Journal of Algorithms and Computation

journal homepage: http://jac.ut.ac.ir

Toughness of the Networks with Maximum Connectivity

D. Moazzami, *

University of Tehran, College of Engineering, Department of Engineering Science.

ABSTRACT

The stability of a communication network composed of processing nodes and communication links is of prime importance to network designers. As the network begins losing links or nodes, eventually there is a loss in its effectiveness. Thus, communication networks must be constructed to be as stable as possible, not only with respect to the initial disruption, but also with respect to the possible reconstruction of the network. For any fixed integers n,p with $p \geq n+1$, Harary constructed classes of graphs $H_{n,p}$ that are n-connected with the minimum number of edges. Thus Harary graphs are examples of graphs with maximum connectivity. This property makes them useful to network designers and thus it is of interest to study the behavior of other stability parameters for the Harary graphs. In this paper we study the toughness of the third case of the Harary graphs.

Keyword: toughness, Harary graph, maximum connectivity, Network.

ARTICLE INFO

Article history:
Received 19, February 2015
Received in revised form 25, July 2015
Accepted 27, October 2015
Available online 26, November 2015

AMS subject Classification: Primary 05C78, 05C70

*Email:*dmoazzami@ut.ac.ir*;

1 Introduction

Let G be a graph with vertex set V. Let A be a subset of V. We define $G - A$ to be the graph induced by the vertices of $V - A$. Also, for any graph G, $\omega(G)$ is the number of components of G.

Throughout this paper we will let p and q be the number of vertices and edges of G respectively. A set of vertices in G is independent if no two of them are adjacent. The largest number of vertices in any such set is called the vertex independence number of G and is denoted by $\beta(G)$ or β.

A cut-set of a graph G is a set of vertices whose removal results in a disconnected graph or the trivial graph K_1. The connectivity of G, $\kappa = \kappa(G)$, is the minimum order of a cut-set in G. A graph G is called n-connected if $\kappa \geq n$.

The toughness of a graph G was defined in [2] as $t(G) = min\{\frac{|A|}{\omega(G-A)}\}$, where the minimum is taken over all cut-sets A of G. A subset A of $V(G)$ is said to be a t-set of G if $t(G) = \frac{|A|}{\omega(G-A)}$. Note that if G is disconnected then the set A may be empty.

Given a graph G, the graph G^r has $V(G^r) = V(G)$ and $uv \in E(G^r)$ if and only if the distance from u to v in G is at most r. Thus, in particular, ${C_p}^r$ has

$$V({C_p}^r) = \{0, 1, \ldots, p-1\} \text{ and } E({C_p}^r) = \{ij : |i - j| \leq r\}.$$

For any fixed integers n,p with $p \geq n+1$, Harary [3] constructed classes of graphs $H_{n,p}$, that are n-connected with the minimum number of edges on p vertices. Thus Harary graphs are examples of graphs which in some sense have the maximum possible connectivity and hence are of interests as possibly having good stability properties.
Also, the Harary graph $H_{n,p}$ with $n = 2r$ is the rth power of the p-cycle, ${C_p}^r$, for which both the toughness have been studied. Harary graphs, $H_{n,p}$ is constructed as follows:

Case1: If n is even then let $n = 2r$. Then $H_{n,p}$ has vertices $0, 1, 2, \ldots, p-1$ and two vertices i and j are adjacent if and only if $|i - j| \leq r$ (where addition is taken module p). Note that this is ${C_p}^r$ and is n-regular.
Case2: (Moazzami and Bafandeh [4]) If n is odd ($n > 1$) and p is even. Let $n = 2r+1$ ($r > 0$). Then $H_{2r+1,p}$ is constructed by drawing $H_{2r,p}$ and adding edges joining vertex i to vertex $i + \frac{p}{2}$ for $1 \leq i \leq \frac{p}{2}$. Again note that this is an n-regular graph.
Case3: If n is odd ($n > 1$) and p is odd. Let $n = 2r+1$ ($r > 0$). Then $H_{2r+1,p}$ is constructed by first drawing $H_{2r,p}$ and adding edges joining vertex i to vertex $i + \frac{p+1}{2}$ for

$0 \leq i \leq \frac{p+1}{2}$. Note that under this definition, vertex 0 is adjacent to both vertices $\frac{p+1}{2}$ and $\frac{p-1}{2}$. Again note that all vertices of $H_{n,p}$ have degree n except vertex 0, which has degree $n+1$. $H_{5,9}$ is shown in Figure 3.

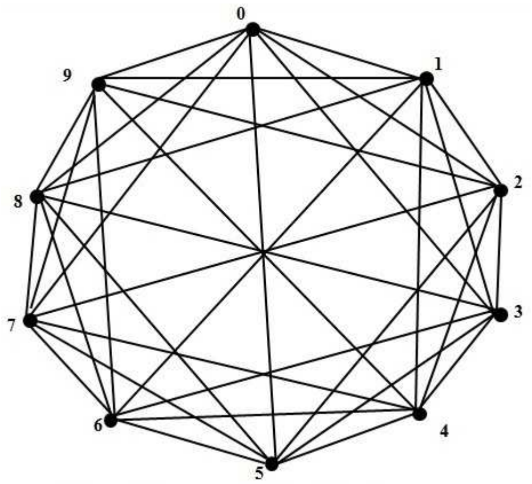

Figure 1. Harary graph with n odd and p even

The following four proposition were proved in [2].
Proposition1: If G is a spanning subgraph of H, then $t(G) \leq t(H)$.
Proposition2: For any graph G, $t(G) \geq \frac{k(G)}{\beta(G)}$.
Proposition3: If G is not complete, then $t(G) \leq \frac{k(G)}{2}$.
Proposition4: If G is not complete, then $t(G) \leq \frac{p-\beta(G)}{\beta(G)}$.

2 Toughness of a graph with maximum connectivity:

In this section we start to calculate the toughness of third case of Harary graphs. Throughout the rest of this paper we will let the connectivity $n = 2r+1$ and the number of vertices $p = k(r+1)+s$ for $0 \leq s < r+1$. So we can see that $p \equiv s \mod (r+1)$ and $k = \lfloor \frac{p}{r+1} \rfloor$. Also we assume that the graph $H_{n,p}$ is not complete, so $n+1 < p$. Note that this implies that $k \geq 2$.

Lemma 1: Let $H_{n,p}$ be the Harary graph with p and n both odd, $n = 2r+1$ and $r > 0$. Then $p \equiv 1 \mod (n+1)$ if and only if $s = 1$ and k is even.

Proof: Let $1 < s < r+1$ and $k = 2q+1$ for some q. Thus $p = k(r+1)+s = q(n+1)+s+r+1$. Since $1 < s+r+1 < n+1$, $p \not\equiv 1 \mod (n+1)$.
Now suppose $k = 2q$ and $1 < s < r+1$. Thus $p = q(n+1)+s$. Since $1 < s < n+1$, $p \not\equiv 1 \mod (n+1)$.

If $s = 0$ then $p = k(r+1)$. Since p is odd we know that k is odd. Thus $p = q(n+1)+r+1$. Since $1 < r+1 < (n+1)$, then $p \not\equiv 1 \bmod (n+1)$.
Finally, consider the case when $s = 1$. If k is odd, then $p = q(n+1) + r + 2$. Since $1 < r + 2 < n + 1$, then $p \not\equiv 1 \bmod (n + 1)$. If k is even, then $p = q(n + 1) + 1$. Thus $p \equiv 1 \bmod (n+1)$. □

Lemma 2: Let $H_{n,p}$ be the Harary graph with p and n both odd, $n = 2r + 1$ and $r > 0$. Then

$$\beta(H_{n,p}) = \begin{cases} k & \text{if } p \not\equiv 1 \ mod\ (n+1) \\ k-1 & \text{if } p \equiv 1 \ mod\ (n+1) \end{cases}$$

Proof: let $G = H_{n,p}$. Since at least r consecutive vertices must be between any two members of an independent set and $s < r + 1$, then $\beta(G) \leq k$. Consider the set

$$B = \{0, r + 1, 2(r + 1), 3(r + 1), \ldots, (k - 1)(r + 1)\}$$

Let $0 \leq s < r$ and assume $k = 2q + 1$ for some q. Since vertex i is adjacent to vertex $i + \frac{p+1}{2} = i + q(r + 1) + \frac{s+r}{2} + 1$, for any $0 \leq i \leq \frac{p-1}{2}$, and $\frac{r}{2} + 1 \leq \frac{s+r}{2} + 1 < r + 1$, and vertex $\frac{p-1}{2} \notin B$ then vertex $t(r+1) \in B$, $0 \leq t \leq k-1$, is not adjacent to vertex $x(r+1)$ for any $0 \leq x \leq k - 1$. Thus the set B is an independent set and $\beta(G) = k$.

Assume $s = r \neq 0$ and $k = 2q + 1$ for some q. Consider the set

$$C = \{0, r + 1, 2(r + 1), \ldots, q(r + 1), (q + 1)(r + 1) + 1, \ldots, (k - 1)(r + 1) + 1\}$$

If $s = r$, then $\frac{s+r}{2} + 1 = r + 1$. Since vertex i is adjacent to $i + \frac{p+1}{2} = i + (q + 1)(r + 1)$ for any $0 \leq i \leq \frac{p-1}{2}$, and vertex $\frac{p-1}{2} \notin C$, then vertex $t(r + 1) \in C$, $0 \leq t \leq q$, is not adjacent to $x(r + 1) + 1$ for any $q + 1 \leq x \leq k - 1$. Thus C is an independent set and hence the independence number is at least k. Therefore $\beta(G) = k$.

Suppose $1 < s < r + 1$ and $k = 2q$ for some q. Consider the set B. Since vertex i is adjacent to vertex $i + \frac{p+1}{2} = i + q(r + 1) + \frac{s+1}{2}$ for any $0 \leq i \leq \frac{p-1}{2}$ and $1 < \frac{s+1}{2} < r + 1$, and vertex $\frac{p-1}{2} \notin B$, then vertex $t(r+1) \in B$, $0 \leq t \leq k-1$, is not adjacent to $x(r+1)$ for any $0 \leq x \leq k - 1$. Thus in this case B is again an independent set. Therefore $\beta(G) = k$.

Now let $s = 1$ and $k = 2q$ for some q. Assume $\beta(G) = k$. Since $s = 1$, $m = 2q(r + 1) + 1$. Hence there are r consecutive vertices between two members of an independent set and so sets of the format B or C are the only possible independent sets. But we also need to consider edges of the form $\{i, i + \frac{p+1}{2}\}$. Hence vertex $t(r+1)$, $1 \le t \le q$, is adjacent to vertex $x(r+1)+1$ for any $q+1 \le x \le k-2$, and vertex 0 is adjacent to vertices $\frac{p-1}{2} = q(r+1)$ and $\frac{p-1}{2} = q(r+1)+1$. First consider the set B. In B vertex 0 is adjacent to $q(r+1)$ and this is a contradiction to the definition of independent set. Now, consider the set C. In C vertex $t(r+1)$, $1 \le t \le q$, is adjacent to vertex $x(r+1)+1$, $q+1 \le x \le k-2$, and again this is a contradiction to the definition of independent set. Hence $\beta(G) \ne k$ and thus $\beta(G) < k$.

Finally consider the set

$C = \{0, r+1, 2(r+1), \ldots, (q-1)(r+1), q(r+1)+2, (q+1)(r+1)+2 \ldots, (k-2)(r+1)+2\}$

Since vertex i is adjacent to $i + \frac{p+1}{2} = i + q(r+1) + 1$ for any $0 \le i \le \frac{p-1}{2}$, and vertex $\frac{p-1}{2} \notin D$ it can be seen that vertex $t(r+1) \in D$, $0 \le t \le q-1$, is not adjacent to vertex $x(r+1)+2$, $q \le x \le k-2$. Thus D is an independent set. Hence $k-1 \le \beta(G) < k$. Therefore since $\beta(G)$ and k are integers, we can conclude that $\beta(G) = k-1$.

Theorem1: Let $H_{n,p}$ be the Harary graph with p an n odd, and $n = 2r+1$, then

$$r \le t(H_{n,p}) \le \begin{cases} r + \frac{s}{k} & \text{if } p \not\equiv 1 \ mod\ (n+1) \\ \frac{kr+s+1}{k-1} & \text{if } p \equiv 1 \ mod\ (n+1) \end{cases}$$

Proof: Let $G = H_{n,p}$. By proposition 4, $t(G) \le \frac{p-\beta(G)}{\beta(G)}$. Thus by lemma 2, if $p \not\equiv 1$ mod $(n+1)$, then $t(G) \le \frac{p-k}{k} = \frac{k(r+1)+s-k}{k} = r + \frac{s}{k}$, and if $p \equiv 1$ mod $(n+1)$, then $t(G) \le \frac{p-(k-1)}{k-1} = \frac{k(r+1)+s-k+1}{k-1} = \frac{kr+s+1}{k-1}$

Since $V(H_{2r,p}) = V(G)$ and $E(H_{2r,p}) \subseteq E(G)$, then $H_{2r,p}$ is a spanning subgraph of G. By proposition 1, we have $t(H_{2r,p}) \le t(G)$. Thus we have $r \le t(G)$.

Corollary1: If p and n are odd $s = 0$, then $t(H_{n,p}) = r$.

Corollary2: If p and n are odd, $s = 1$ and k is odd, then $r \le t(H_{n,p}) \le r + \frac{1}{k}$.

Corollary3: If p and n are odd, $s = 1$ and k is even, then $r + \frac{1}{k} \le t(H_{n,p}) \le \frac{kr+2}{k-1}$.

Lemma 3: Let $H_{n,p}$ be the Harary graph with $n = 2r+1$, p and k both odd, $r \ge 2$, $1 < s < r+1$ and $s < k$. Then there is a cut-set A with kr elements such that

$\omega(H_{n,p} - A) = k$.

Proof: We may assume $H_{n,p}$ is labeled by $0, 1, 2, \ldots, p-1$. Let $s < k$, so $s = k - l$ for some l. Thus $p = s(r+2) + l(r+1)$. Since k is odd then $k = 2q+1$ for some q.

Case1: If r is odd then s is odd and l is even. Then $s = k - l \geq 1$ and $l = 2t$ for some t. Hence $k - l = 2q + 1 - 2t \geq 1$ which implies that $q \geq t$. Define the sets W_i for $1 \leq i \leq 2q+1$ as follows:

$$W_i = \begin{cases} \{ir+i\} & 1 \leq i \leq t \\ \{ir-t+2i-1, ir-t+2i\} & t+1 \leq i \leq q \\ \{ir-t+q+i\} & q+1 \leq i \leq q+t \\ \{ir-2t+2i-1, ir-2t+2i\} & q+t+1 \leq i \leq 2q+1 \end{cases}$$

Let W be the union of the sets W_i, $1 \leq i \leq 2q+1$ and $A = V(G) - W$. The number of vertices in W is equal to $t + 2(q-t) + t + 2(q-t+1) = 2(2q+1) - 2t = 2k - l = k + s$, so $|A| = p - k - s = kr$. Now, we can see that for any $1 \leq i \leq 2q+1$, the elements in W_i differ from those in W_{i+1} by at least $r+1$. Hence, no vertex in W_i is adjacent to a vertex in W_j, $1 \leq i < j \leq 2q+1$, by an edge in the copy of $H_{2r,p}$ in G. Thus we only need consider edges of the form $\{x, x + \frac{p+1}{2}\}$. In fact, we need to consider only such edges when x is at most $\frac{p-1}{2}$. Hence, since $\frac{p-1}{2} = qr + 2q - t + \frac{r}{2} + \frac{1}{2} < (q+1)r - t + q + (q+1)$, we need to consider only vertices in W_i for $1 \leq i \leq q$.

So consider $W_i = \{ir+i\}$ for $1 \leq i \leq t$. Then

$ir + i + \frac{p+1}{2} = (q+i)r - t + q + (q+i) + 1 + \frac{r+1}{2} > (q+i)r - t + q + (q+i) = jr - t + q + j$, for $j = q+i$.

Also, since $r \geq 2$,

$ir + i + \frac{p+1}{2} < (q+i)r - t + q + (q+i) + 1 + r = (q+i+1)r - t + q + (q+i+1) = (j+1)r - t + q + (j+1)$.

Therefore the set $\{ir + i + \frac{p+1}{2}\}$ is strictly between W_j and W_{j+1} for $j = q+i$, and so it is contained in A.

Finally, consider $W_i = \{ir-t+2i-1, ir-t+2i\}$ for $t+1 \leq i \leq q$. Then

$ir-t+2i-1+\frac{p+1}{2}=(q+i)r-2t+2(q+i)+\frac{r+1}{2}>(q+i)r-2t+2(q+i)=jr-2t+2j$, for $j=q+i$.

Also,

$ir-t+2i+\frac{p+1}{2}=(q+i)r-2t+2(q+i)+1+r=(q+i+1)r-2t+2(q+i+1)-1=(j+1)r-2t+2(j+1)-1$.

Hence the set $\{ir-t+2i-1+\frac{p+1}{2}, ir-t+2i+\frac{p+1}{2}\}$ is strictly between W_j and W_{j+1} for $j=q+i$ and so it is contained in A. Also note that $0+\frac{p+1}{2}=qr+2q-t+1+\frac{r}{2}+\frac{1}{2}\notin W_j$ for any j and so is in A. Therefore the W_i, $1\le i\le 2q+1=k$ are the components of $H_{n,p}-A$, so $\omega(H_{n,p}-A)=k$.

Case2: If r is odd, then s is even and l is odd. Hence $s=2h$ for some h. Define

$$W_i=\begin{cases}\{ir+2i-1, ir+2i\} & 1\le i\le h\\ \{ir+i+h\} & h+1\le i\le q\\ \{ir+2i-q+h-1, ir+2i-q+h\} & q+1\le i\le q+h\\ \{ir+i+2h\} & q+h+1\le i\le 2q+1\end{cases}$$

Let W be the union of the sets W_i, $1\le i\le 2q+1$ and $A=V(G)-W$. The number of vertices in W is equal to $2h+(q-h)+2h+q-h+1=2q+1+2h=k+s$, so $|A|=p-k-s=kr$. Now, we can see that for any $1\le i\le 2q+1$, the elements in W_i differ from those in W_{i+1} by at least $r+1$. Hence, no vertex in W_i is adjacent to a vertex in W_j, $1\le i<j\le 2q+1$, by an edge in the copy of $H_{2r,p}$ in G. Thus we only need consider edges of the form $\{x, x+\frac{p+1}{2}\}$. In fact, we need to consider only such edges when x is at most $\frac{p-1}{2}$. Hence, since $\frac{p-1}{2}=qr+q+h+\frac{r}{2}<(q+1)r+2(q+1)-q+h-1$, we need only consider vertices in W_i for $1\le i\le q$.

So consider $W_i=\{ir+2i-1, ir+2i\}$ for $1\le i\le h$. Then

$ir+2i-1+\frac{p+1}{2}=(q+i)r+2(q+i)-q+h+\frac{r}{2}>(q+i)r+2(q+i)-q+h=jr+2j-q+h$, for $j=q+i$.

Also, since $r\ge 2$

$ir+2i+\frac{p+1}{2}=(q+i)r+2(q+i)-q+h+\frac{r}{2}+1<(q+i+1)r+2(q+i+1)-q+h-1=(j+1)r+2(j+1)-q+h-1$.

Therefore the set $\{ir+2i-1+\frac{p+1}{2}, ir+2i+\frac{p+1}{2}\}$ is strictly between W_j and W_{j+1} for $j=q+i$, and so it is contained in A.

Finally, consider $W_i=\{ir+i+h\}$ for $h+1\leq i\leq q$. Then

$ir+i+h+\frac{p+1}{2}=(q+i)r+(q+i)+2h+\frac{r}{2}+1>(q+i)r+(q+i)+2h=jr+j+2h$, for $j=q+i$.

Again we have, $ir+i+h+\frac{p+1}{2}<(q+i)r+(q+i)+2h+\frac{r}{2}+1<(q+i+1)r+(q+i+1)+2h=(j+1)r+(j+1)+2h$.

Hence the set $\{ir+i+h+\frac{p+1}{2}\}$ is strictly between W_j and W_{j+1} for $j=q+i$ and so it is contained in A. Also $0+\frac{p+1}{2}=qr+q+h\frac{r}{2}+1\in A$ Therefore the W_i, $1\leq i\leq 2q+1=k$ are the components of $H_{n,p}-A$, so $\omega(H_{n,p}-A)=k$.

Theorem 2: Let $H_{n,p}$ be the Harary graph with $n=2r+1$, p and k both odd, $1<s<r+1$ and $s<k$. Then $t(H_{n,p})=r$.

Proof: First we see that $r\geq 2$, since if $r=1$, $1<s<2$, a contradiction. By Theorem 1, we have $r\leq t(H_{n,p})$. By lemma 3, there is a cut-set A of $H_{n,p}$ with kr elements such that $\omega(H_{n,p}-A)=k$. Therefore, the toughness attains the lower bound using A, so $t(H_{n,p})=r$.

Now we consider the cases when $s\geq k$ and p and k are odd. Again as before the following lemmas are required in the proofs of Lemmas 6,7 and 8.

Lemma 4: Let $H_{n,p}$ be the Harary graph with $n=2r+1$, p and k both odd, $1<s<r+1$ and $s\geq k$, where $s=ak+b$, for some a and b.

Case 1: For $0<b<k$,
(i)If r is even, then

$$r\geq\begin{cases}4 & \text{a odd}\\ 8 & \text{a even}\end{cases}$$

(ii)If r is odd, then

$$r \geq \begin{cases} 5 & \text{a odd} \\ 7 & \text{a even} \end{cases}$$

Case 2: For $b = 0$, $s = ak$,

(i) If r is even, then $r \geq 6$,
(ii) If r is odd, then $r \geq 3$.

Proof:

Case 1. Let r be even and $0 < b < k$. Hence s is even.
Subcase (i). If a is even then b is even and the minimum values for b and a are 2. Since k is odd and $b < k$, the minimum value for k is 3. Therefore the minimum value for s is $2(3) + 2 = 8$. Since r is even and $s < r + 1$, we have $r \geq 8$.
Subcase (ii). If a is odd then b is odd and the minimum value for b is 1. Since a and k are odd and $b < k$, the minimum values for k and a are 3 and 1 respectively. Therefore the minimum value for s is $1(3) + 1 = 4$. Since r is even and $s < r + 1$, we have $r \geq 4$.

Case 2. Let r be odd and $0 < b < k$. Hence s is odd.
Subcase (i). If a is even then b is odd and the minimum value for b is 1. Since a is even and k is odd and $b < k$, the minimum values for k and a are 3 and 2 respectively. Therefore the minimum value for s is $2(3) + 1 = 7$. Since r is odd and $s < r + 1$, we have $r \geq 7$.
Subcase (ii). If a is odd then b is even and the minimum value for b is 2. Since a and k are odd and $b < k$, the minimum values for a and k are 1 and 3 respectively. Therefore the minimum value for s is $1(3) + 2 = 5$. Since r is odd and $s < r + 1$, we have $r \geq 5$.

Case 3. Suppose r is even and $b = 0$. Then s is even and so a is even. Since $k > 1$ and k is odd, the minimum values for a and k are 2 and 3 respectively. Therefore the minimum value for s is $2(3) = 6$. Since r is even and $s < r + 1$, we have $r \geq 6$.

Case 4. Suppose r is odd and $b = 0$. Then s is odd and so a is odd. Therefore the minimum values for a and k are 1 and 3 respectively. So the minimum value for s is $1(3) = 3$. Since r is odd and $s < r + 1$, we have $r \geq 3$.

Lemma 5: Let $H_{n,p}$ be the Harary graph with $n = 2r+1$, p and k both odd, $1 < s < r+1$

and $s \geq k$. Where $s = ak + b$ for some a and b. If $0 < b < k$ then $a + 1 \leq \frac{r}{2}$ with equality possible only if r is even and a is odd. If $b = 0$, so that $s = ak$, then $a + 1 \leq \lceil r/2 \rceil$.

Proof.

Case 1. Let r be even and $0 < b < k$. Hence s is even.
Subcase (i). If a is even then b is even and the minimum value for b is 2. Since $b < k$ and k is odd, the minimum value for k is 3. Thus $3a + 2 \leq ak + b = s$. Since $2a + 2 < 3a + 1$, we have $2a + 2 < r$. Therefore $a + 1 < \frac{r}{2}$.
Subcase (ii). If a is odd then b is odd and the minimum values for b and k are 1 and 3 respectively. Thus $3a + 1 \leq ak + b = s$. Since a is odd, we have $2a + 2 \leq 3a + 1$ and so $2a + 2 \leq r$. Therefore $a + 1 \leq \frac{r}{2}$.

Case 2. Let r be odd and $0 < b < k$. Hence s is odd.
Subcase (i). If a is even then b is odd and the minimum value for b is 1, and since k is odd and $b < k$, the minimum value for k is 3. Thus $3a + 2 \leq ak + b = s$. Since a is even, we have $2a + 2 < 3a + 1$. Thus $2a + 2 < r$. Therefore $a + 1 < \frac{r}{2}$.
Subcase (ii). If a is odd then b is even and the minimum value for b is 2. Since k is odd and $b < k$, the minimum value for k is 3. Thus $3a + 2 \leq ak + b = s$. Since $2a + 2 < 3a + 2$ and $s \leq r$, we have $a + 1 < \frac{r}{2}$.

Case 3. Let r be even and $b = 0$. Thus a is even. Since a is even, we have $2a + 2 \leq 3a$. Since k is odd and $k > 1$, the minimum value for k is 3. Hence $3a \leq ak = s \leq r$. Therefore $a + 1 \leq \frac{r}{2}$.

Case 4. Let r be odd and $b = 0$. Thus a is odd. Therefore we have $2a + 1 \leq 3a$. Since k is odd and $k > 1$, the minimum value for k is 3. Hence $3a \leq ak = s \leq r$, and so $a + 1 \leq \frac{r+1}{2}$.

Lemma 6: Let $H_{n,p}$ be the Harary graph with $n = 2r + 1$, p and k both odd, $1 < s < r + 1$ and $s = ak$ for some a. Then there is a cut-set A with kr elements such that $\omega(H_(n, p) - A) = k$.

Proof. Let $s = ak$ and $k = 2q + 1$ for some q. Thus

$m = 2qr + 2q(a + 1) + r + a + 1.$

Let $W_i = \{ir + (i - 1)a + i, \ldots, ir + ia + i\}$ for $1 \leq i \leq 2q + 1$ and let W be the union of the sets, W_i, $1 \leq i \leq 2q + 1$, and $A = V(G) - W$. The number of vertices in W is equal to $(2q + 1)(a + 1) = ka + k = s + k$, so $|A| = p - k - s = kr$. Now we can see that for any

$1 \leq i \leq 2q+1$, the elements in W_i differ from those in W_{i+1} by at least $r+1$. Hence, no vertex in W_i is adjacent to a vertex in W_j, $1 \leq i < j \leq 2q+1$, by an edge in the copy of $H_{2r,p}$ in G. Thus we need only consider edges of the form $\{x, x+\frac{p+1}{2}\}$. In fact, we need to consider only such edges when x is at most $\frac{p-1}{2}$. Hence, since by Lemma 5, $\frac{a}{2} < \frac{r}{2}$, so

$$\tfrac{p-1}{2} = qr + qa + q + \tfrac{r}{2} + \tfrac{a}{2} < (q+1)r + qa + q + 1,$$

we need to consider only vertices in W_i for $1 \leq i \leq q$. So consider

$$W_i = \{ir + (i-1)a + i, \ldots, ir + ia + i\}$$

for $1 \leq i \leq q$. Then

$$ir + (i-1)a + i + \tfrac{p+1}{2} = (q+i)r + (q+i)a + (q+i) + \tfrac{r}{2} - \tfrac{a}{2} + 1 > (q+i)r + (q+i)a + q + i = jr + ja + j, \text{ for } j = q+i.$$

Also,
$$ir + ia + i + \tfrac{p+1}{2} < (q+i+1)r + (q+i)a + (q+i+1) = (j+1)r + ja + j + 1.$$

Hence, the set $\{ir + (i-1)a + i + \frac{p+1}{2}, \ldots, ir + ia + i + \frac{p+1}{2}\}$ is strictly between W_j and W_{j+1} for $j = q+i$, and so it is contained in A. Also

$$0 + \tfrac{p+1}{2} = qr + qa + q + \tfrac{r}{2} + \tfrac{a}{2} + 1 \in A$$

Therefore the W_i, $1 \leq i \leq 2q+1 = k$ are the components of $H_{n,p} - A$, so $\omega(H_{n,p} - A) = k$.

Theorem 3: Let $H_{n,p}$ be the Harary graph with $n = 2r+1$, p and k both odd, $1 < s < r+1$ and $s = ak$ for some a. Then $t(H_{n,p}) = r$.

Proof: By Theorem 1, we have $r \leq t(H_{n,p})$. Set A of Lemma 6 achieves this lower bound, since $|A| = kr$ and $\omega(H_{n,p} - A) = k$. Therefore $t(H_{n,p}) = r$.

Lemma 7: Let $H_{n,p}$ be the Harary graph with $n = 2r+1$, p and k both odd, $1 < s < r+1$ and $s > k$ where $s = ak + b$ for some a and b, $0 < b < k$. Then there is a cut-set A with kr elements such that $\omega(H_(n,p) - A) = k$.

Proof: Let $s > k$, and $s = ak + b$ for $0 < b < k$. Thus

$$p = kr + (k-b)(a+1) + b(a+2).$$

Write $p = k(r+1)+s$. If r is even, then s is even. In this case a is even if and only if b is even. If r is odd, then s is odd. In this case a is odd if and only if b is even. Thus we have the following two cases.

Case1: Let r and a both be even, or r and a both odd. Hence b is even. Therefore $b = 2h$ for some h. Since k is odd, $k = 2q+1$ for some q. Hence $k - b = 2q + 1 - 2h \geq 1$ which implies that $q \geq h$. Define the sets W_i for $1 \leq i \leq 2q+1$ as follows:

$$W_i = \begin{cases} \{ir + (i-1)a + 2i - 1, \ldots, ir + ia + 2i\} & 1 \leq i \leq h \\ \{ir + (i-1)a + i + h, \ldots, ir + ia + i + h\} & h+1 \leq i \leq q \\ \{ir + (i-1)a + 2i - q + h - 1, \ldots, ir + ia + 2i - q + h\} & q+1 \leq i \leq q+h \\ \{ir + (i-1)a + i + 2h, \ldots, ir + ia + i + 2h\} & q+h+1 \leq i \leq 2q+1 \end{cases}$$

Let W be the union of the sets W_i, $1 \leq i \leq 2q+1$ and $A = V(G) - W$. The number of vertices in W is equal to

$$h(a+2)+(q-h)(a+1)+h(a+2)+(q-h+1)(a+1) = (2q+1)a+2h+k = ka+2h+k = k+s,$$

so $|A| = p - k - s = kr$. Now, we can see that for any $1 \leq i \leq 2q+1$, the elements in W_i differ from those in W_{i+1} by at least $r+1$. Hence, no vertex in W_i is adjacent to a vertex in W_j, $1 \leq i < j \leq 2q+1$, by an edge in the copy of $H_{2r,p}$ in G. Thus we only need consider edges of the form $\{x, x + \frac{p+1}{2}\}$. In fact, we need to consider only such edges when x is at most $\frac{p-1}{2}$. By Lemma 5, $\frac{a+1}{2} < \frac{r}{2}$ Hence, $\frac{p-1}{2} = qr+qa+q+h+\frac{r}{2}+\frac{a}{2} < (q+1)r+qa+2(q+1)-q+h-1$. Thus we need to consider only vertices in W_i for $1 \leq i \leq q$. So consider $W_i = \{ir+(i-1)a+2i-1, \ldots, ir+ia+2i\}$ for $1 \leq i \leq h$. By Lemma 5, $\frac{r}{2} - \frac{a}{2} > \frac{a}{2} + 1 > 1$. Hence

$ir + (i-1)a + 2i - 1 + \frac{p+1}{2} = (q+i)r + (q+i)a + 2(q+i) - q + h + \frac{r}{2} - \frac{a}{2} > (q+i)r + (q+i)a + 2(q+i) - q + h = jr + ja + 2j - q + h$, for $j = q+i$.

Also, by Lemma 4 and 5, $\frac{r}{2} + \frac{a}{2} < r$ Hence

$ir+ia+2i+\frac{p+1}{2} = (q+i)r+(q+i)a+2(q+i)-q+h+\frac{r}{2}+\frac{a}{2}+1 < (q+i)r+(q+i)a+2(q+i+1)-q+h-1+r = (q+i+1)r+(q+i)a+2(q+i+1)-q+h-1 = (j+1)r+(j-1)a+2j-q+h-1$

Therefore the set $\{ir+(i-1)a+2i-1+\frac{p+1}{2}, \ldots, ir+ia+2i+\frac{p+1}{2}\}$ is strictly between

W_j and W_{j+1} for $j = q + i$, and so it is contained in A.

Finally, consider $W_i = \{ir + (i-1)a + i + h, \ldots, ir + ia + i + h\}$ for $h + 1 \leq i \leq q$. By Lemma 5, $\frac{r}{2} - \frac{a}{2} + 1 > 0$. Hence

$ir + (i-1)a + i + h + \frac{p+1}{2} = (q+i)r + (q+i)a + (q+i) + 2h + \frac{r}{2} - \frac{a}{2} + 1 > (q+i)r + (q+i)a + (q+i) + 2h = jr + ja + j - q + 2h$, for $j = q + i$.

Also, by Lemma 4 and 5, $\frac{r}{2} + \frac{a}{2} < r$ Hence

$ir + ia + i + h + \frac{p+1}{2} = (q+i)r + (q+i)a + (q+i) + 2h + \frac{r}{2} + \frac{a}{2} + 1 < (q+i+1)r + (q+i)a + (q+i+1) + 2h = (j+1)r + ja + (j+1) + 2h$.

Therefore the set $\{ir + (i-1)a + i + h + \frac{p+1}{2}, \ldots, ir + ia + i + h + \frac{p+1}{2}\}$ is strictly between W_j and W_{j+1} for $j = q + i$, and so it is contained in A. Also note that $0 + \frac{p+1}{2} \in A$. Therefore the W_i, $1 \leq i \leq 2q+1 = k$ are the components of $H_{n,p} - A$, so $\omega(H_{n,p} - A) = k$.

Case2: If r is even and a is odd, or r is odd and a is even, then b is odd and hence $k - b$ is even. Therefore $k - b = 2t$ for some t. Define the sets W_i for $1 \leq i \leq 2q + 1$ as follows:

$$W_i = \begin{cases} \{ir + (i-1)a + i, \ldots, ir + ia + i\} & 1 \leq i \leq t \\ \{ir + (i-1)a + 2i - t - 1, \ldots, ir + ia + 2i - t\} & t + 1 \leq i \leq q \\ \{ir + (i-1)a + i + q - t, \ldots, ir + ia + i + q - t\} & q + 1 \leq i \leq q + t \\ \{ir + (i-1)a + 2i - 2t - 1, \ldots, ir + ia + 2i - 2t\} & q + t + 1 \leq i \leq 2q + 1 \end{cases}$$

Let W be the union of the sets W_i, $1 \leq i \leq 2q + 1$ and $A = V(G) - W$. The number of vertices in W is equal to

$$t(a+1)+(q-t)(a+2)+t(a+1)+(q-t+1)(a+2) = (2q+1)a+2(2q+1)-2t = ka+k+b = k+s,$$

so $|A| = p - k - s = kr$. Now, we can see that for any $1 \leq i \leq 2q + 1$, the elements in W_i differ from those in W_{i+1} by at least r. Hence, no vertex in W_i is adjacent to a vertex in W_j, $1 \leq i < j \leq 2q + 1$, by an edge in the copy of $H_{2r,p}$ in G. Thus we only need consider edges of the form $\{x, x + \frac{p+1}{2}\}$. In fact, we need to consider only such edges when x is at most $\frac{p-1}{2}$. By Lemma 5, $\frac{a}{2} + \frac{1}{2} < \frac{r}{2}$ Hence, $\frac{p-1}{2} = qr + qa + 2q - t + \frac{r}{2} + \frac{a}{2} + \frac{1}{2} < (q+1)r + qa + 2q + 1 - t$. Thus we need to consider only vertices in W_i for $1 \leq i \leq q$. So consider $W_i = \{ir + (i-1)a + i, \ldots, ir + ia + i\}$ for

$1 \le i \le t$. By Lemma 5, $\frac{r}{2} - \frac{a-3}{2} > 0$. Hence

$ir + (i-1)a + i + \frac{p+1}{2} = (q+i)r + (q+i)a + (q+i) + q - t\frac{r}{2} - \frac{a-3}{2} > (q+i)r + (q+i)a + (q+i) + q - t = jr + ja + j + q - t$, for $j = q + i$.

Also, by Lemma 4 and 5, $\frac{a+1}{2} + \frac{r}{2} < r$ Hence

$ir + ia + i + \frac{p+1}{2} = (q+i+1)r + (q+i)a + 2(q+i+1) + q - t = (j+1)r + ja + j + 1 + q - t$

Therefore the set $\{ir + (i-1)a + i + \frac{p+1}{2}, \dots, ir + ia + \frac{p+1}{2}\}$ is strictly between W_j and W_{j+1} for $j = q + i$, and so it is contained in A.

Finally, consider $W_i = \{ir + (i-1)a + 2i - t - 1, \dots, ir + ia + 2i - t\}$ for $t + 1 \le i \le q$. By Lemma 5, $\frac{r}{2} - \frac{a-1}{2} \ge \frac{a+3}{2} > 0$. Hence

$ir + (i-1)a + 2i - t - 1 + \frac{p+1}{2} = (q+i)r + (q+i)a + 2(q+i) - 2t + \frac{r}{2} - \frac{a-1}{2} > (q+i)r + (q+i)a + 2(q+i) - 2t = jr + ja + 2j - 2t$, for $j = q + i$.
Also, by Lemma 4 and 5, $\frac{r}{2} + \frac{a+1}{2} < r$ Hence

$ir + ia + 2i - t + \frac{p+1}{2} < (q+i+1)r + (q+i)a + 2(q+i+1) - 2t - 1 = (j+1)r + ja + 2(j+1) - 2t - 1.$

Hence the set $\{ir + (i-1)a + 2i - t - 1 + \frac{p+1}{2}, \dots, ir + ia + 2i - t + \frac{p+1}{2}\}$ is strictly between W_j and W_{j+1} for $j = q + i$, and so it is contained in A. Also note that $0 + \frac{p+1}{2} \in A$. Therefore the W_i, $1 \le i \le 2q + 1 = k$ are the components of $H_{n,p} - A$, so $\omega(H_{n,p} - A) = k$.

Theorem 4: Let $H_{n,p}$ be the Harary graph with $n = 2r+1$, p and k both odd, $1 < s < r+1$ and $s > k$, where $s = ak + b$ for some a and b, $0 < b < k$. Then $t(H_{n,p}) = r$.

Proof: By Theorem 1, we have $r \le t(H_{n,p})$. Set A of Lemma 7 achieves this lower bound, since $|A| = kr$ and $\omega(H_{n,p} - A) = k$. Therefore $t(H_{n,p}) = r$.

Lemma 8: Let $H_{n,p}$ be the Harary graph with $n = 2r + 1$, p odd, $r \ge 2$ $1 < s < r + 1$ and $s < k$ and k even. Then there is a cut-set A with $kr + 1$ elements such that $\omega(H_(n, p) - A) = k$.

Proof: We may assume $H_{n,p}$ is labeled by $0, 1, 2, \dots, p - 1$. Let $s < k$, then $s = k - l$ for some l. Since p is odd and k is even, then s is odd. Hence $l = 2t + 1$ and $k = 2q$, for some t and q. Thus $s = k - l = 2q - 2t - 1 > 1$, which implies that $q > t + 1$. Define the sets W_i for $1 \le i \le 2q + 1$ as follows:

$$W_i = \begin{cases} \{ir+2i-1, ir+2i\} & 1 \le i \le q-t-1 \\ \{ir+i+q-t-1\} & q-t \le i \le q+1 \\ \{ir+2i-t-3, ir+2i-t-2\} & q+2 \le i \le 2q-t \\ \{ir+i+2q-2t-2\} & 2q-t+1 \le i \le 2q \end{cases}$$

Let W be the union of the sets W_i, $1 \le i \le 2q$ and $A = V(G) - W$. The number of vertices in W is equal to

$$2(q-t-1)+t+2+2(q-t-1)+t = 4q-(2t+1)-1 = 2k-l-1 = k+s-1,$$

so $|A| = p-k-s+1 = kr+1$. Now, we can see that for any $1 \le i \le 2q$, the elements in W_i differ from those in W_{i+1} by at least $r+1$. Hence, no vertex in W_i is adjacent to a vertex in W_j, $1 \le i < j \le 2q$, by an edge in the copy of $H_{2r,p}$ in G. Thus we only need consider edges of the form $\{x, x+\frac{p+1}{2}\}$. In fact, we need to consider only such edges when x is at most $\frac{p-1}{2}$. Hence, since $\frac{p-1}{2} = qr+2q-t-1 < (q+1)r+(q+1)+q-t-1$, we need to consider only vertices in W_i for $1 \le i \le q$. So consider $W_i = \{ir+2i-1, \ldots, ir+2i\}$ for $1 \le i \le q-t-1$. Then

$ir+2i-1+\frac{p+1}{2} = (q+i)r+2(q+i)-t-1 > (q+i)r+2(q+i)-t-2 = jr+2j-t-2$, for $j = q+i$.

Also, since $r \ge 2$

$$ir+2i+\frac{p+1}{2} < (q+i+1)r+2(q+i+1)-t-3 = (j+1)+2(j+1)r-t-3.$$

Therefore the set $\{ir+2i-1+\frac{p+1}{2}, ir+2i+\frac{p+1}{2}\}$ is strictly between W_j and W_{j+1} for $j = q+i$, and so it is contained in A.

Finally, consider $W_i = \{ir+i+q-t-1\}$ for $q-t \le i \le q$. Then

$ir+i+q-t-1+\frac{p+1}{2} = (q+i)r+(q+i)+2q-2t-1 > (q+i)r+(q+i)+2q-2t-2 = jr+j+2q-2t-2$, for $j = q+i$.

Also, since $r \ge 2$

$$ir+i+q-t-1+\frac{p+1}{2} < (q+i+1)r+(q+i+1)+2q-2t-2 = (j+1)r+j+1+2q-2t-2.$$

Hence the set $\{ir + i + q - t - 1\frac{p+1}{2}\}$ is strictly between W_j and W_{j+1} for $j = q + i$, and so it is contained in A. Therefore the W_i, $1 \leq i \leq 2q$ are the components of $H_{n,p} - A$, so $\omega(H_{n,p} - A) = k$.

Theorem 5: Let $H_{n,p}$ be the Harary graph with $n = 2r+1$, p odd, k even, $1 < s < r+1$ and $s < k$. Then $r \leq t(H_{n,p}) \leq r + \frac{1}{k}$.

Proof: First note that if $r = 1$ then $1 < s < 2$, a contradiction. Hence we have $r \geq 2$. By Theorem 1, we have $r \leq t(H_{n,p})$. By selecting Set A of Lemma 8 we have $|A| = kr+1$ and $\omega(H_{n,p} - A) = k$. Therefore $r \leq t(H_{n,p}) \leq r + \frac{1}{k}$.

Lemma 9: Let $H_{n,p}$ be the Harary graph with $n = 2r + 1$, p odd, k even, $k > 2$, $1 < s < r+1$ and $s > k$, where $s = ak + b$ for some a and b, $0 < b < k$. Then

$$r \geq \begin{cases} 5 & \text{a odd} \\ 9 & \text{a even} \end{cases}$$

Proof: Write $s = ak + b$, for some $0 < b < k$. Since p is odd and k is even, then s is odd. Hence b is odd, and the minimum value for b is 1. Since k is even and $k > 2$, then the minimum value for k is 4.

Case 1. Suppose a is odd. Hence the minimum value for a is 1. Thus the minimum value for s is $1(4) + 1 = 5$. Since $s < r + 1$, $r \geq 5$.

Case 2. Suppose a is even. Hence the minimum value for a is 2. Thus the minimum value for s is $2(4) + 1 = 9$. Since $s < r + 1$, $r \geq 9$.
Note that if r is even then the bounds in the previous lemma can be increased by 1..

Lemma 10: Let $H_{n,p}$ be the Harary graph with $n = 2r + 1$, p odd, k even, $k > 2$, $1 < s < r+1$ and $s > k$, where $s = ak + b$ for some a and b, $0 < b < k$. Then $a + 1 < \frac{r}{2}$

Proof: Write $s = ak + b$, for some $0 < b < k$. Since p is odd and k is even, then s is odd. Hence b is odd. Since k is even and $k > 2$ and b is odd, then the minimum value for k and b are 4 and 1 respectively. Thus $2a + 2 < 4a + 1 \leq ak + b = s$. Hence, since $s \leq r$, we have $a + 1 < \frac{r}{2}$.

Since k is even and p is odd, then p is not a multiple of k, $s \neq ak$. Hence we have our final lemma.

Lemma 11: Let $H_{n,p}$ be the Harary graph with $n = 2r+1$, p odd, k even, $k > 2$, $1 < s < r+1$ and $s > k$, where $s = ak+b$ for some a and b, $0 < b < k$. Then there is a cut-set A with $kr+1$ elements such that $\omega(H_(n,p) - A) = k$.

Proof: Let $s > k$, then $s = ak+b$, for $0 < b < k$. Since p is odd and k is even, then s is odd. Hence b is odd. Thus $b = 2t-1$ and $k = 2q$ for some t and q.

Case1: Suppose $a+1 < q$. Define the sets W_i for $1 \leq i \leq 2q$ as follows:

$$W_i = \begin{cases} \{ir+(i-1)a+2i-1, \ldots, ir+ia+2i\} & 1 \leq i \leq a+t-1 \\ \{ir+ia+i+t-1, \ldots, ir+(i+1)a+i+t-1\} & a+t \leq i \leq q-1 \\ \{ir+qa+t+i-1\} & q \leq i \leq q+1 \\ \{ir+(i-2)a+2(i-1)+t-q-1, \ldots, & \\ ir+(i-1)a+2(i-1)+t-q\} & q+2 \leq i \leq q+a+t \\ \{ir+(i-1)a+(i-1)+2t-1, \ldots, & \\ ir+ia+(i-1)+2t-1\} & q+a+t+1 \leq i \leq 2q \end{cases}$$

Let W be the union of the sets W_i, $1 \leq i \leq 2q$ and $A = V(G) - W$. The number of vertices in W is equal to

$$2(a+t-1)(a+2)+2+2(q-a-t)(a+1) = k-1+ka+2t-1 = k-1+s,$$

so $|A| = p-k-s+1 = kr+1$. Now, we can see that for any $1 \leq i \leq 2q$, the elements in W_i differ from those in W_{i+1} by at least $r+1$. Hence, no vertex in W_i is adjacent to a vertex in W_j, $1 \leq i < j \leq 2q$, by an edge in the copy of $H_{2r,p}$ in G. Thus we only need consider edges of the form $\{x, x+\frac{p+1}{2}\}$. In fact, we need to consider only such edges when x is at most $\frac{p-1}{2}$. Hence, since $\frac{p-1}{2} = qr+qa+q+t-1 < (q+1)r+qa+q+t$, we need to consider only vertices in W_i for $1 \leq i \leq q$. So consider $W_i = \{ir+(i-1)a+2i-1, \ldots, ir+ia+2i\}$ for $1 \leq i \leq a+t-1$. Then
$ir+(i-1)a+2i-1+\frac{p+1}{2} = (q+i)r+(q+i-1)a+2(q+i-1)-q+t+1 > (q+i)r+(q+i-1)a+2(q+i-1)-q+t = jr+(j-1)a+2(j-1)-q+t$, for $j = q+i$.

Also, by Lemma 9 and 10, $r-a-1 > 0$, and so

$ir + ia + 2i + \frac{p+1}{2} < (q+i)r + (q+i)a + 2(q+i) - q + t < (q+i+1)r + (q+i-1)a + 2(q+i) + t - q - 1 = (j+1)r + (j-1)a + 2j + t - q - 1.$

Therefore the set $\{ir + (i-1)a + 2i - 1 + \frac{p+1}{2}, \dots, ir + ia + 2i + \frac{p+1}{2}\}$ is strictly between W_j and W_{j+1} for $j = q+i$, and so it is contained in A.

Now, consider $W_i = \{ir+ia+i+t-1, \dots, ir+(i+1)a+i+t-1\}$ for $a+t \le i \le q-1$. Then

$ir+ia+i+t-1+\frac{p+1}{2} = (q+i)r+(q+i)a+(q+i-1)+2t > (q+i)r+(q+i)a+(q+i-1)+2t-1 = jr + ja + j - 1 + 2t - 1$, for $j = q+i$.

Also, by Lemma 10, $a < r$, and so

$ir + (i+1)a + i + t - 1 + \frac{p+1}{2} < (q+i)r + (q+i+1)a + q + i - 1 + 2t < (q+i+1)r + (q+i)a + (q+i) + 2t - 1 = (j+1)r + ja + j + 2t - 1.$

Hence the set $\{ir + ia + i + t - 1 + \frac{p+1}{2}, \dots, ir + (i+1)a + i + t - 1 + \frac{p+1}{2}\}$ is strictly between W_j and W_{j+1} for $j = q+i$, and so it is contained in A.

Finally, consider $W_i = \{ir + qa + t + i - 1\}$ for $i = q$. Then $qr + qa + t + q - 1 + \frac{p+1}{2} = \frac{p-1}{2} + \frac{p+1}{2} = p$, Therefore $\{qr + qa + t + q - 1 + \frac{p+1}{2}\}$ is strictly between W_{2q} and W_1 and so it is contained in A. Therefore the W_i, $1 \le i \le 2q$ are the components of $H_{n,p} - A$, so $\omega(H_{n,p} - A) = k$.

Case2: Suppose $a + t - 1 = z(q-1)$, for some integer z and so $q-1$ divides $a + \frac{b-1}{2}$. Define the sets W_i for $1 \le i \le 2q$ as follows:

$$W_i = \begin{cases} \{ir + (i-1)a + (i-1)z + i, \dots, ir + ia + iz + i\} & 1 \le i \le q-1 \\ \{ir + (q-1)a + (q-1)z + i\} & q \le i \le q+1 \\ \{ir + (i-3)a + (i-3)z + i, \dots, ir + (i-2)a + (i-2)z + i\} & q+2 \le i \le 2q \end{cases}$$

Let W be the union of the sets W_i, $1 \le i \le 2q$, and $A = V(G) - W$. The number of vertices in W is equal to

$2(q-1)(a+z+1) + 2 = k - 1 + ak + b = k - 1 + s,$

so $|A| = m - k - s + 1 = kr + 1$. Now, we can see that for any $1 \le i \le 2q$, the elements in W_i differ from those in W_{i+1} by at least $r + 1$. Hence, no vertex in W_i is adjacent to a vertex in W_j, $1 \le i < j \le 2q$, by an edge in the copy of $H_{p,2r}$, in G. Thus we need only consider edges of the form $\{x, x + \frac{p+1}{2}\}$. In fact, we need to consider only such edges when x is at most $\frac{p-1}{2}$. Hence, since

$$\tfrac{p-1}{2} = qr + (q-1)a + z(q-1) < (q+1)r + (q-1)a + z(q-1) + q + 1.$$

We need to consider only vertices in W_i for $1 \le i \le q$. So consider $\{ir + (i-1)a + (i-1)z + i, \ldots, ir + ia + iz + i\}$ for $1 \le i \le q - 1$. Then

$$ir + (i-1)a + (i-1)z + i + \tfrac{p+1}{2} = (q+i)r + (q+i-2)a + (q+i-2)z + q + i + 1 > (q+i)r + (q+i-2)a + (q+i-2)z + q + i = jr + (j-2)a + (j-2)z + j, \text{ for } j = q + i.$$

Also, since $a + z = a + \frac{a+t-1}{q-1} < a + a + b < ak + b = s \le r$,

$$ir + ia + iz + i + \tfrac{p+1}{2} = (q+i)r + (q+i-1)a + (q+i-1)z + q + i + 1 < (q+i+1)r + (q+i-2)a + (q+i-2)z + q + i + 1 = (j+1)r + (j-2)a + (j-2)z + j + 1.$$

Therefore the set $\{ir + (i-1)a + (i-1)z + i + \frac{p+1}{2}, \ldots, ir + ia + iz + i + \frac{p+1}{2}\}$ is strictly between Wj and W_{j+1} for $j = q + i$, and so it is contained in A.

Finally, consider $\{ir+(q-1)a+(q-1)z+i\}$ for $i = q$. Then $qr+(q-1)a+(q-1)z+q+\frac{p+1}{2} = \frac{p-1}{2} + \frac{p+1}{2} = m$. Hence it is contained in A. Therefore the W_i, $1 \le i \le 2q$, are the components of $H_{p,n} - A$, so $\omega(H_{p,n} - A) = k$.

Case3: Suppose that $a + t - 1 = z(q-1) + c$, for $0 < c < q - 1$ and so $q - 1$ does not divide $a + \frac{b-1}{2}$. Define the sets W_i for $1 \le i \le 2q$ as follows:

$$W_i = \begin{cases} \{ir + (i-1)a + (i-1)z + 2i - 1, \ldots, ir + ia + iz + 2i\} & 1 \le i \le c \\ \{ir + (i-1)a + (i-1)z + i + c, \ldots, ir + ia + iz + i + c\} & c + 1 \le i \le q - 1 \\ \{ir + (q-1)a + (q-1)z + i + c\} & q \le i \le q + 1 \\ \{ir + (i-3)a + (i-3)z + 2i - q + c - 2, \ldots, & \\ ir + (i-2)a + (i-2)z + 2i - q + c - 1\} & q + 2 \le i \le q + c + 1 \\ \{ir + (i-3)a + (i-3)z + i + 2c, \ldots, & \\ ir + (i-2)a + (i-2)z + i + 2c\} & q + c + 2 \le i \le 2q \end{cases}$$

Let W be the union of the sets W_i, $1 \leq i \leq 2q$, and $A = V(G) - W$. The number of vertices in W is equal to

$2c(a+z+2)+2+2(q-c-1)(a+z+1) = k+ka+b-1 = k+s-1$, so $|A| = m-k-s+1 = kr+1$.

Now, we can see that for any $1 \leq i \leq 2q$, the elements in W_i differ from those in W_{i+1} by at least $r+1$. Hence, no vertex in W_i is adjacent to a vertex in W_j, $1 \leq i < j \leq 2q$, by an edge in the copy of $H_{p,2r}$, in G. Thus we need only consider edges of the form $\{x, x+\frac{p+1}{2}\}$. In fact, we need to consider only such edges when x is at most $\frac{p-1}{2}$. Hence, since

$\frac{p-1}{2} = qr + (q-1)a + (q-1)z + q + c < (q+1)r + (q-1)a + (q-1)z + q + c + 1.$

We need to consider only vertices in W_i for $1 \leq i \leq q$. So consider $\{ir + (i-1)a + (i-1)z + 2i - 1, \ldots, ir + ia + iz + 2i\}$ for $1 \leq i \leq c$. Then

$ir + (i-1)a + (i-1)z + 2i + \frac{p+1}{2} = (q+i)r + (q+i-2)a + (q+i-2)z + 2(q+i) - q + c > (q+i)r + (q+i-2)a + (q+i-2)z + 2(q+i) - q + c - 1 = jr + (j-2)a + (j-2)z + 2j - q + c - 1$, for $j = q + i$.

Also, since

$a + z + 1 = a + \frac{a}{q-1} + \frac{t-1}{q-1} - \frac{c}{q-1} + 1 < a + \frac{a}{q-1} + \frac{t-1}{q-1} + 1 \leq a + a + b + 1 \leq ak + b = s \leq r.$

Thus

$ir + ia + iz + 2i + \frac{p+1}{2} = (q+i)r + (q+i-1)a + (q+i-1)z + 2(q+i) - q + c + 1 < (q+i+1)r + (q+i-2)a + (q+i-2)z + 2(q+i+1) - q + c - 2 = (j+1)r + (j-2)a + (j-2)z + 2(j+1) + c - q - 2.$

Therefore the set $\{ir + (i-1)a + (i-1)z + 2i + \frac{p+1}{2}, \ldots, ir + ia + iz + 2i + \frac{p+1}{2}\}$ for $1 \leq i \leq c$ is strictly between Wj and W_{j+1} for $j = q + i$, and so it is contained in A.

Now, consider $\{ir+(i-1)a+(i-1)z+i+c, \ldots, ir+ia+iz+i+c\}$ for $c+1 \leq i \leq q-1$. Then

$ir + (i-1)a + (i-1)z + i + c + \frac{p+1}{2} > (q+i)r + (q+i-2)a + (q+i-2)z + q + i + 2c = jr + (j-2)a + (j-2)z + j + 2c$, for $j = q + i$.

Also, by Lemma 10, $r > a + z$, and so

$ir + ia + iz + i + c + \frac{p+1}{2} < (q+i+1)r + (q+i-2)a + (q+i-2)z + (q+i+1) + 2c =$

$(j+1)r+(j-2)a+(j-2)z+(j+1)+2c$.

Hence the set $\{ir+(i-1)a+(i-1)z+i+c+\frac{p+1}{2},\ldots,ir+ia+iz+i+c+\frac{p+1}{2}\}$ is strictly between W_j and W_{j+1} for $j=q+i$ and so it is contained in A.

Finally, consider $\{ir+(q-1)a+(q-1)z+i+c\}$ for $i=q$. Then $qr+(q-1)a+(q-1)z+q+c+\frac{p+1}{2}=\frac{p-1}{2}+\frac{p+1}{2}=p$. Hence it is contained in A. Therefore the W_i, $1\le i\le 2q$, are the components of $H_{p,n}-A$ so $\omega(H_{p,n}-A)=k$.

Finally, we have the following theorem.

Theorem 6: Let $H_{n,p}$ be the Harary graph with $n=2r+1$, p odd, k even, $k>2$, $1<s<r+1$ and $s>k$, where $s=ak+b$ for some a and b, $0<b<k$. Then $r\le t(H_{n,p})=r+\frac{1}{k}$.

Proof: By Theorem 1, we have $r\le t(H_{n,p})$. Also by Lemma 11, there is a cut-set A of $H_{n,p}$ with $kr+1$ elements. The number of components of $H_{n,p}-A$ is k. Hence $r\le\frac{|A|}{\omega(H_{n,p}-A)}=r+\frac{1}{k}$, and the theorem follows.

Conclusion: We can summarize what we have proved about the Harery graphs as follows, where $n=2r$ or $n=2r+1$ and $p=k(r+1)+s$ for $0\le s<r+1$:
If n is even, then $t(H_{n,p})=r$
If n is odd, p even, k odd, then $t(H_{n,p})=r$
If n is odd, p and k both even, then $t(H_{n,p})=r+\frac{1}{k}$
If n is odd, p and k both odd, then $t(H_{n,p})=r$
If n is odd, p odd, k even, then $t(H_{n,p})=r+\frac{1}{k}$

References

[1] J.A. Bondy and U.S.R. Murty, "Graph Theory", Graduate Text in Mathematics, Springer, 2008.

[2] V. *Chvátal*, Tough graphs and hamiltonian circuits. Discrete Mathematics, 5, 215-228 (1973).

[3] F. Harary, The maximum connectivity of a graph, Proc. Nat. Acad. Sci., 48 (1962), 1142-1146.

[4] D. Moazzami and B. Bafandeh Mayvan, " Toughness of a Harary graph ", submited.