{p_{11},p_{22},p_{43}\}}$. Variables s_{ij} are now renamed to s^1_{ij} variables, written as a 25 long vector, referencing 25 columns partitioned into groups of five, as row one, row two,...,row five, read from left to right, i.e. the first five rows are row sums of the signal matrix, the next five rows are column sums, and the last three rows would implement $s^1_{11} = s^1_{22} = s^1_{43} = \alpha_{\{p_{11},p_{22},p_{43}\}}$. The column vector of -1s beside the coefficient matrix is the right hand side, coded as a variable. The identity matrix below the coefficient matrix maps the s^1_{ij} variables onto the p_{ij} variables, and the last row constrains contributions from each solution set of assignment constraints (to p_{ij}

variables) to be a convex combination of solutions from all systems. Similar use of assignment constraints derived from signal matrices is shown in [10].
Now refer again to Figure 6 and observe that there are six straddling permutation solutions in all, shown Figure 8, i.e. two solutions for each signal matrix respectively.
These six solutions are expected. That is, there are precisely six ways to complete a tour that uses path $1-2-4$. Observe that the first two straddling permutations correspond to straddling tours $6-1-2-4-3-5, 6-1-2-4-5-3$ and that they satisfy the assignment constraints associated with the first signal matrix created from $\{p_{11}, p_{22}, p_{43}\}$. Likewise $6-3-1-2-4-5, 6-5-1-2-4-3$ and $6-3-5-1-2-4, 6-5-3-1-2-4$ are the only straddling tours corresponding to straddling permutations that satisfy the assignment constraints associated with the second and third signal matrices created from $\{p_{12}, p_{23}, p_{44}\}$ and $\{p_{13}, p_{24}, p_{45}\}$ respectively. The image of the solution set of the extended formulation in Figure 7 is the convex hull of all three solution sets. The complete set of extrema is the set of six permutations, i.e. straddling permutations corresponding to straddling tours that use arc $(1, 2)$ in G_6 and arc $(2, 4)$ in G_6^c.

4.2 How to Model All Straddling Permutations for All Arc Pairings

Until otherwise indicated, the following sets of arc pairings illustrate Characterization 3.4. The next arc pairing is arc $(1, 2)$ in G_6 and arc $(2, 5)$ in G_6^c. Observe that vertex two is common. Therefore these arcs must be sequenced consecutively. A complete family of permissible 3-doublets is $\{p_{53}, p_{22}, p_{11}\}$, $\{p_{54}, p_{23}, p_{12}\}$, and $\{p_{55}, p_{24}, p_{13}\}$. The next arc pairing is arc $(1, 2)$ in G_6 and arc $(3, 1)$ in G_6^c. Observe again that vertex one is common, and these arcs must be sequenced consecutively. A complete family of permissible 3-doublets is $\{p_{23}, p_{12}, p_{31}\}$, $\{p_{24}, p_{13}, p_{32}\}$, and $\{p_{25}, p_{14}, p_{33}\}$. The next arc pairing is arc $(1, 2)$ in G_6 and arc $(4, 1)$ in G_6^c, and vertex one is common. The family of permissible 3-doublets are $\{p_{23}, p_{12}, p_{41}\}$, $\{p_{24}, p_{13}, p_{42}\}$, and $\{p_{25}, p_{14}, p_{43}\}$. Arcs $(4, 2)$ and $(5, 2)$ both in G_6^c cannot be paired with arc $(1, 2)$ in G_6 since this leads to sequencing two distinct arcs as entering the same vertex. Arc $(5, 3)$ in G_6^c is the last arc that can be paired with arc $(1, 2)$ in G_6, in the case of no common vertex. Table 2 below illustrates the coding of all permissible 4-doublets associated with these straddling tours. Note that these arcs do not share any common vertex and therefore can be sequenced either as $(1, 2)$ and $(5, 3)$, or $(5, 3)$ and $(1, 2)$, but never sequenced consecutively.
Observe that the complete family of straddling tours is interpreted directly from each of the 4-doublets in Table 2. That is, each 4-doublet uniquely defines each 5-permutation due to linear dependency. Continuing, repeat these codings for all remaining sets of arc pairings and hence generate all 3,4-doublets. For completeness, there are 53 remaining arc pairings. The list below is written the way it might be programmed, to sequence each arc from G_6 with every possible arc from G_6^c.

G_6: (1,3). G_6^c: (2,4),(2,5),(4,1),(4,2),(5,2)
G_6: (2,1). G_6^c: (1,4),(1,5),(4,2),(5,2),(5,3)

G_6: (2,3). G_6^c: (1,4),(1,5),(3,1),(4,1),(4,2),(5,2)
G_6: (3,2). G_6^c: (1,4),(1,5),(2,4),(2,5),(4,1),(5,3)
G_6: (3,4). G_6^c: (1,5),(2,5),(4,1),(4,2),(5,2),(5,3)
G_6: (3,5). G_6^c: (1,4),(2,4),(4,1),(4,2),(5,2)
G_6: (4,3). G_6^c: (1,4),(1,5),(2,4),(2,5),(3,1),(5,2)
G_6: (4,5). G_6^c: (1,4),(2,4),(3,1),(5,2),(5,3)
G_6: (5,1). G_6^c: (1,4),(2,4),(2,5),(4,2)
G_6: (5,4). G_6^c: (1,5),(2,5),(3,1),(4,1),(4,2)

Six remaining Characterizations 3.1, 3.2, 3.3, 3.5, 3.6 and 3.7 are now illustrated. Regarding

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{34} & s_{35} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & s_{54} & s_{55} \end{bmatrix} \quad \begin{aligned} s_{34} + s_{35} &= \alpha_{\{p_{11},p_{22},p_{43}\}} \\ s_{54} + s_{55} &= \alpha_{\{p_{11},p_{22},p_{43}\}} \\ s_{34} + s_{54} &= \alpha_{\{p_{11},p_{22},p_{43}\}} \\ s_{35} + s_{55} &= \alpha_{\{p_{11},p_{22},p_{43}\}} \end{aligned}$$

$$\text{Signal Matrix for } s_{11} = s_{22} = s_{43} = \alpha_{\{p_{11},p_{22},p_{43}\}}$$
$$s_{34}, s_{35}, s_{54}, s_{55} \geqslant 0,\ 0 \leqslant \alpha_{\{p_{11},p_{22},p_{43}\}} \leqslant 1$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ s_{31} & 0 & 0 & 0 & s_{35} \\ 0 & 0 & 0 & 1 & 0 \\ s_{51} & 0 & 0 & 0 & s_{55} \end{bmatrix} \quad \begin{aligned} s_{31} + s_{35} &= \alpha_{\{p_{12},p_{23},p_{44}\}} \\ s_{51} + s_{55} &= \alpha_{\{p_{12},p_{23},p_{44}\}} \\ s_{31} + s_{51} &= \alpha_{\{p_{12},p_{23},p_{44}\}} \\ s_{35} + s_{55} &= \alpha_{\{p_{12},p_{23},p_{44}\}} \end{aligned}$$

$$\text{Signal Matrix for } s_{12} = s_{23} = s_{44} = \alpha_{\{p_{12},p_{23},p_{44}\}}$$
$$s_{31}, s_{35}, s_{51}, s_{55} \geqslant 0,\ 0 \leqslant \alpha_{\{p_{12},p_{23},p_{44}\}} \leqslant 1$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ s_{31} & s_{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ s_{51} & s_{52} & 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} s_{31} + s_{32} &= \alpha_{\{p_{13},p_{24},p_{45}\}} \\ s_{51} + s_{52} &= \alpha_{\{p_{13},p_{24},p_{45}\}} \\ s_{31} + s_{51} &= \alpha_{\{p_{13},p_{24},p_{45}\}} \\ s_{35} + s_{52} &= \alpha_{\{p_{13},p_{24},p_{45}\}} \end{aligned}$$

$$\text{Signal Matrix for } s_{13} = s_{24} = s_{45} = \alpha_{\{p_{13},p_{24},p_{45}\}}$$
$$s_{31}, s_{32}, s_{51}, s_{52} \geqslant 0,\ 0 \leqslant \alpha_{\{p_{13},p_{24},p_{45}\}} \leqslant 1$$

Figure 6: Sample Signal Matrices

$$\left[\begin{array}{c|c|c|c|c|c|c} S^1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & S^2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & S^3 & -1 & 0 \\ \mathrm{I} & 0 & \mathrm{I} & 0 & \mathrm{I} & 0 & -\mathrm{I} \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \left[\begin{array}{c} s^1_{11\{p_{11},p_{22},p_{43}\}} \\ s^1_{12\{p_{11},p_{22},p_{43}\}} \\ \cdots \\ s^1_{55\{p_{11},p_{22},p_{43}\}} \\ \hline \alpha_{\{p_{11},p_{22},p_{43}\}} \\ \hline s^2_{11\{p_{12},p_{23},p_{44}\}} \\ s^2_{12\{p_{12},p_{23},p_{44}\}} \\ \cdots \\ s^2_{55\{p_{12},p_{23},p_{44}\}} \\ \hline \alpha_{\{p_{12},p_{23},p_{44}\}} \\ \hline s^3_{11\{p_{13},p_{24},p_{45}\}} \\ s^3_{12\{p_{13},p_{24},p_{45}\}} \\ \cdots \\ s^3_{55\{p_{13},p_{24},p_{45}\}} \\ \hline \alpha_{\{p_{13},p_{24},p_{45}\}} \\ \hline p_{11} \\ p_{12} \\ \cdots \\ p_{55} \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline 1 \end{array}\right]$$

$$s^k_{ij\{p_{ui},p_{vj},p_{wx}\}},\ \alpha_{\{p_{ui},p_{vj},p_{wx}\}} \geqslant 0$$

Figure 7: Straddling Permutations Encoded in Mutually Disjoint Orthogonal Subspaces

$$\underset{6-1-2-4-3-5}{\begin{bmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&0&1&0\\0&0&1&0&0\\0&0&0&0&1 \end{bmatrix}} \quad \underset{6-1-2-4-5-3}{\begin{bmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&0&0&1\\0&0&1&0&0\\0&0&0&1&0 \end{bmatrix}} \quad \underset{6-3-1-2-4-5}{\begin{bmatrix} 0&1&0&0&0\\0&0&1&0&0\\1&0&0&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{bmatrix}}$$

$$\underset{6-5-1-2-4-3}{\begin{bmatrix} 0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&1\\0&0&0&1&0\\1&0&0&0&0 \end{bmatrix}} \quad \underset{6-3-5-1-2-4}{\begin{bmatrix} 0&0&1&0&0\\0&0&0&1&0\\1&0&0&0&0\\0&0&0&0&1\\0&1&0&0&0 \end{bmatrix}} \quad \underset{6-5-3-1-2-4}{\begin{bmatrix} 0&0&1&0&0\\0&0&0&1&0\\0&1&0&0&0\\0&0&0&0&1\\1&0&0&0&0 \end{bmatrix}}$$

Figure 8: Six Straddling Permutations, extrema of the image polytope, a sub-polytope of P

Characterization 3.1, let arc $(6,1)$ be a first arc in G_6 and observe arc $(1,4)$ as an arc in G^c_6. These arcs share a common vertex and therefore must be sequenced consecutively leading to

Table 2: Family of 4-doublets

Arc Pairing $(1,2)$	$(5,3)$	Permissible 4-doublets & Tours
2^{nd}	4^{th}	$p_{34} = p_{53} = 1$, $p_{22} = p_{11} = 1$ $\Rightarrow \{p_{34}, p_{53}, p_{22}, p_{11}\} \Rightarrow 6-1-2-5-3-4$
2^{nd}	5^{th}	$p_{35} = p_{54} = 1$, $p_{22} = p_{11} = 1$ $\Rightarrow \{p_{35}, p_{54}, p_{22}, p_{11}\} \Rightarrow 6-1-2-4-5-3$
3^{rd}	5^{th}	$p_{35} = p_{54} = 1$, $p_{23} = p_{12} = 1$ $\Rightarrow \{p_{35}, p_{54}, p_{23}, p_{12}\} \Rightarrow 6-4-1-2-5-3$
4^{th}	2^{nd}	$p_{32} = p_{51} = 1$, $p_{24} = p_{13} = 1$ $\Rightarrow \{p_{32}, p_{51}, p_{24}, p_{13}\} \Rightarrow 6-5-3-1-2-4$
5^{th}	2^{nd}	$p_{32} = p_{51} = 1$, $p_{25} = p_{14} = 1$ $\Rightarrow \{p_{32}, p_{51}, p_{25}, p_{14}\} \Rightarrow 6-5-3-4-1-2$
5^{th}	3^{rd}	$p_{33} = p_{52} = 1$, $p_{25} = p_{14} = 1$ $\Rightarrow \{p_{33}, p_{52}, p_{25}, p_{14}\} \Rightarrow 6-4-5-3-1-2$

the creation of a single 2-doublet $\{p_{11}, p_{42}\}$. Arc $(1,4)$ can only play the role of a second arc, since if it were allowed to play the role of a third arc, for example, this implies that arc $(6,1)$ is a second arc which is not allowed, by definition. Regarding arc $(1,5)$ in G_6^c, the 2-doublet $\{p_{11}, p_{52}\} results$. Consider the arc pairing $(6,1)$ in G_6 and $(2,4)$ in G_6^c. The complete list of 3-doublets is $\{p_{11}, p_{43}, p_{22}\}$, $\{p_{11}, p_{44}, p_{23}\}$, and $\{p_{11}, p_{45}, p_{24}\}$. Repeat these codings for all remaining arc pairings (below) and generate all 2,3-doublets.

G_6: (6,1). G_6^c: (1,4),(1,5),(2,4),(2,5),(4,2),(5,2),(5,3)
G_6: (6,5). G_6^c: (1,4),(2,4),(3,1),(4,1),(4,2),(5,2),(5,3)

Regarding Characterization 3.3, let arc $(6,2)$ be a first arc in G_6^c. Observe arc $(1,3)$ in G_6. The complete list of 3-doublets is $\{p_{21}, p_{33}, p_{12}\}$, $\{p_{21}, p_{34}, p_{13}\}$, and $\{p_{21}, p_{35}, p_{14}\}$. Repeat these codings for all remaining arc pairings (below) and generate all 2,3-doublets.

G_6^c: (6,2).
G_6: (1,3),(2,1),(2,3),(3,4),(3,5),(4,3),(4,5),(5,1),(5,4)
G_6^c: (6,3).
G_6: (1,2),(2,1),(3,2),(3,4), (3,5),(4,5),(5,1),(5,4)
G_6^c: (6,4).
G_6: (1,2),(1,3),(2,1),(2,3),(3,2),(3,5),(4,3),(4,5),(5,1)

Regarding Characterization 3.7, let arc $(1,6)$ be a sixth arc in G_6. Observe arc $(2,4)$ in G_6^c. The complete list of 3-doublets is $\{p_{15}, p_{42}, p_{21}\}$, $\{p_{15}, p_{43}, p_{22}\}$, and $\{p_{15}, p_{44}, p_{23}\}$. For arc pairing

$(1,6)$ in G_6 and $(2,5)$ in G_6^c, the complete set of 3-doublets is $\{p_{15}, p_{52}, p_{21}\}$, $\{p_{15}, p_{53}, p_{22}\}$, and $\{p_{15}, p_{54}, p_{23}\}$. Now consider the arc pairing $(1,6)$ in G_6 and $(3,1)$ in G_6^c. These arcs share a common vertex and must be sequenced consecutively leading to the creation of a single 2-doublet $\{p_{15}, p_{34}\}$. Similar to the example in Characterization 3.1, while arc $(3,1)$ might appear to be allowed to play the role of a second, third and fourth arc, it can only play the role of a fifth arc. Otherwise this implies that arc $(1,6)$ is a third, fourth and fifth arc which is not allowed. Repeat these codings for all remaining arc pairings (below) and generate all 2,3-doublets.

G_6: (1,6). G_6^c: (2,4),(2,5),(3,1),(4,1),(4,2),(5,2),(5,3)
G_6: (5,6). G_6^c: (1,4),(1,5),(2,4),(2,5),(3,1),(4,1),(4,2)

Regarding Characterization 3.5, let arc $(2,6)$ be a sixth arc in G_6^c and observe arc $(1,2)$ as an arc in G_6. These arcs share a common vertex and must be sequenced consecutively leading to the creation of a single 2-doublet $\{p_{25}, p_{14}\}$. While arc $(1,2)$ might appear to be allowed to play the role of a second, third and fourth arc, it can only play the role of a fifth arc. Otherwise this implies that arc $(2,6)$ is a third, fourth and fifth arc which is not allowed. Let arc $(2,6)$ be a sixth arc in G_6^c and observe arc $(1,3)$ as an arc in G_6. The complete list of 3-doublets is $\{p_{25}, p_{32}, p_{11}\}$, $\{p_{25}, p_{33}, p_{12}\}$ and $\{p_{25}, p_{34}, p_{13}\}$. Repeat these codings for all remaining arc pairings (below) and generate all 2,3-doublets.

G_6^c: (2,6).
G_6: (1,2),(1,3),(3,2),(3,4),(3,5),(4,3),(4,5),(5,1),(5,4)
G_6^c: (3,6).
G_6: (1,2),(1,3),(2,1),(2,3),(4,3),(4,5),(5,1),(5,4)
G_6^c: (4,6).
G_6: (1,2),(1,3),(2,1),(2,3),(3,2),(3,4),(3,5),(5,1),(5,4)

Regarding Characterizations 3.2 and 3.6, treated together, there are 12 arc pairings (below).

G_6: (6,1). G_6^c: (2,6),(3,6),(4,6)
G_6: (6,5). G_6^c: (2,6),(3,6),(4,6)
G_6^c: (6,2). G_6: (1,6),(5,6)
G_6^c: (6,3). G_6: (1,6),(5,6)
G_6^c: (6,4). G_6: (1,6),(5,6)

Paired arcs share a common vertex. The complete list of 2-doublets is: $\{p_{11}, p_{25}\}$, $\{p_{11}, p_{35}\}$, $\{p_{11}, p_{45}\}$, $\{p_{51}, p_{25}\}$, $\{p_{51}, p_{35}\}$, $\{p_{51}, p_{45}\}$, $\{p_{21}, p_{15}\}$, $\{p_{21}, p_{55}\}$, $\{p_{31}, p_{15}\}$, $\{p_{31}, p_{55}\}$, $\{p_{41}, p_{15}\}$, and $\{p_{41}, p_{55}\}$.
Imagine now that Equation 7 is modified to incorporate all solution sets associated with all 2,3,4-doublets illustrated above. That is, create all permissible 2,3,4-doublets, associated signal matrices and assignment constraints, incorporated into an extended formulation. Every

straddling permutation is therefore mapped into the space spanned by the set of p_{ui} variables and therefore defines the convex hull of all straddling permutations. Summarizing, there exist two tours in G_6 $6-1-2-3-4-5, 6-5-4-3-2-1$, three tours in G_6^c $6-2-5-3-1-4, 6-3-1-5-2-4, 6-2-4-1-5-3$ and 115 straddling tours that would be modelled by an extended formulation of this example.

5 Discussion and Applications

The model and techniques presented in this paper are specific to sets of permutations in correspondence with sets of tours not in a graph, and extreme point solutions of subsets of assignment constraints. In general, models can be based upon tours 'in' or 'not in' a graph (if they *can be* modelled) and may or may not invoke/combine assignment constraints. There may be utility simply knowing that a model *can be* created.

These techniques might be applicable to the graph isomorphism problem. Recall that graph G is isomorphic to graph H if and only if there exists permutation matrix P such that $P^TGP = H$. The graph isomorphism complete class is neither in **P**, nor **NP**-complete and it would be interesting to investigate the possibility of modelling the set of permutations satisfying either $P^TGP = H$ or $P^TGP \neq H$. If it's possible to build a compact formulation for either model, then graph isomorphism is in **P**. A polyhedral approach toward developing a linear form of P^TGP might be developed from [26]. As an aside, the automorphism group of a graph is the set of all permutations such that $P^TGP = G$. Observe that the convex hull of these permutations is both a p-set polytope and permutation polytope[3], perhaps relevant in some way. It's harder to comment on the **coNP** approach i.e. modelling the set of all permutations such that $P^TGP \neq G$ (or H above). There appear to be few relevant studies on the **coNP**-graph non-isomorphism complete class.

While the model presented here is focused on tours as permutations, observe that paths can be coded as subsequences of permutations. If these techniques are modified to be path specific, then it's not necessary to require that *all* variables be integral. For example, if all that's required are integral paths that connect two vertices in a specified way, then observe how 2-doublets of the form $\{p_{ai}, p_{bj}\}$ code for paths from node a to node b of length $|j-i|$. If an application requires that these paths be excluded, then append the constraint $p_{ai} + p_{bj} \leq 1$. No matter that fractional extrema may be introduced, it's perhaps useful in an application. Alternatively it's possible to create the union of three compact formulations, each of the form $p_{ai} = p_{bj} = 0$, and $p_{ai} = 0$ & $p_{bj} = 1$, and $p_{ai} = 1$ & $p_{bj} = 0$ accomplishing the same task as $p_{ai} + p_{bj} \leq 1$, but maintaining integrality. It's also possible to force extreme solutions to maintain paths of length $|j-i|$ by assigning $p_{ai} = p_{bj} = 1$.

These modelling techniques therefore allow for a variety of linear constraints and might be adaptable to many practical applications, for example, linear programming models of mesh networks. See [27, 28, 29]. Observe that a dedicated link is an arc (u,v) in a path in G, and can be coded by assigning $p_{ui-1} = p_{vi} = 1$, $i = 2, ..., n$. In the case of two dedicated links, this is the example presented in this paper where two arcs are specified in a path i.e. an arc in both G and G^c, coded via 2,3,4-doublets. All of these ideas and many more can be generalized to code for a variety of models of networks.

References

[1] E. Swart, Personal communication (ongoing).

[2] G. Woeginger, Website of: Gerhard J Woeginger, The P-versus-NP page. Retrieved November 7, 2012, http://www.win.tue.nl/ gwoegi/P-versus-NP.htm (2012).

[3] M. Yannakakis, Expressing combinatorial optimization problems by linear programs, Proceedings of the Twentieth Annual ACM Symposium on Theory of Computing (1988) 223–228.

[4] V. Kaibel, K. Pashkovich, D. Theis, Website of: Cornell University Library. Symmetry matters for sizes of extended formulations. Retrieved January 20, 2013, http://arxiv.org/abs/0911.3712v3 to appear in: SIAM Journal on Discrete Mathematics (2012).

[5] E. Balas, Projection with a minimal system of inequalities, Computational Optimization and Applications 2241 (1998) 189–193.

[6] E. Balas, Projection and lifting in combinatorial optimization, Computational Combinatorial Optimization. Schlo? Dagstuhl 2000. Lecture Notes in Computer Science. Springer Berlin 2241 (2001) 26–56.

[7] E. Balas, Projection lifting and extended formulation in integer and combinatorial optimization, Annals of Operations Research 140 (2005) 125–161.

[8] S. Gismondi, Dueling cubes, Utilitas Mathematica 54 (1998) 241–251.

[9] S. Gismondi, An $O(n^3)$ sized external representation of a factorial faceted factorial extreme point polytope, Utilitas Mathematica 63 (2003) 109–114.

[10] S. Gismondi, E. Swart, A model of the coNP-complete non-Hamilton tour decision problem, Mathematical Programming Series A 100 (2004) 471–483.

[11] M. Conforti, L. Wolsey, Compact formulations as a union of polyhedra, Mathematical Programming Series A 114 (2008) 277–289.

[12] S. Fiorini, S. Masser, S. Pokutta, H. R. Tiwary, R. de Wolf, Linear vs. semidefinite extended formulations: exponential separation and strong lower bounds, Proceedings of the Forty Fourth Symposium on Theory of Computing (2012) 95–106.

[13] L. Billera, A. Sarangarajan, All 0/1 polytopes are traveling salesman polytopes, Combinatorica 16 (1996a) 175–188.

[14] L. Billera, A. Sarangarajan, The combinatorics of permutation polytopes, Discrete Mathematics & Theoretical Computer Science. Formal power series and algebraic combinatorics. New Brunswick NJ. 1994. DIAMAC. American Mathematical Society. Providence RI 24 (1996b) 1–23.

[15] V. M. Demidenko, A criterion for the adjacency of vertices of polytopes generated by subsets of the symmetric group (Russian). Translation, Mathematical Notes 80 (2006) 791–805.

[16] M. Guralnick, D. Perkinson, Permutation polytopes and indecomposable elements in permutation groups, Journal of Combinatorial Theory Series A 113 (2006) 1243–1256.

[17] S. Onn, Geometry complexity and combinatorics of permutation polytopes, Journal of Combinatorial Theory Series A 64 (1993) 31–49.

[18] V. Sarvanov, On optimization of permutations, BSSR Ser. F$\ddot{i}$z.-Mat. Navuk 139 (1979) 9–11.

[19] D. Applegate, R. Bixby, V. Chvàtal, W. Cook, TSP cuts which do not conform to the template paradigm., Computational Combinatorial Optimization. Schlofl Dagstuhl 2000. Lecture Notes in Computer Science. Springer Berlin 2241 (2001) 261–303.

[20] D. Applegate, R. Bixby, V. Chvàtal, W. Cook, Implementing the Dantzig-Fulkerson-Johnson algorithm for large traveling salesman problems, Mathematical Programming Series B. ISMP 2003 Copenhagen 97 (2003) 91–153.

[21] D. Applegate, R. Bixby, V. Chvàtal, W. Cook, The Traveling Salesman Problem: A Computational Study, Princeton Series in Applied Mathematics, Princeton University Press, Princeton, NJ, 2006.

[22] S. Cook, Website of: Clay Mathematics Institute. The P versus NP problem. Retrieved January 31, 2013, http://www.claymath.org/millennium/P_vs_NP/pvsnp.pdf (2006).

[23] E. Lawler, J. Lenstra, A. Rinnooy Kan, S. D.B., The Traveling Salesman Problem, John Wiley and Sons, New York, 1985.

[24] M. Sipser, The history and status of the P versus NP question, Proceedings of the Twenty Fourth Annual ACM Symposium on Theory of Computing (1992) 603–618.

[25] M. Todd, The many facets of linear programming, Mathematical Programming Series B 91 (2002) 417–436.

[26] S. Gismondi, Subgraph isomorphism and the Hamilton tour decision problem using a linearized form of PGP^t, Utilitas Mathematica 76 (2008) 229–248.

[27] J. Hu, Traffic grooming in wavelength-division-multiplexing ring networks: a linear programming solution, Journal of Optical Networking 1 (2002) 397–408.

[28] Y. Yang, J. Wang, R. Kravets, Designing routing metrics for mesh networks, in: IEEE Workshop on Wireless Mesh Networks (WiMesh).

[29] K. Srinivasan, K. Chatha, G. Konjevod, Linear-programming-based techniques for synthesis of network-on-chip architectures, Very Large Scale Integration (VLSI) Systems, IEEE Transactions on 14 (2006) 407–420.