**Journal of Algorithms and Computation**

journal homepage: http://jac.ut.ac.ir

# Two different inverse eigenvalue problems for nonsymmetric tridiagonal matrices

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**ABSTRACT**

Inverse eigenvalue problems (IEPs) of tridiagonal matrices are among the most popular IEPs, this is due to the widespread application of this matrix. In this paper, two different IEPs with different eigen information including eigenvalues and eigenvectors are presented on the nonsymmetric tridiagonal matrix. A recursive relation of characteristic polynomials of the leading principal submatrices of the required matrix is presented to solve the problems. The application of the problems in graph and perturbation theory is studied. The necessary and sufficient conditions for solvability of the problems are obtained. The algorithms and numerical examples are given to show the applicability of the proposed scheme.

**ARTICLE INFO**

*Article history:*
Received 26, August 2019
Received in revised form 14, October 2020
Accepted 11 November 2020
Available online 30, December 2020
Research paper

*Keyword:* Inverse eigenvalue problem, Tridiagonal matrix, Principal submatrix.

AMS subject Classification: 65F18, 65F15.

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# 1 Introduction

Inverse eigenvalue problems (IEPs) have received much attention in a wide range and different case of studies. The following matrix, denoted by Jacobi matrix

$$J_n = \begin{bmatrix} a_1 & b_1 & 0 & \dots & 0 \\ b_1 & a_2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{n-1} \\ 0 & \dots & 0 & b_{n-1} & a_n \end{bmatrix}, \tag{1}$$

has very important role in variety of applications, for this reason, different IEPs with different eigen information of this matrix are investigated.
For example, we can mention works done by [4, 7, 9, 12, 13, 14, 18, 19, 21]. In these papers, different applications of Jacobi matrices in Finite element, Schrodinger equation, Signal processing and nonlinear Control theory are studied. In [2, 3, 5, 11, 6, 10] the relations between IEPs and graphs are studied. A widely used method to solve IEPs, is recurrence relations. This method has been studied for some graphs in [1, 8, 15, 16, 17]. In [20], the construction of block matrices containing Jacobi matrices has been discussed in detail.
In this work, we generalized the matrix discussed in mentioned works to asymmetric tridiagonal matrix which is as follows:

$$A_n = \begin{bmatrix} a_1 & b_1 & 0 & \dots & 0 \\ c_1 & a_2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{n-1} \\ 0 & \dots & 0 & c_{n-1} & a_n \end{bmatrix}. \tag{2}$$

In this case, we can implement the perturbation and directed graph problems on this matrix.
In the sequel, in section 2, we introduce the problem statement and some of the preliminary concepts that will be used throughout the paper. In section 3, main results are discussed. Section 4 presents some examples to illustrate the efficiency of the proposed scheme. And finally, in section 5, we conclude the paper.

# 2 Preliminaries

The graph $G$ is called a weighted graph, if each edge of graph $G$ has an associated numerical value, called a weight. A graph is connected when there is a path between every pair of vertices. Let $G$ be a weighted connected directed graph with vertices $\{v_1, ..., v_n\}$,

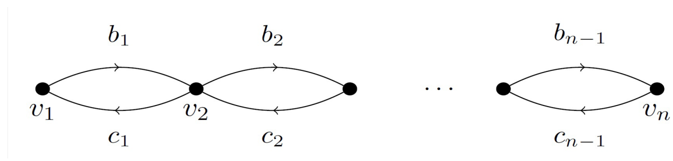


Figure 1: A weighted directed path graph

and edge set $\{e_1, ..., e_{2n}\}$. Figure 1 shows a weighted directed path graph with weights $b_1, ..., b_{n-1}, c_1, ..., c_{n-1}$.
The matrix of the weighted graph $G$ with $n$ vertices is an $n \times n$ square matrix $A_n$, such that the entry $a_{ij}$ is the weight of the edge between vertices $v_i$ and $v_j$ and $a_{ij} = 0$ if there is no edge between them. A nonsymmetric tridiagonal matrix $A_n$ which corresponds to a directed path graph is given by Eq.(2), such that the diagonal entries are real and $b_i, c_i \in \mathbb{R} - \{0\}, i = 1, ..., n-1$. If the graph $G$ is directed, in the general form, then $A_n$ is nonsymmetric.
The characteristic polynomial of matrix $A_{n\times n}$ is

$$P_n(\lambda) = \det(A_n - \lambda I_n), \tag{3}$$

where $\lambda$ is a scalar and $I_n$ is an $n \times n$ identity matrix.
We denote the $i$-th leading principal submatrix of $A_n$ by $A_i$, and eigenvalues of $A_i$ by $\lambda_1^{(i)}, \lambda_2^{(i)}, ..., \lambda_i^{(i)}$, and $X_n = (x_1, ..., x_n)^T$ is an eigenvector of $A_n$. The symbol $\sigma(A_n)$ denotes the collection of all eigenvalues of $A_n$.
We define principal backward submatrices of $A_n$ by

$$\widetilde{A}_i = \begin{bmatrix} a_{n-i+1} & b_{n-i+1} & 0 & ... & 0 \\ c_{n-i+1} & a_{n-i+2} & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{n-1} \\ 0 & ... & 0 & c_{n-1} & a_n \end{bmatrix}, i = 1, ..., n, \tag{4}$$

and

$$Q_i(\lambda) = \det(\widetilde{A}_i - \lambda I_i), i = 1, ..., n. \tag{5}$$

We will solve the following two IEPs for nonsymmetric tridiagonal matrix that we briefly call them IEPNT [1].
**problem 1.**(IEPNT1). Given $n$ real distinct numbers $\lambda^{(1)}$,$\lambda^{(2)}$,...,$\lambda^{(n)}$, $n-1$ small real numbers $\epsilon_1, ..., \epsilon_{n-1}$,$|\epsilon_i| \leqslant 1$ and a real vector $X_n = (x_1, ..., x_n)^T$, find matrix $A_n$ such that $\lambda^{(i)}, i = 1, ..., n$ is an eigenvalue of $A_i$ ,$c_i = \epsilon_i b_i$, and $(\lambda^{(n)}, X_n)$ is an eigenpair of $A_n$.
**problem 2.**(IEPNT2). Given $2n-1$ real pairwise distinct numbers $\lambda_1^{(1)}$,$\lambda_1^{(2)}$, $\lambda_2^{(2)}$,...,$\lambda_1^{(n)}$, $\lambda_2^{(n)}$, and a real vector $X_n = (x_1, ..., x_n)^T$, find matrix $A_n$ such that $\lambda_1^{(i)}, \lambda_2^{(i)}, i = 1, ..., n$ are eigenvalues of $A_i$ , and $(\lambda_1^{(n)}, X_n)$ is an eigenpair of $A_n$.

[1] Inverse Eigenvalue Problem for Nonsymmetric Tridiagonal Matrix

# 3 Main idea

In this section, we will present necessary proofs to obtain the solution of the problems IEPNT1 and IEPNT2.
In the following lemma, the recurrence relation of $P_j(\lambda), j = 1, ..., n$ is introduced.
**Lemma 3.1** The characteristic polynomial of the matrix $A_n$ and principal submatrices $A_j$ satisfy the following recurrence relations:

$$\begin{aligned} P_0(\lambda) &= 1, \\ P_1(\lambda) &= a_1 - \lambda, \\ P_{j+1}(\lambda) &= (a_{j+1} - \lambda)P_j(\lambda) - b_j c_j P_{j-1}(\lambda), j = 1, ..., n-1. \end{aligned} \tag{6}$$

**proof** It is achieved by expanding the determinant. □
**Corollary 3.2.**In the same way we can show

$$\begin{aligned} Q_0(\lambda) &= 1 \\ Q_1(\lambda) &= a_n - \lambda, \\ Q_{i+1}(\lambda) &= (a_{n-i} - \lambda)Q_i(\lambda) - b_{n-i}c_{n-i}Q_{i-1}(\lambda), i = 1, ..., n-1. \end{aligned} \tag{7}$$

**Lemma 3.3.** For any real nonzero number $\lambda$, $P_j(\lambda)$ and $P_{j+1}(\lambda), j = 1, ..., n-1$ can not be simultaneously zero. **proof** From Lemma 3, if $P_1(\lambda) = P_2(\lambda) = 0$ then $b_1c_1 = 0$ that contradicts the properties of $A_n$. For $1 \leqslant j \leqslant n-1$, if $P_j(\lambda) = P_{j+1}(\lambda) = 0$ then the recurrence relation (6) $P_{j-1} = 0$, by continuing this way $P_2(\lambda) = 0$ which implies that $b_1 = 0$ or $c_1 = 0$ and it is a contradiction. □

## 3.1 The solution of IEPNT1

In this section, we will prove the conditions for the solution to IEPNT1. **Theorem 3.4.** The IEPNT1 has a unique solution if and only if $x_i \neq 0, i = 1, ..., n$. **proof** Let $x_i \neq 0, i = 1, ..., n$. It is easy to see that $a_1 = \lambda^{(1)}$. Since $(\lambda^{(n)}, X_n)$ is an eigenpair of $A_n$, we obtain $(a_1 - \lambda^{(n)})x_1 + b_1x_2 = 0$ or $b_1 = \frac{(\lambda^{(n)} - a_1)}{x_2}x_1$. We note that $a_1 = \lambda^{(1)} \neq \lambda^{(n)}$, $x_i \neq 0, i = 1, ..., n$, hence $b_1$ as a result $c_1 = \epsilon_1 b_1$ are always nonzero.
If $x_1 \neq 0$, we can show that every component of $X_n$ is multiplier of $x_1$. Since $(\lambda^{(n)}, X_n)$ is an eigenpair of $A_n$, so $A_nX_n = \lambda^{(n)}X_n$. By expanding this relation:

$$\begin{aligned} a_1x_1 + b_1x_2 &= \lambda^{(n)}x_1, \\ c_{i-1}x_{i-1} + a_ix_i + b_ix_{i+1} &= \lambda^{(n)}x_i, i = 2, ..., n-1, \\ c_{n-1}x_{n-1} + a_nx_n &= \lambda^{(n)}x_n, \end{aligned}$$

and applying $P_1(\lambda^{(n)})$ yields

$$x_2 = \frac{(\lambda^{(n)} - a_1)}{b_1}x_1 = \frac{-P_1(\lambda^{(n)})}{b_1}x_1.$$

This can be concluded

$$x_i = \frac{(-1)^{i-1} P_{i-1}(\lambda^{(n)})}{\prod\limits_{j=1}^{i-1} b_j} x_1, i = 2, ...n. \tag{8}$$

The Eq.(8) can be verified by induction on $x_i$. Let it to be true for $x_1, ..., x_j$, we prove it for $x_{j+1}$. From Eq.(6) we get

$$\begin{aligned} x_{i+1} &= \frac{1}{b_i}((\lambda^{(n)} - a_i)x_i - c_{i-1}x_{i-1}) \\ &= \frac{1}{b_i}\left[(\lambda^{(n)} - a_i)\frac{(-1)^{i-1}P_{i-1}(\lambda^{(n)})}{\prod\limits_{j=1}^{i-1} b_j}x_1 - c_{i-1}\frac{(-1)^{i-2}P_{i-2}(\lambda^{(n)})}{\prod\limits_{j=1}^{i-2} b_j}x_1\right] \\ &= \frac{(-1)^i P_i(\lambda^{(n)})}{\prod\limits_{j=1}^{i} b_j}x_1. \end{aligned}$$

Therefore

$$b_{i-1} = \frac{(-1)^{i-1}P_{i-1}(\lambda^{(n)})x_1}{x_i \prod\limits_{j=1}^{i-2} b_j}, i = 3, ..., n, \tag{9}$$

so, every component of eigenvector $X_n$ is a multiplier of $x_1$. The distinctness of $\lambda^{(i)}$'s results in $P_{i-1}(\lambda^{(n)}) \neq 0$, therefore all of $b_i$'s and $c_i$'s are nonzero. We note that similar to Eq.(8), we can show that if $(\lambda^{(n)}, X_n)$ is an eigenpair of $A_n$, then $x_n \neq 0$ and components of eigenvector $X_n$ are obtained as

$$x_i = \frac{(-1)^{n-i}Q_{n-i}(\lambda^{(n)})}{\prod\limits_{j=1}^{n-i} c_{n-j}} x_n \quad , \quad i = 1, ..., n-1, \tag{10}$$

or

$$c_{i-1} = \frac{(-1)^{n-i+1}Q_{n-i+1}(\lambda^{(n)})x_n}{x_{i-1} \prod\limits_{j=1}^{n-(i-2)} c_{n-j}}, i = 2, ..., n. \tag{11}$$

This means that every component of $X_n$ is also a multiplier of $x_n$ and we can compute the solution in terms of $c_{i-1}$.
To obtain $a_i$, since $\lambda^{(i)}$ is eigenvalue of $A_i$, so for $i = 2, ..., n$ one has

$$P_i(\lambda^{(i)}) = (a_i - \lambda^{(i)})P_{i-1}(\lambda^{(i)}) - b_{i-1}c_{i-1}P_{i-2}(\lambda^{(i)}) = 0, \tag{12}$$

and so

$$a_i = \frac{\epsilon_{i-1} b_{i-1}^2 P_{i-2}(\lambda^{(i)})}{P_{i-1}(\lambda^{(i)})} + \lambda^{(i)}. \tag{13}$$

From by Lemma (3), $P_{i-1}(\lambda^{(i)}) \neq 0$ and so $a_i$ is always valid. It is clear that solution is unique.

Conversely, let there exist a solution to IEPNT1 and $X_n$ be an eigenvector of $A_n$, this means that $b_i$'s and $c_i$'s are nonzero. Since $(\lambda^{(n)}, X_n)$ is an eigenpair, by Eq.s (8) and (10), $x_i \neq 0, i = 1, \cdots, n$ which completes the proof. □

The following algorithm is presented to solve IEPNT1.

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**Algorithm 1** ( Solving IEPNT1)

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1: Input: Distinct real numbers $\lambda^{(1)}$, $\lambda^{(2)}$,...,$\lambda^{(n)}$,
$\epsilon_1, ..., \epsilon_{n-1}, |\epsilon_i| \leqslant 1, i = 1, \cdots, n-1$,
real vector $X_n = (x_1, ..., x_n)^T$,

2: If $x_i = 0, i = 1, \cdots, n$, then problem can not be solved by this algorithm, stop.

3: Set $a_1 = \lambda^{(1)}$.

4: Set
$$P_0(\lambda) = 1,$$
$$P_1(\lambda) = a_1 - \lambda.$$

5: Set
$$b_1 = \frac{(\lambda^{(n)} - a_1)}{x_2} x_1,$$
$$c_1 = \epsilon_1 b_1,$$
$$a_2 = \frac{\epsilon_1 b_1^2}{P_1(\lambda^{(2)})} + \lambda^{(2)},$$
$$P_2(\lambda) = (a_2 - \lambda) P_1(\lambda) - b_1 c_1.$$

6: For $i = 3, ..., n$
$$b_{i-1} = \frac{(-1)^{i-1} P_{i-1}(\lambda^{(n)}) x_1}{x_i \prod_{j=0}^{i-2} b_j},$$
$$c_{i-1} = \epsilon_{i-1} b_{i-1},$$
$$a_i = \frac{\epsilon_{i-1} b_{i-1}^2 P_{i-2}(\lambda^{(i)})}{P_{i-1}(\lambda^{(i)})} + \lambda^{(i)},$$
$$P_i(\lambda) = (a_i - \lambda) P_{i-1}(\lambda) - b_{i-1} c_{i-1} P_{i-2}(\lambda).$$
End For.

7: Output: $A_n$.

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## 3.2 The solution of IEPNT2

**Theorem 3.5.** The IEPNT2 has a unique solution if and only if $x_i \neq 0, i = 1, ..., n$.
**proof** Parts of the proof are the same as proof of Theorem (3.1). If $x_i \neq 0, i = 1, ..., n$,

with a similar argument $a_1 = \lambda_1^{(1)}$, $b_1 = \frac{(\lambda_1^{(n)} - a_1)}{x_2} x_1$ , $b_1$ is always nonzero and

$$b_{i-1} = \frac{(-1)^{i-1} P_{i-1}(\lambda_1^{(n)}) x_1}{x_i \prod_{j=1}^{i-2} b_j}, i = 3, ..., n,$$

$$c_{i-1} = \frac{(-1)^{n-i+1} Q_{n-i+1}(\lambda_1^{(n)}) x_n}{x_{i-1} \prod_{j=1}^{n-(i-2)} c_{n-j}}, i = 2, ..., n.$$

By the assumption of this theorem, $x_i \neq 0, i = 1, ..., n,$, hence $P_{i-1}(\lambda_1^{(n)})$ and $Q_{n-i+1}(\lambda_1^{(n)})$ are nonzero as a result $b_{i-1}, c_{i-1}$ are nonzero. To obtain $b_{i-1}, c_{i-1}, a_i$ one has

$$P_i(\lambda_1^{(i)}) = (a_i - \lambda_1^{(i)}) P_{i-1}(\lambda_1^{(i)}) - b_{i-1} c_{i-1} P_{i-2}(\lambda_1^{(i)}) = 0, \tag{14}$$

$$P_i(\lambda_2^{(i)}) = (a_i - \lambda_2^{(i)}) P_{i-1}(\lambda_2^{(i)}) - b_{i-1} c_{i-1} P_{i-2}(\lambda_2^{(i)}) = 0, \tag{15}$$

from Eq.(14) and substitute $b_{i-1}$ from Eq.(9) we obtain

$$c_{i-1} = \frac{(-1)^{i-1} (a_i - \lambda_1^{(i)}) P_{i-1}(\lambda_1^{(i)}) x_i \prod_{j=1}^{i-2} b_j}{x_1 P_{i-1}(\lambda_1^{(n)}) P_{i-2}(\lambda_1^{(i)})}, \tag{16}$$

from Eqs. (14), (15),(16) and (9) we have

$$a_i = \frac{\lambda_2^{(i)} P_{i-2}(\lambda_1^{(i)}) P_{i-1}(\lambda_2^{(i)}) - \lambda_1^{(i)} P_{i-1}(\lambda_1^{(i)}) P_{i-2}(\lambda_2^{(i)})}{P_{i-2}(\lambda_1^{(i)}) P_{i-1}(\lambda_2^{(i)}) - P_{i-1}(\lambda_1^{(i)}) P_{i-2}(\lambda_2^{(i)})}, \tag{17}$$

and it has a unique solution if $P_{i-2}(\lambda_1^{(i)}) P_{i-1}(\lambda_2^{(i)}) - P_{i-1}(\lambda_1^{(i)}) P_{i-2}(\lambda_2^{(i)}) \neq 0$. By substituting $a_i$ from Eq.(17) in Eq.(16) we obtain $c_i$.
Conversely, let IEPNT2 has a solution with unique values of the entries of $A_n$ and $X_n$ be an eigenvector of $A_n$. With similar argument to the proof of Theorem (3.1), $x_1$ and $x_n$ are nonzero. As a result, by Eq.s (9) and (11) we obtain that $x_i \neq 0, i = 1, ..., n$ which completes the proof. □
The following algorithm is presented to solve IEPNT2.

**Algorithm 2** ( Solving IEPNT2)

1: Input: Pairwise distinct real numbers $\lambda_1^{(1)}$, $\lambda_1^{(2)}$, $\lambda_2^{(2)}$,...,$\lambda_1^{(n)}$,$\lambda_2^{(n)}$,
real vector $X_n = (x_1, ..., x_n)^T$.

2: If for an index $i, x_i = 0, i = 1, \cdots, n$ then problem can not be solved by this algorithm, stop.

3: Set $a_1 = \lambda_1^{(1)}, b_1 = \frac{(\lambda_1^{(n)} - a_1)}{x_2} x_1$.

4: Set

$$P_0(\lambda) = 1,$$
$$P_1(\lambda) = a_1 - \lambda.$$

5: Set

$$a_2 = \frac{\lambda_2^{(2)} P_1(\lambda_2^{(2)}) - \lambda_1^{(2)} P_1(\lambda_1^{(i)})}{\lambda_1^{(2)} - \lambda_2^{(2)}},$$
$$c_1 = \frac{(a_2 - \lambda_1^{(2)}) P_1(\lambda_1^{(2)})}{b_1},$$
$$P_2(\lambda) = (a_2 - \lambda) P_1(\lambda) - b_1 c_1.$$

6: For $i = 3, ..., n$

$$b_{i-1} = \frac{(-1)^{i-1} P_{i-1}(\lambda_1^{(n)}) x_1}{x_i \prod_{j=0}^{i-2} b_j},$$

If $P_{i-2}(\lambda_1^{(i)}) P_{i-1}(\lambda_2^{(i)}) - P_{i-1}(\lambda_1^{(i)}) P_{i-2}(\lambda_2^{(i)}) \neq 0$

Set

$$a_i = \frac{\lambda_2^{(i)} P_{i-2}(\lambda_1^{(i)}) P_{i-1}(\lambda_2^{(i)}) - \lambda_1^{(i)} P_{i-1}(\lambda_1^{(i)}) P_{i-2}(\lambda_2^{(i)})}{P_{i-2}(\lambda_1^{(i)}) P_{i-1}(\lambda_2^{(i)}) - P_{i-1}(\lambda_1^{(i)}) P_{i-2}(\lambda_2^{(i)})},$$
$$c_{i-1} = \frac{(-1)^{i-1} (a_i - \lambda_1^{(i)}) P_{i-1}(\lambda_1^{(i)}) x_i \prod_{j=1}^{i-1} b_j}{x_1 P_{i-1}(\lambda_1^{(n)}) P_{i-2}(\lambda_1^{(i)})},$$

Set $P_i(\lambda) = (a_i - \lambda) P_{i-1}(\lambda) - b_{i-1} c_{i-1} P_{i-2}(\lambda)$.

else, Stop.

"The problem can not be solved by this algorithm."

End For.

7: Output: $A_n$.

# 4 Numerical Experiments

In this section, we present the numerical results of IEPNT1and IEPNT2. The computational results are provided by MATLAB software. **Example 4.1.**The real distinct numbers
$\lambda^{(1)} = 3, \lambda^{(2)} = 4, \lambda^{(3)} = 1, \lambda^{(4)} = 6.1, \lambda^{(5)} = 3.5, \lambda^{(6)} = 10, \lambda^{(7)} = 4.5, \lambda^{(8)} = 12, \lambda^{(9)} = 5, \lambda^{(10)} = 9$
$\epsilon_1 = 0.0570,\ \epsilon_2 = 0.5860, \epsilon_3 = 0.4900, \epsilon_4 = 0.1110, \epsilon_5 = 0.0180, \epsilon_6 = 0.1120, \epsilon_7 = 0.2210,$
$\epsilon_8 = -0.1720, \epsilon_9 = 0.0710, \epsilon_{10} = 0.0430$

and vector

$$X_{10} = (0.1910,\ 0.2866,\ \text{-}\ 0.0955,\ 0.3821, 0.1433, 0.3343, 0.4776,\ 0.3057,\ 0.2961,\ 0.4298)^T$$

are given. We make $A_{10}$, such that $(\lambda^{(10)}, X_{10})$ be an eigenpair of $A_{10}$.
By Algorithm 1, we get:

$$A_{10} = \begin{bmatrix} 3.0000 & 3.9986 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.2279 & 3.0886 & -\ 17.2844 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\ 10.1287 & 108.2097 & 32.3931 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 15.8726 & 9.1874 & 10.0783 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.1187 & 3.8492 & 0.9293 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1673 & 9.9599 & -\ 0.7221 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\ 0.0809 & 4.5106 & 7.1023 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.5696 & 10.5058 & -\ 4.0863 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.7028 & 4.8965 & 2.3271 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.1652 & 8.8862 \end{bmatrix}$$

We compute the spectra of all of the principal submatrices of $A_{10}$ to verify the results.

$\sigma(A_{10}) = \{\mathbf{9.0000},\ \text{-}\ 1.0626,\ 2.1463,\ 11.6420, 114.6586,\ 9.9995,\ 8.0560,\ 3.4968,\ 3.2741,\ 4.8834\}$,
$\sigma(A_9) = \{\mathbf{5.0000},\ \text{-}\ 1.0626,\ 11.6497,\ 114.6586,\ 9.9997,\ 8.0560,\ 2.1463,\ 3.4968,\ 3.2633\}$,
$\sigma(A_8) = \{\mathbf{12.0000},\ \text{-}\ 1.0626,\ 8.0561,\ 2.1463,\ 3.4967,\ 3.0142,\ 10.0021, 114.6586\}$,
$\sigma(A_7) = \{\mathbf{4.5000},\ 114.6586,\ 10.0105,\ 8.0562,\ \text{-}\ 1.0626,\ 2.1463,\ 3.4966\}$,
$\sigma(A_6) = \{\mathbf{10.0000},\ 8.0560,\ 114.6586,\ \text{-}\ 1.0626,\ 2.1463,\ 3.4966\}$,
$\sigma(A_5) = \{\mathbf{3.5000},\ 114.6586,\ 8.0831,\ \text{-}\ 1.0605,\ 2.1537\}$,
$\sigma(A_4) = \{\mathbf{6.1000}, 114.6542,\ \text{-}\ 0.3751,\ 3.1067\}$,
$\sigma(A_3) = \{\mathbf{1.0000},\ 3.4487,\ 109.8497\}$,
$\sigma(A_2) = \{\mathbf{4.0000},\ 2.0886\}$,
$\sigma(A_1) = \{\mathbf{3.0000}\}$.

**Example 4.2.** The real distinct numbers
$\lambda_1^{(1)} = 3, \lambda_1^{(2)} = 4, \lambda_2^{(2)} = 1, \lambda_1^{(3)} = 6.2, \lambda_2^{(3)} = 3.4, \lambda_1^{(4)} = 10.6, \lambda_2^{(4)} = 4.1,$
$\lambda_1^{(5)} = 12.6\, \lambda_2^{(5)} = 5\lambda_1^{(6)} = 3.2, \lambda_2^{(6)} = 7, \lambda_1^{(7)} = 8, \lambda_2^{(7)} = 12, \lambda_1^{(8)} = 6,$
$\lambda_2^{(8)} = 11, \lambda_1^{(9)} = 5.7,\ \lambda_2^{(9)} = 4.3, \lambda_1^{(10)} = 4.9,\ \lambda_2^{(10)} = 6.7$, and vector

$$X_{10} = (0.1910,\ 0.2866,\ \text{-}\ 0.0955,\ 0.3821, 0.1433, 0.3343, 0.4776,\ 0.3057,\ 0.2961,\ 0.4298)^T$$

are given. We make $A_{10}$, such that $(\lambda^{(10)}, X_{10})$ be an eigenpair of $A_{10}$.
By Algorithm 2, we get:

$$A_{10} = \begin{bmatrix} 3.0000 & 3.9986 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5002 & 2.0000 & -\ 7.7027 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\ 0.6520 & 4.7951 & 0.4629 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 57.0344 & 5.5257 & 36.3414 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0631 & 1.4623 & 0.2585 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\ 1.3369 & 7.0478 & -\ 1.1022 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4.4735 & 13.0106 & -\ 17.5632 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.2309 & 5.6657 & -\ 1.1629 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4794 & 3.9818 & 0.2915 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\ 3.1497 & 6.9727 \end{bmatrix}$$

We compute the spectra of all of the principal submatrices of $A_{10}$ to verify the results.

$\sigma(A_{10}) = \{\mathbf{4.9000}, \mathbf{6.7000}$, - 4.3153, 0.3270,2.5983,11.0194,3.1989,8.6365,5.0680,5.3289,$\}$,
$\sigma(A_9) = \{\mathbf{5.7000}, \mathbf{4.3000}$, - 4.3153,0.3270,2.5983,11.0200,8.6323,3.1989,5.0278,$\}$,
$\sigma(A_8) = \{\mathbf{11.0000}, \mathbf{6.0000}$, - 4.3153,0.3270,12.5983,3.1990,8.6734,5.0249,$\}$,
$\sigma(A_7) = \{\mathbf{8.0000}, \mathbf{12.0000}$, - 4.3153,0.3269,3.1988,5.0223,12.6088,$\}$,
$\sigma(A_6) = \{\mathbf{3.2000}, \mathbf{7.0000}$, 12.5894, - 4.3150,0.3272,5.0293,$\}$,
$\sigma(A_5) = \{\mathbf{12.6000}, \mathbf{5.0000}$, 3.1896, 0.3220, - 4.3285,$\}$,
$\sigma(A_4) = \{\mathbf{4.1000}, \mathbf{10.6000}$, - 1.1049, 1.7257,$\}$,
$\sigma(A_3) = \{\mathbf{3.4000}, \mathbf{6.2000}, 0.1951\}$,
$\sigma(A_2) = \{\mathbf{4.0000}, \mathbf{1.0000}\}$,
$\sigma(A_1) = \{\mathbf{3.0000}\}$.

# 5 Conclusion

Two different IEPs for asymmetric tridiagonal matrices were studied. The recurrence relation of the leading principal minors and the relation of obtaining the component $x_i$ of the given eigenvector $X_n$ from the entries of leading principal minor $A_i$, are central in obtaining the solution. In the first IEP, assuming $c_i = e_i b_i, 0 < e_i \leqslant 1$, the system considered by [13] is exposed by perturbation. In Theorem (3.1), the conditions of solvability of the first problem is obtained and its solution is formulated as an algorithm in Algorithm (1). The same problem was investigated by Qifang in [14]. Qifang applied simpler conditions on the problem by considering complex data. More precisely, it suffices to consider the Theorem (3.1) to be applicable in Schrdinger equations and complex data.
In the second IEP, we discuss the more general case of the first IEP. In this IEP, $c_i$'s are not necessarily a multiple of $b_i$ and can be any arbitrary nonzero number. In this case, it is applicable for the weighted directed graphs. In Theorem (3.2), the conditions under which this problem has a solution are obtained and we also drive a numerical algorithm for its solution. This problem applies to matrices whose entire entire spectral information is unattainable.

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