More On $\lambda_\kappa$–closed sets in generalized topological spaces

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ABSTRACT

In this paper, we introduce $\lambda_\kappa$–closed sets and study its properties in generalized topological spaces.

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1 Introduction

The theory of generalized topology was introduced by Császár in [1]. The properties of generalized topology, basic operators, generalized neighborhood systems and some con-
structions for generalized topologies have been studied by the same author in [1, 2, 3, 4, 5, 6]. It is well known that generalized topology in the sense of Császár [1] is a generalization of topology on a nonempty set. On the other hand, many important collections of sets related with topology on a set form a generalized topology. In this paper we define several subsets in a generalized topological spaces and study their properties.

A nonempty family $\mu$ of subsets of a set $X$ is said to be a generalized topology [2] if $\emptyset \in \mu$ and arbitrary union of elements of $\mu$ is again in $\mu$. The pair $(X, \mu)$ is called a generalized topological space and elements of $\mu$ are called $\mu$-open sets. $A \subset X$ is $\mu$-closed if $X - A$ is $\mu$-open. By a space $(X, \mu)$, we always mean a generalized topological space. If $X \in \mu$, $(X, \mu)$ is called a strong [3] space. Clearly, $(X, \mu)$ is strong if and only if $\emptyset$ is $\mu$-closed if and only if $c_\mu(\emptyset) = \emptyset$. In a space $(X, \mu)$, if $\mu$ is closed under finite intersection, $(X, \mu)$ is called a quasi-topological space [5]. Clearly, every strong, quasi-topological space is a topological space. For $A \subset X$, $c_\mu(A)$ is the smallest $\mu$-closed set containing $A$ and $i_\mu(A)$ is the largest $\mu$-open set contained in $A$. Moreover, $X - c_\mu(A) = i_\mu(X - A)$, for every subset $A$ of $X$. A subset $A$ of a space $(X, \mu)$ is said to be $\alpha$-open [4] (resp. $\sigma$-open [4], $\pi$-open [4], b-open [7], $\beta$-open [4]) if $A \subset i_\mu c_\mu i_\mu(A)$ (resp. $A \subset c_\mu i_\mu(A)$, $A \subset i_\mu c_\mu(A)$, $A \subset i_\mu c_\mu(A) \cup c_\mu i_\mu(A)$, $A \subset c_\mu i_\mu c_\mu(A)$). A subset $A$ of a space $(X, \mu)$ is said to be $\alpha$-closed (resp. $\sigma$-closed, $\pi$-closed, b-closed, $\beta$-closed) if $X - A$ is $\alpha$-open (resp. $\sigma$-open, $\pi$-open, b-open, $\beta$-open). Let $(X, \mu)$ be a space and $\zeta = \{\mu, \alpha, \sigma, \pi, b, \beta\}$. For $\kappa \in \zeta$, we consider the space $(X, \kappa)$, throughout the paper. For $A \subset \mathcal{M}_\kappa = \cup\{B \subset X \mid B \in \mu\}$, the subset $\Lambda_\kappa(A)$ is defined by $\Lambda_\kappa(A) = \cap\{G \mid A \subset G, G \in \kappa\}$. The proof of the following lemma is clear.

**Lemma 1.1.** Let $A, B$ and $B_\alpha, \alpha \in \Delta$ be subsets of $\mathcal{M}_\kappa$ in a space $(X, \kappa)$. Then the following properties are hold.

(a) $B \subset \Lambda_\kappa(B)$.
(b) If $A \subset B$ then $\Lambda_\kappa(A) \subset \Lambda_\kappa(B)$.
(c) $\Lambda_\kappa(\Lambda_\kappa(B)) = \Lambda_\kappa(B)$.
(d) If $A \in \kappa$, then $A = \Lambda_\kappa(A)$.
(e) $\Lambda_\kappa(\cup\{B_\alpha \mid \alpha \in \Delta\}) = \cup\{\Lambda_\kappa(B_\alpha) \mid \alpha \in \Delta\}$.
(f) $\Lambda_\kappa(\cap\{B_\alpha \mid \alpha \in \Delta\}) \subset \cap\{\Lambda_\kappa(B_\alpha) \mid \alpha \in \Delta\}$.

**2 More on $\Lambda_\kappa$-closed sets**

In a space $(X, \kappa)$, a subset $B$ of $\mathcal{M}_\kappa$ is called a $\Lambda_\kappa$-set if $B = \Lambda_\kappa(B)$. We state the following theorem without proof.

**Theorem 2.1.** For subsets $A$ and $A_\alpha, \alpha \in \Delta$ of $\mathcal{M}_\kappa$ in a space $(X, \kappa)$, the following hold.

(a) $\Lambda_\kappa(A)$ is a $\Lambda_\kappa$-set.
(b) If $A \in \kappa$, then $A$ is a $\Lambda_\kappa$-set.
(c) If $A_\alpha$ is a $\Lambda\kappa$-set for each $\alpha \in \triangle$, then $\cap \{ A_\alpha \mid \alpha \in \triangle \}$ is a $\Lambda\kappa$-set.

(d) If $A_\alpha$ is a $\Lambda\kappa$-set for each $\alpha \in \triangle$, then $\cup \{ A_\alpha \mid \alpha \in \triangle \}$ is a $\Lambda\kappa$-set.

A subset $A$ of $M_\kappa$ in a space $(X, \kappa)$ is said to be a $\lambda\kappa$-closed set if $A = T \cap C$, where $T$ is a $\Lambda\kappa$-set and $C$ is a $\kappa$-closed set. The complement of a $\lambda\kappa$-closed set is called a $\lambda\kappa$-open set. We denote the collection of all $\lambda\kappa$-open (resp., $\lambda\kappa$-closed) set of $X$ by $\lambda\kappa O(X)$ (resp., $\lambda\kappa C(X)$). The following theorem gives the characterization of $\lambda\kappa$-closed sets.

**Lemma 2.2.** Let $A \subset M_\kappa$ be a subset in a space $(X, \kappa)$. Then the following are equivalent.

(a) $A$ is a $\lambda\kappa$-closed set.

(b) $A = T \cap c_\kappa(A)$, where $T$ is a $\Lambda\kappa$-set.

(c) $A = \Lambda\kappa(A) \cap c_\kappa(A)$.

Let $(X, \kappa)$ be a space. A point $x \in M_\kappa$ is called a $\lambda\kappa$-cluster point of $A$ if for every $\lambda\kappa$-open set $U$ of $M_\kappa$ containing $x$ we have $A \cap U \neq \emptyset$. The set of all $\lambda\kappa$-cluster points of $A$ is called the $\lambda\kappa$-closure of $A$ and is denoted by $c_\lambda(A)$.

Lemma 2.3 gives some properties of $c_\lambda$, the easy proof of which is omitted.

**Lemma 2.3.** Let $(X, \kappa)$ be a space and $A, B \subset M_\kappa$. Then the following properties hold.

(a) $A \subset c_\lambda(A)$.

(b) $c_\lambda(A) = \cap \{ F \mid A \subset F \text{ and } F \text{ is } \lambda\kappa \text{ - closed} \}$.

(c) If $A \subset B$, then $c_\lambda(A) \subset c_\lambda(B)$.

(d) $A$ is a $\lambda\kappa$-closed set if and only if $A = c_\lambda(A)$.

(e) $c_\lambda(A)$ is a $\lambda\kappa$-closed set.

Let $(X, \kappa)$ be a space and $A \subset M_\kappa$. A point $x \in M_\kappa$ is said to be a $\kappa$-limit point of $A$ if for each $\kappa$-open set $U$ containing $x$, $U \cap \{ A \setminus \{ x \} \} \neq \emptyset$. The set of all $\kappa$-limit points of $A$ is called a $\kappa$-derived set of $A$ and is denoted by $D_\kappa(A)$.

Let $(X, \kappa)$ be a space and $A \subset M_\kappa$. A point $x \in M_\kappa$ is said to be a $\lambda\kappa$-limit point of $A$ if for each $\lambda\kappa$-open set $U$ containing $x$, $U \cap \{ A \setminus \{ x \} \} \neq \emptyset$. The set of all $\lambda\kappa$-limit points of $A$ is called a $\lambda\kappa$-derived set of $A$ and is denoted by $D_\lambda(A)$.

Theorem 2.4 gives some properties of $\lambda\kappa$-derived sets and Theorem 2.5 gives the characterization of $\lambda\kappa$-derived sets.

**Theorem 2.4.** Let $(X, \kappa)$ be a space and $A, B \subset M_\kappa$. Then the following hold.

(a) $D_\lambda(A) \subset D_\kappa(A)$.

(b) If $A \subset B$, then $D_\lambda(A) \subset D_\lambda(B)$.

(c) $D_\lambda(A) \cup D_\lambda(B) \subset D_\lambda(A \cup B)$ and $D_\lambda(A \cap B) \subset D_\lambda(A) \cap D_\lambda(B)$.

(d) $D_\lambda(D_\lambda(A) - A \subset D_\lambda(A)$.

(e) $D_\lambda(A \cup D_\lambda(A)) \subset A \cup D_\lambda(A)$.

**Proof.** (a) Since every $\kappa$-open set is a $\lambda\kappa$-open set, it follows.

(b) Let $x \in D_\lambda(A)$. Let $U$ be any $\lambda\kappa$-open set containing $x$. Then $U \cap \{ A \setminus \{ x \} \} \neq \emptyset$ and so $V \cap \{ B \setminus \{ x \} \} \neq \emptyset$, since $A \subset B$. Therefore, $x \in D_\lambda(B)$.

(c) Since $A \cap B \subset A, B$ we have $D_\lambda(A \cap B) \subset D_\lambda(A) \cap D_\lambda(B)$. Since $A, B \subset A \cup B$,
we have $D_{\lambda_\kappa}(A) \cup D_{\lambda_\kappa}(B) \subset D_{\lambda_\kappa}(A \cup B)$.

(d) Let $x \in D_{\lambda_\kappa}(A) - A$ and $U$ be a $\lambda_\kappa$-open set containing $x$. Then $U \cap (D_{\lambda_\kappa}(A) - \{x\}) \neq \emptyset$. Let $y \in U \cap (D_{\lambda_\kappa}(A) - \{x\})$. Since $y \in D_{\lambda_\kappa}(A)$ and $x \neq y \in U$, $U \cap (A - \{y\}) \neq \emptyset$. Let $z \in U \cap (A - \{y\})$. Then $z \in U \cap (A - \{y\})$ implies that $z \in U$ and $z \in A - \{y\}$ and so $z \neq y$. Since $x \notin A$, $z \in U \cap (A - \{x\})$ and so $U \cap (A - \{x\}) \neq \emptyset$. Therefore, $x \in D_{\lambda_\kappa}(A)$.

(e) Let $x \in D_{\lambda_\kappa}(A \cup D_{\lambda_\kappa}(A))$. If $x \in A$, the result is clear. Suppose $x \notin A$. Since $x \in D_{\lambda_\kappa}(A \cup D_{\lambda_\kappa}(A)) - A$, then for $\lambda_\kappa$-open set $U$ containing $x$, $U \cap ((A \cup D_{\lambda_\kappa}(A)) - \{x\}) \neq \emptyset$. Thus $U \cap (A - \{x\}) \neq \emptyset$ or $U \cap (D_{\lambda_\kappa}(A) - \{x\}) \neq \emptyset$. Now it follows from (d) that $U \cap (A - \{x\}) \neq \emptyset$. Hence, $x \in D_{\lambda_\kappa}(A)$. Therefore, in all the cases $D_{\lambda_\kappa}(A \cup D_{\lambda_\kappa}(A)) \subset A \cup D_{\lambda_\kappa}(A)$.

**Theorem 2.6.** Let $(X, \kappa)$ be space and $A \subset X$. Then $c_{\lambda_\kappa}(A) = A \cup D_{\lambda_\kappa}(A)$.

**Proof.** Since $D_{\lambda_\kappa}(A) \subset c_{\lambda_\kappa}(A)$, $A \cup D_{\lambda_\kappa}(A) \subset c_{\lambda_\kappa}(A)$. On the other hand, let $x \in c_{\lambda_\kappa}(A)$. If $x \in A$, the proof is complete. If $x \notin A$, then each $\lambda_\kappa$-open set $U$ containing $x$ intersects $A$ at a point distinct from $x$. Therefore, $x \in D_{\lambda_\kappa}(A)$. Thus, $c_{\lambda_\kappa}(A) \subset A \cup D_{\lambda_\kappa}(A)$ and so $c_{\lambda_\kappa}(A) = A \cup D_{\lambda_\kappa}(A)$ which completes the proof.

Let $(X, \kappa)$ be a space and $A \subset X$. Then $i_{\lambda_\kappa}(A)$ is the union of all $\lambda_\kappa$-open set contained in $A$.

Theorem 2.6 gives some properties of $i_{\lambda_\kappa}$.

**Theorem 2.6.** Let $(X, \kappa)$ be a space and $A, B \subset X$. Then the following hold.

(a) $A$ is a $\lambda_\kappa$-open set if and only if $A = i_{\lambda_\kappa}(A)$.

(b) $i_{\lambda_\kappa}(i_{\lambda_\kappa}(A)) = i_{\lambda_\kappa}(A)$.

(c) $i_{\lambda_\kappa}(A) = A - D_{\lambda_\kappa}(X - A)$.

(d) $X - i_{\lambda_\kappa}(A) = c_{\lambda_\kappa}(X - A)$.

(e) $X - c_{\lambda_\kappa}(A) = i_{\lambda_\kappa}(X - A)$.

(f) $A \subset B$ then $i_{\lambda_\kappa}(A) \subset i_{\lambda_\kappa}(B)$.

(g) $i_{\lambda_\kappa}(A) \cup i_{\lambda_\kappa}(B) \subset i_{\lambda_\kappa}(A \cup B)$ and $i_{\lambda_\kappa}(A) \cap i_{\lambda_\kappa}(B) \supset i_{\lambda_\kappa}(A \cap B)$.

**Proof.** (c) If $x \in A - D_{\lambda_\kappa}(X - A)$, then $x \notin D_{\lambda_\kappa}(X - A)$ and so, there exists a $\lambda_\kappa$-open set $U$ containing $x$ such that $U \cap (X - A) = \emptyset$. Then $x \in U \subset A$ and hence $x \in i_{\lambda_\kappa}(A)$. That is, $A - D_{\lambda_\kappa}(X - A) \subset i_{\lambda_\kappa}(A)$. On the other hand, if $x \in i_{\lambda_\kappa}(A)$, then $x \notin D_{\lambda_\kappa}(X - A)$, since $i_{\lambda_\kappa}(A)$ is a $\lambda_\kappa$-open set and $i_{\lambda_\kappa}(A) \cap (X - A) = \emptyset$. Hence, $i_{\lambda_\kappa}(A) = A - D_{\lambda_\kappa}(X - A)$.

(d) $X - i_{\lambda_\kappa}(A) = X - (A - D_{\lambda_\kappa}(X - A)) = (X - A) \cup D_{\lambda_\kappa}(X - A) = c_{\lambda_\kappa}(X - A)$.

Let $(X, \kappa)$ be a space and $A \subset X$. Then $b_{\kappa}(A) = A - i_{\kappa}(A)$ is said to be $\kappa$-border of $A$.

Let $(X, \kappa)$ be a space and $A \subset X$. Then $b_{\lambda_\kappa}(A) = A - i_{\lambda_\kappa}(A)$ is said to be $\lambda_\kappa$-border of $A$.

Theorem 2.7 gives some properties of $b_{\lambda_\kappa}$.

**Theorem 2.7.** Let $(X, \kappa)$ be a space and $A \subset X$. Then the following hold.

(a) $b_{\lambda_\kappa}(A) \subset b_{\kappa}(A)$.

(b) $A = i_{\lambda_\kappa}(A) \cup b_{\lambda_\kappa}(A)$.

(c) $i_{\lambda_\kappa}(A) \cap b_{\lambda_\kappa}(A) = \emptyset$.
Hence (i) \( F \) is a \( \lambda \)-open set if and only if \( b_{\lambda}(A) = \emptyset \).

Proof. (f) If \( x \in i_{\lambda}(b_{\lambda}(A)) \), then \( x \in b_{\lambda}(A) \). On the other hand, since \( b_{\lambda}(A) \subset A \), \( x \in i_{\lambda}(b_{\lambda}(A)) \subset i_{\lambda}(A) \). Hence \( x \in i_{\lambda}(A) \cap b_{\lambda}(A) \) which contradicts \( c \). Thus, \( i_{\lambda}(b_{\lambda}(A)) = \emptyset \).

(h) \( b_{\lambda}(A) = A - i_{\lambda}(A) = A - (X - c_{\lambda}(X - A)) = A \cap c_{\lambda}(X - A) \).

(i) \( b_{\lambda}(A) = D_{\lambda}(X - A) \).

Let \( (X, \kappa) \) be a space and \( A \subset X \). Then \( F_{\kappa}(A) = c_{\kappa}(A) - i_{\kappa}(A) \) is said to be the \( \kappa \)-frontier of \( A \).

Let \( (X, \kappa) \) be a space and \( A \subset X \). Then \( F_{\lambda}(A) = c_{\lambda}(A) - i_{\lambda}(A) \) is said to be the \( \lambda \)-frontier of \( A \).

Theorem 2.8 gives some properties of \( F_{\lambda} \).

Theorem 2.18 Let \( (X, \kappa) \) be a space and \( A \subset X \). Then the following hold.

(a) \( F_{\lambda}(A) \subset F_{\kappa}(A) \).

(b) \( c_{\lambda}(A) = i_{\lambda}(A) \cup F_{\lambda}(A) \).

(c) \( i_{\lambda}(A) \cap F_{\lambda}(A) = \emptyset \).

(d) \( b_{\lambda}(A) \subset F_{\lambda}(A) \).

(e) \( F_{\lambda}(A) = b_{\lambda}(A) \cup D_{\lambda}(A) \).

(f) \( A \) is a \( \lambda \)-open set if and only if \( F_{\lambda}(A) = D_{\lambda}(A) \).

(g) \( F_{\lambda}(A) = c_{\lambda}(A) \cap c_{\lambda}(X - A) \).

(h) \( F_{\lambda}(A) = F_{\lambda}(X - A) \).

(i) \( F_{\lambda}(A) \) is a \( \lambda \)-closed set.

(j) \( F_{\lambda}(i_{\lambda}(A)) \subset F_{\lambda}(A) \).

(k) \( F_{\lambda}(c_{\lambda}(A)) \subset F_{\lambda}(A) \).

(l) \( i_{\lambda}(A) \cap F_{\lambda}(A) = A - F_{\lambda}(A) \).

Proof. (b) \( i_{\lambda}(A) \cup F_{\lambda}(A) = i_{\lambda}(A) \cup (c_{\lambda}(A) - i_{\lambda}(A)) = c_{\lambda}(A) \).

(c) \( i_{\lambda}(A) \cap F_{\lambda}(A) = i_{\lambda}(A) \cap (c_{\lambda}(A) - i_{\lambda}(A)) = \emptyset \).

(e) Since \( i_{\lambda}(A) \cup F_{\lambda}(A) = i_{\lambda}(A) \cup b_{\lambda}(A) \cup D_{\lambda}(A) \), \( F_{\lambda}(A) = b_{\lambda}(A) \cup D_{\lambda}(A) \).

(g) \( F_{\lambda}(A) = c_{\lambda}(A) - i_{\lambda}(A) = c_{\lambda}(A) \cap c_{\lambda}(X - A) \).

(i) \( c_{\lambda}(F_{\lambda}(A)) = c_{\lambda}(c_{\lambda}(A) \cap c_{\lambda}(X - A)) \subset c_{\lambda}(c_{\lambda}(A)) \cap c_{\lambda}(c_{\lambda}(X - A)) = F_{\lambda}(A) \).

Hence \( F_{\lambda}(A) \) is a \( \lambda \)-closed set.

(j) \( F_{\lambda}(F_{\lambda}(A)) = c_{\lambda}(F_{\lambda}(A)) \cap c_{\lambda}(X - F_{\lambda}(A)) \subset c_{\lambda}(F_{\lambda}(A)) = F_{\lambda}(A) \).

(l) \( F_{\lambda}(c_{\lambda}(A)) = c_{\lambda}((c_{\lambda}(A)) - i_{\lambda}(c_{\lambda}(A)) = c_{\lambda}(A) - i_{\lambda}(c_{\lambda}(A)) \subset c_{\lambda}(A) - i_{\lambda}(A) = F_{\lambda}(A) \).

(m) \( A - F_{\lambda}(A) = A - (c_{\lambda}(A) - i_{\lambda}(A)) = i_{\lambda}(A) \).
Let \((X, \kappa)\) be a space and \(A \subset X\). Then \(E_\kappa(A) = i_\kappa(X - A)\) is said to be \(\kappa\)–exterior of \(A\).

Let \((X, \kappa)\) be a space and \(A \subset X\). Then \(E_\lambda_\kappa(A) = i_\lambda_\kappa(X - A)\) is said to be \(\lambda_\kappa\)–exterior of \(A\).

Theorem 2.9 gives some properties of \(E_\lambda_\kappa\).

**Theorem 2.9.** Let \((X, \kappa)\) be a space and \(A \subset X\). Then the following hold.

(a) \(E_\kappa(A) \subset E_\lambda_\kappa(A)\) where \(E_\kappa(A)\) denotes the exterior of \(A\).

(b) \(E_\lambda_\kappa(A)\) is \(\lambda_\kappa\)–open.

(c) \(E_\lambda_\kappa(A) = i_\lambda_\kappa(X - A) = X - c_\lambda_\kappa(A)\).

(d) \(E_\lambda_\kappa(E_\lambda_\kappa(A)) = i_\lambda_\kappa(c_\lambda_\kappa(A))\).

(e) If \(A \subset B\), then \(E_\lambda_\kappa(A) \supset E_\lambda_\kappa(B)\).

(f) \(E_\lambda_\kappa(A \cup B) \subset E_\lambda_\kappa(A) \cup E_\lambda_\kappa(B)\).

(g) \(E_\lambda_\kappa(A \cup B) \supset E_\lambda_\kappa(A) \cap E_\lambda_\kappa(B)\).

(h) \(E_\lambda_\kappa(X) = \emptyset\).

(i) \(E_\lambda_\kappa(\emptyset) = X\).

(j) \(E_\lambda_\kappa(A) = E_\lambda_\kappa(X - E_\lambda_\kappa(A))\).

(k) \(i_\lambda_\kappa(A) \subset E_\lambda_\kappa(E_\lambda_\kappa(A))\).

(l) \(X = i_\lambda_\kappa(A) \cup E_\lambda_\kappa(A) \cup F_\lambda_\kappa(A)\).

**Proof.**

(d) \(E_\lambda_\kappa(E_\lambda_\kappa(A)) = E_\lambda_\kappa(X - c_\lambda_\kappa(A)) = i_\lambda_\kappa(X - (X - i_\lambda_\kappa(A))) = i_\lambda_\kappa(c_\lambda_\kappa(A))\).

(j) \(E_\lambda_\kappa(X - E_\lambda_\kappa(A)) = E_\lambda_\kappa(X - i_\lambda_\kappa(X - A)) = i_\lambda_\kappa(X - (X - i_\lambda_\kappa(X - A))) = i_\lambda_\kappa(i_\lambda_\kappa(X - A)) = i_\lambda_\kappa(X - A) = E_\lambda(A)\).

(k) \(i_\lambda_\kappa(A) \subset i_\lambda_\kappa(c_\lambda_\kappa(A)) = i_\lambda_\kappa(X - i_\lambda_\kappa(X - A)) = i_\lambda_\kappa(X - E_\lambda_\kappa(A)) = E_\lambda_\kappa(E_\lambda_\kappa(A))\).
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