Further results on total mean cordial labeling of graphs

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ABSTRACT

A graph $G = (V, E)$ with $p$ vertices and $q$ edges is said to be a total mean cordial graph if there exists a function $f : V(G) \rightarrow \{0, 1, 2\}$ such that $f(xy) = \left\lceil \frac{f(x) + f(y)}{2} \right\rceil$ where $x, y \in V(G)$, $xy \in E(G)$, and the total number of 0, 1 and 2 are balanced. That is $|ev_f(i) - ev_f(j)| \leq 1$, $i, j \in \{0, 1, 2\}$ where $ev_f(x)$ denotes the total number of vertices and edges labeled with $x$ ($x = 0, 1, 2$). In this paper, we investigate the total mean cordial labeling of $C_n^2$, ladder $L_n$, book $B_m$ and some more graphs.

Article history:
Received 28, September 2015
Received in revised form 20, January 2016
Accepted 8, March 2016
Available online 18, March 2016

Keyword: cycle, path, union of graphs, star, ladder

AMS subject Classification: Primary 05C78,

1 Introduction

Graphs considered here are finite, undirected and simple. The symbols $V(G)$ and $E(G)$ will denote the vertex set and edge set of a graph $G$. The order and size of a graph $G$ are respectively denoted by $p$ and $q$. Origin of graph labeling is graceful labeling which was introduced by Rosa [8] in 1967. In 1980, Cahit [4] introduced the cordial labeling of graphs. Cordial labeling behavior of several graphs were studied in [10]. Somasundaram and Ponraj introduced the notion of mean labeling of graphs in [11]. Also in [1, 2, 3, 9], mean labeling behavior of several graphs were studied. Motivated by this labeling,
Ponraj, Ramasamy and Sathish Narayanan [7] introduced the concept of total mean cordial labeling of graphs and studied about the total mean cordial labeling behavior of path, cycle, wheel and some more standard graphs.

In this paper we investigate $C^{(2)}_n$, ladder $L_n$, book $B_m$ and some more graphs. If $x$ is any real number. Then the symbol $\lfloor x \rfloor$ stands for the largest integer less than or equal to $x$ and $\lceil x \rceil$ stands for the smallest integer greater than or equal to $x$. Terms and definitions not defined here are used in the sense of Harary [6] and Gallian [5].

2 Total mean cordial labeling

Definition 2.1. A total mean cordial labeling of a graph $G = (V, E)$ is a function $f : V(G) \rightarrow \{0, 1, 2\}$ such that $f(xy) = \lceil \frac{f(x) + f(y)}{2} \rceil$ where $x, y \in V(G)$, $xy \in E(G)$, and the total number of 0, 1 and 2 are balanced. That is $|ev_f(i) - ev_f(j)| \leq 1$, $i, j \in \{0, 1, 2\}$ where $ev_f(x)$ denotes the total number of vertices and edges labeled with $x$ ($x = 0, 1, 2$).

If there exists a total mean cordial labeling on a graph $G$, we will call $G$ is total mean cordial.

The following results are frequently used in the subsequent section.

Theorem 2.1. [7] Any path $P_n$ is total mean cordial.

The above theorem 2.1 is used in theorem 3.1. So we recall the structure of the total mean cordial labeling $g$ of a path.

Let $P_m : v_1, v_2, \ldots, v_m$ be the path. When $m = 3t$. Assign the label 2 to the vertices $v_1, v_2, \ldots, v_t$ and assign the integer 1 to the vertices $v_{t+1}, v_{t+2}, \ldots, v_{2t}$. Finally assign the label 0 to the vertices $v_{2t+1}, v_{2t+2}, \ldots, v_{3t}$.

In the case of $m = 3t + 1$, assign the label to the vertices $v_i$ ($1 \leq i \leq 3t$) as in the case of $m = 3t$. Then assign 0 to the vertex $v_{3t+1}$.

For $m = 3t + 2$, we define a map $g : V(P_m) \rightarrow \{0, 1, 2\}$ by

$$
\begin{align*}
g(v_i) &= 0, \quad 1 \leq i \leq t + 1 \\
g(v_{t+1+i}) &= 1, \quad 1 \leq i \leq t \\
g(v_{2t+1+i}) &= 2, \quad 1 \leq i \leq t \\
g(v_{3t+2}) &= 1.
\end{align*}
$$

Theorem 2.2. [7] The cycle $C_n$ is total mean cordial if and only if $n \neq 3$.

The theorem 3.2 is based on theorem 2.2. So we recall the total mean cordial labeling of a cycle $C_n$. Let $C_n : u_1u_2 \ldots u_nu_1$ be the cycle. If $n = 3$, then we have $ev_f(0) = ev_f(1) = ev_f(2) = 2$. But this is an impossible one. Assume $n > 3$. When $n \equiv 0 \ (mod \ 3)$, let
n = 3t, t ∈ ℤ⁺. It is easy to see that $C_6$ is total mean cordial. Take $t ≥ 3$. Define $f : V(C_n) → \{0, 1, 2\}$ by

$$
\begin{align*}
f(u_i) &= 0, \quad 1 ≤ i ≤ t \\
f(u_{t+i}) &= 2, \quad 1 ≤ i ≤ t \\
f(u_{2t+i}) &= 1, \quad 1 ≤ i ≤ t - 2.
\end{align*}
$$

$f(u_{3t-1}) = 0$ and $f(u_{3t}) = 1$. When $n ≡ 1, 2 (mod 3)$, The labeling $g$ defined in theorem 2.1 satisfy total mean cordial condition of $C_n$.

## 3 Main Results

The union of two graphs $G_1$ and $G_2$ is the graph $G_1 ∪ G_2$ with $V(G_1 ∪ G_2) = V(G_1) ∪ V(G_2)$ and $E(G_1 ∪ G_2) = E(G_1) ∪ E(G_2)$.

**Theorem 3.1.** Let $G$ be a $(p, q)$ total mean cordial graph and $n ≠ 3$ then $G ∪ P_n$ is total mean cordial.

**Proof.** Let $v_1, v_2, . . . , v_p$ be the vertices of $G$ and $P_n : u_1, u_2, . . . , u_n$ be the path. Let $f$ be a total mean cordial labeling of $G$ and $g$ be the total mean cordial labeling of $P_n$ as defined in theorem 2.1. The induced total mean cordial labeling $h$ by $f$ and $g$ of $G ∪ P_n$ as follows:

**Case 1.** $n ≤ 8$.

**Subcase 1.** $p + q ≡ 0 (mod 3)$.

Let $p + q = 3t$. Assign the labels to the vertices of $P_n$ $(n ≤ 8)$ as in theorem 2.1. One can easily check that the labeling $f$ together with $g$ forms a total mean cordial labeling $h$ of $G ∪ P_n$.

**Subcase 2.** $p + q ≡ 1 (mod 3)$.

Put $p + q = 3t + 1$. Suppose $ev_f(0) = ev_f(1) = t$, $ev_f(2) = t + 1$ then the following table 1 establishes that $G ∪ P_n$ $(n ≤ 8)$ is total mean cordial.

<table>
<thead>
<tr>
<th></th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$u_4$</th>
<th>$u_5$</th>
<th>$u_6$</th>
<th>$u_7$</th>
<th>$u_8$</th>
<th>$ev_h(0)$</th>
<th>$ev_h(1)$</th>
<th>$ev_h(2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$t + 1$</td>
<td>$t$</td>
<td>$t + 1$</td>
</tr>
<tr>
<td>$P_2$</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$t + 1$</td>
<td>$t + 2$</td>
<td>$t + 1$</td>
</tr>
<tr>
<td>$P_3$</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$t + 3$</td>
<td>$t + 2$</td>
<td>$t + 3$</td>
</tr>
<tr>
<td>$P_4$</td>
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<td>0</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>$t + 3$</td>
<td>$t + 3$</td>
<td>$t + 4$</td>
</tr>
<tr>
<td>$P_5$</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
<td>$t + 4$</td>
<td>$t + 4$</td>
<td>$t + 4$</td>
</tr>
<tr>
<td>$P_6$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td></td>
<td>$t + 5$</td>
<td>$t + 5$</td>
<td>$t + 4$</td>
</tr>
<tr>
<td>$P_7$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>$t + 5$</td>
<td>$t + 5$</td>
<td>$t + 6$</td>
</tr>
</tbody>
</table>

Table 1:
If \( ev_f(0) = ev_f(2) = t, \ ev_f(1) = t + 1 \) then the following table 2 shows the total mean cordial condition of \( G \cup P_n \ (n \leq 8) \).

<table>
<thead>
<tr>
<th></th>
<th>( u_1 )</th>
<th>( u_2 )</th>
<th>( u_3 )</th>
<th>( u_4 )</th>
<th>( u_5 )</th>
<th>( u_6 )</th>
<th>( u_7 )</th>
<th>( u_8 )</th>
<th>( ev_h(0) )</th>
<th>( ev_h(1) )</th>
<th>( ev_h(2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_1 )</td>
<td>0</td>
<td>t + 1</td>
<td>t + 1</td>
<td>t</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( P_2 )</td>
<td>0</td>
<td>2</td>
<td>t + 1</td>
<td>t + 2</td>
<td>t + 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( P_3 )</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>t + 3</td>
<td>t + 2</td>
<td>t + 3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( P_4 )</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>t + 3</td>
<td>t + 3</td>
<td>t + 4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( P_5 )</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>t + 4</td>
<td>t + 4</td>
<td>t + 4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( P_6 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>t + 5</td>
<td>t + 5</td>
<td>t + 4</td>
<td></td>
</tr>
<tr>
<td>( P_7 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>t + 5</td>
<td>t + 5</td>
<td>t + 5</td>
<td></td>
</tr>
<tr>
<td>( P_8 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>t + 6</td>
<td>t + 5</td>
<td>t + 5</td>
</tr>
</tbody>
</table>

Table 2:

If \( ev_f(1) = ev_f(2) = t, \ ev_f(0) = t + 1 \) then the following table 3 establishes that \( G \cup P_n \ (n \leq 8) \) is total mean cordial.

<table>
<thead>
<tr>
<th></th>
<th>( u_1 )</th>
<th>( u_2 )</th>
<th>( u_3 )</th>
<th>( u_4 )</th>
<th>( u_5 )</th>
<th>( u_6 )</th>
<th>( u_7 )</th>
<th>( u_8 )</th>
<th>( ev_h(0) )</th>
<th>( ev_h(1) )</th>
<th>( ev_h(2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_1 )</td>
<td>1</td>
<td>t + 1</td>
<td>t + 1</td>
<td>t</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( P_2 )</td>
<td>1</td>
<td>2</td>
<td>t + 1</td>
<td>t + 1</td>
<td>t + 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( P_3 )</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>t + 2</td>
<td>t + 2</td>
<td>t + 3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( P_4 )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>t + 4</td>
<td>t + 3</td>
<td>t + 3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( P_5 )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>t + 4</td>
<td>t + 4</td>
<td>t + 4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( P_6 )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>t + 5</td>
<td>t + 4</td>
<td>t + 5</td>
<td></td>
</tr>
<tr>
<td>( P_7 )</td>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>t + 6</td>
<td>t + 5</td>
<td>t + 5</td>
</tr>
<tr>
<td>( P_8 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>t + 6</td>
<td>t + 5</td>
<td>t + 5</td>
</tr>
</tbody>
</table>

Table 3:

**Subcase 3.** \( p + q \equiv 2 \pmod{3} \).

Put \( p + q = 3t + 2 \). Suppose \( ev_f(0) = ev_f(1) = t + 1, \ ev_f(2) = t \) then the following table 4 establishes that \( G \cup P_n \ (n \leq 8) \) is total mean cordial.

<table>
<thead>
<tr>
<th></th>
<th>( u_1 )</th>
<th>( u_2 )</th>
<th>( u_3 )</th>
<th>( u_4 )</th>
<th>( u_5 )</th>
<th>( u_6 )</th>
<th>( u_7 )</th>
<th>( u_8 )</th>
<th>( ev_h(0) )</th>
<th>( ev_h(1) )</th>
<th>( ev_h(2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_1 )</td>
<td>2</td>
<td>t + 1</td>
<td>t + 1</td>
<td>t + 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( P_2 )</td>
<td>0</td>
<td>2</td>
<td>t + 2</td>
<td>t + 2</td>
<td>t + 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( P_3 )</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>t + 3</td>
<td>t + 3</td>
<td>t + 3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( P_4 )</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>t + 4</td>
<td>t + 3</td>
<td>t + 4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( P_5 )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>t + 4</td>
<td>t + 4</td>
<td>t + 5</td>
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<td></td>
</tr>
<tr>
<td>( P_6 )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>t + 5</td>
<td>t + 5</td>
<td>t + 5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( P_7 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>t + 6</td>
<td>t + 6</td>
<td>t + 5</td>
</tr>
<tr>
<td>( P_8 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>t + 6</td>
<td>t + 6</td>
<td>t + 5</td>
</tr>
</tbody>
</table>

Table 4:

Suppose \( ev_f(0) = ev_f(2) = t + 1, \ ev_f(1) = t \) then with the help of the following table 5, we can understand that \( G \cup P_n \ (n \leq 8) \) is total mean cordial.
Subcase 4. Let $P \equiv 0 \pmod{3}$ and $n \equiv 0 \pmod{3}$. Let $p + q = 3t_1$ and $n = 3t_2$. In this case $ev_h(0) = t_1 + 2t_2 - 1$, $ev_h(1) = ev_h(2) = t_1 + 2t_2$.

Subcase 5. $p + q \equiv 1 \pmod{3}$ and $n \equiv 1 \pmod{3}$. Let $p + q = 3t_1 + 1$ and $n = 3t_2$. If $ev_f(0) = t_1 + 1$, $ev_f(1) = ev_f(2) = t_1$ then $ev_h(0) = ev_h(1) = ev_h(2) = t_1 + 2t_2$. If $ev_f(1) = t_1 + 1$, $ev_f(0) = ev_f(2) = t_1$ then relabel the vertex $u_{t+2}$ by 0. Here also $ev_h(0) = ev_h(1) = ev_h(2) = t_1 + 2t_2$. If $ev_f(2) = t_1 + 1$, $ev_f(0) = ev_f(1) = t_1$ then relabel the vertices $u_{2t}$, $u_{2t+2}$ by 2, 0 respectively. Hence $ev_h(0) = ev_h(1) = ev_h(2) = t_1 + 2t_2$.

Subcase 5. $p + q \equiv 1 \pmod{3}$ and $n \equiv 1 \pmod{3}$.
Let $p + q = 3t_1 + 1$ and $n = 3t_2 + 1$. If $ev_f(0) = t_1 + 1$, $ev_f(1) = ev_f(2) = t_1$ then relabel the vertex $u_1$ by 2. Hence $ev_h(0) = t_1 + 2t_2$, $ev_h(1) = ev_h(2) = t_1 + 2t_2 + 1$. Suppose $ev_f(1) = t_1 + 1$, $ev_f(0) = ev_f(2) = t_1$ then $ev_h(0) = ev_h(1) = t_1 + 2t_2 + 1$, $ev_h(2) = t_1 + 2t_2$.

If $ev_f(2) = t_1 + 1$, $ev_f(0) = ev_f(1) = t_1$ then $ev_h(0) = ev_h(2) = t_1 + 2t_2 + 1$, $ev_h(1) = t_1 + 2t_2$.

**Subcase 6.** $p + q \equiv 1 \pmod{3}$ and $n \equiv 2 \pmod{3}$.

Let $p + q = 3t_1 + 1$ and $n = 3t_2 + 2$. If $ev_f(0) = t_1 + 1$, $ev_f(1) = ev_f(2) = t_1$ Here $ev_h(0) = t_1 + 2t_2 + 2$, $ev_h(1) = ev_h(2) = t_1 + 2t_2 + 1$. Suppose $ev_f(1) = t_1 + 1$, $ev_f(0) = ev_f(2) = t_1$ then $ev_h(0) = ev_h(2) = t_1 + 2t_2 + 1$, $ev_h(1) = t_1 + 2t_2 + 2$. If $ev_f(2) = t_1 + 1$, $ev_f(0) = ev_f(1) = t_1$ then $ev_h(0) = ev_h(1) = t_1 + 2t_2 + 1$, $ev_h(2) = t_1 + 2t_2 + 2$.

**Subcase 7.** $p + q \equiv 2 \pmod{3}$ and $n \equiv 0 \pmod{3}$.

Let $p + q = 3t_1 + 2$ and $n = 3t_2$. If $ev_f(0) = t_1$, $ev_f(1) = ev_f(2) = t_1 + 1$ then relabel the vertex $u_{t+2}$ by 0. In this case $ev_h(2) = t_1 + 2t_2 + 1$, $ev_h(0) = ev_h(1) = t_1 + 2t_2$. Suppose $ev_f(1) = t_1$, $ev_f(0) = ev_f(2) = t_1 + 1$ then $ev_h(0) = ev_h(1) = t_1 + 2t_2$, $ev_h(2) = t_1 + 2t_2 + 1$.

If $ev_f(2) = t_1$, $ev_f(0) = ev_f(1) = t_1 + 1$ then $ev_h(0) = ev_h(2) = t_1 + 2t_2$, $ev_h(1) = t_1 + 2t_2 + 1$.

**Subcase 8.** $p + q \equiv 2 \pmod{3}$ and $n \equiv 1 \pmod{3}$.

Let $p + q = 3t_1 + 2$ and $n = 3t_2 + 1$. If $ev_f(0) = t_1$, $ev_f(1) = ev_f(2) = t_1 + 1$ then $ev_h(0) = ev_h(1) = ev_h(2) = t_1 + 2t_2 + 1$. Suppose $ev_f(1) = t_1$, $ev_f(0) = ev_f(2) = t_1 + 1$ then relabel the vertices $u_1, u_{t+3}$ by 1, 0 respectively. In this case $ev_h(0) = ev_h(1) = ev_h(2) = t_1 + 2t_2 + 1$. If $ev_f(2) = t_1$, $ev_f(0) = ev_f(1) = t_1 + 1$ then relabel the vertices $u_1, u_{t+3}$ by 2, 0 respectively. Here also $ev_h(0) = ev_h(1) = ev_h(2) = t_1 + 2t_2 + 1$.

**Subcase 9.** $p + q \equiv 2 \pmod{3}$ and $n \equiv 2 \pmod{3}$.

Let $p + q = 3t_1 + 2$ and $n = 3t_2 + 2$. If $ev_f(0) = t_1$, $ev_f(1) = ev_f(2) = t_1 + 1$ then $ev_h(0) = t_1 + 2t_2 + 1$, $ev_h(1) = ev_h(2) = t_1 + 2t_2 + 2$. Suppose $ev_f(1) = t_1$, $ev_f(0) = ev_f(2) = t_1 + 1$ then $ev_h(0) = ev_h(2) = t_1 + 2t_2 + 2$, $ev_h(1) = t_1 + 2t_2 + 1$. If $ev_f(2) = t_1$, $ev_f(0) = ev_f(1) = t_1 + 1$ then $ev_h(0) = ev_h(1) = t_1 + 2t_2 + 2$, $ev_h(2) = t_1 + 2t_2 + 1$.

The graph $C_n^{(t)}$ denotes the one point union of $t$ copies of the cycle $C_n : u_1, u_2, \ldots, u_n, u_1$.

**Theorem 3.2.** The graph $C_n^{(2)}$ is total mean cordial.

**Proof.** Let the two copies of the cycles be $u_1u_2 \ldots u_nu_1$ and $v_1v_2 \ldots v_nv_1$. Assume $u_i = v_j$.

It is clear that $p + q = 4n - 1$.

**Case 1.** $n \leq 8$.

**Figure 1** shows that $C_n^{(2)}$ $(3 \leq n \leq 8)$ is Total Mean Cordial.

**Case 2.** $n > 8$.

Treat $u_i$ as $u_1$, $u_{i+1}$ as $u_2$ and so on. Similarly treat $v_j$ as $v_1$, $v_{j+1}$ as $v_2$ and so on. Assign the labels to the vertices of the two copies of the cycle $C_n$ as in theorem 2.2.

**Subcase 1.** $n \equiv 0 \pmod{3}$.

Let $n = 3t$. In this case $ev_f(0) = 4t - 1$, $ev_f(1) = ev_f(2) = 4t$.

**Subcase 2.** $n \equiv 1 \pmod{3}$. 

Let $n = 3t + 1$. Relabel the vertices $u_2$, $u_{t+2}$ and $u_{t+4}$ by 2, 0, 0 respectively. Then we have $ev_f(0) = ev_f(1) = ev_f(2) = 4t + 1$.

**Subcase 3.** $n \equiv 2 \pmod{3}$.

Let $n = 3t+2$. Relabel the vertex $u_{t+2}$ by 0. Then $ev_f(0) = 4t+3$, $ev_f(1) = ev_f(2) = 4t+2$.

Thus $f$ is a total mean cordial labeling of $C_{2n}^2$.

The book $B_m$ is the graph $S_m \times P_2$ where $S_m$ is the star with $m+1$ vertices.

**Theorem 3.3.** The book $B_m$ is total mean cordial.

**Proof.** Let the vertex set of $B_m$ be $V(B_m) = \{u, v, u_i, v_i : 1 \leq i \leq m\}$ and the edge set of $B_m$ be $E(B_m) = \{uv, uu_i, vv_i, u_i v_i : 1 \leq i \leq m\}$. It is clear that $p + q = 5m + 3$.

**Case 1.** $m \equiv 0 \pmod{6}$.

Let $m = 6t$ and $t > 0$. Define a function $f : V(B_m) \rightarrow \{0, 1, 2\}$ by $f(u) = 0$, $f(v) = 2$,

\[
\begin{align*}
  f(u_i) &= 0, \quad 1 \leq i \leq 5t \\
  f(u_{5t+i}) &= 1, \quad 1 \leq i \leq t \\
  f(v_i) &= 2, \quad 1 \leq i \leq 4t \\
  f(v_{4t+i}) &= 1, \quad 1 \leq i \leq 2t.
\end{align*}
\]

In this case, $ev_f(0) = ev_f(1) = ev_f(2) = 10t + 1$.

**Case 2.** $m \equiv 1 \pmod{6}$.

Let $m = 6t - 5$ and $t > 0$. Define a map $f : V(B_m) \rightarrow \{0, 1, 2\}$ by $f(u) = 0$, $f(v) = 2$,

\[
\begin{align*}
  f(u_i) &= 0, \quad 1 \leq i \leq 5t - 4 \\
  f(u_{5t-4+i}) &= 1, \quad 1 \leq i \leq t - 1 \\
  f(v_i) &= 2, \quad 1 \leq i \leq 4t - 3 \\
  f(v_{4t-3+i}) &= 1, \quad 1 \leq i \leq 2t - 2.
\end{align*}
\]

In this case, $ev_f(0) = ev_f(2) = 10t - 7$, $ev_f(1) = 10t - 8$.

**Case 3.** $m \equiv 2 \pmod{6}$.

Let $m = 6t - 4$ and $t > 0$. Define a function $f : V(B_m) \rightarrow \{0, 1, 2\}$ by $f(u) = 0$, $f(v) = 2$,
Here, $ev_f(0) = 10t - 5$, $ev_f(1) = ev_f(2) = 10t - 6$.

**Case 4.** $m \equiv 3 \pmod{6}$.

Let $m = 6t - 3$ and $t > 0$. When $t = 1$, the vertex labeling of $B_3$ given in Figure 2 is a total mean cordial labeling.

![Figure 2](image)

Assume $t > 1$. Define a function $f : V(B_m) \rightarrow \{0, 1, 2\}$ by $f(u) = 0$, $f(v) = 2$,

\[
\begin{align*}
  f(u_i) &= 0, \quad 1 \leq i \leq 5t - 3 \\
  f(u_{5t-3+i}) &= 1, \quad 1 \leq i \leq t - 1 \\
  f(v_i) &= 2, \quad 1 \leq i \leq 4t - 3 \\
  f(v_{4t-3+i}) &= 1, \quad 1 \leq i \leq 2t - 1.
\end{align*}
\]

and $f(v_{6t-3}) = 0$. In this case, $ev_f(0) = ev_f(1) = ev_f(2) = 10t - 4$.

**Case 5.** $m \equiv 4 \pmod{6}$.

Let $m = 6t - 2$ and $t > 0$. Define a function $f : V(B_m) \rightarrow \{0, 1, 2\}$ by $f(u) = 0$, $f(v) = 2$,

\[
\begin{align*}
  f(u_i) &= 0, \quad 1 \leq i \leq 5t - 2 \\
  f(u_{5t-2+i}) &= 1, \quad 1 \leq i \leq t \\
  f(v_i) &= 2, \quad 1 \leq i \leq 4t - 1 \\
  f(v_{4t-1+i}) &= 1, \quad 1 \leq i \leq 2t - 1.
\end{align*}
\]

In this case, $ev_f(0) = 10t - 3$, $ev_f(1) = ev_f(2) = 10t - 2$.

**Case 6.** $m \equiv 5 \pmod{6}$.

Let $m = 6t - 1$ and $t > 0$. Define a function $f : V(B_m) \rightarrow \{0, 1, 2\}$ by $f(u) = 0$, $f(v) = 2$,

\[
\begin{align*}
  f(u_i) &= 0, \quad 1 \leq i \leq 5t - 1 \\
  f(u_{5t-1+i}) &= 1, \quad 1 \leq i \leq t \\
  f(v_i) &= 2, \quad 1 \leq i \leq 4t \\
  f(v_{4t+i}) &= 1, \quad 1 \leq i \leq 2t - 1.
\end{align*}
\]

In this case, $ev_f(0) = ev_f(1) = 10t - 1$, $ev_f(2) = 10t$. Hence $B_m$ is a total mean cordial
The graph $L_n = P_n \times P_2$ is called ladder.

**Theorem 3.4.** The ladder $L_n$ is total mean cordial.

**Proof.** Let $V(L_n) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(L_n) = \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n - 1\} \cup \{u_iv_i : 1 \leq i \leq n\}$. Clearly $p + q = 5n - 2$.

**Case 1.** $n \equiv 0 \pmod{6}$.
Let $n = 6t$ and $t > 0$. Define a vertex labeling $f : V(L_n) \to \{0, 1, 2\}$ as follows:

\[
\begin{align*}
  f(u_i) &= 0, \quad 1 \leq i \leq 5t \\
  f(u_{5t+i}) &= 1, \quad 1 \leq i \leq t \\
  f(v_i) &= 2, \quad 1 \leq i \leq 5t \\
  f(v_{5t+i}) &= 1, \quad 1 \leq i \leq t.
\end{align*}
\]

In this case, $ev_f(0) = ev_f(1) = 10t - 1$, $ev_f(2) = 10t$.

**Case 2.** $n \equiv 1 \pmod{6}$.
Let $n = 6t + 1$ and $t > 0$. Define a vertex labeling $f : V(L_n) \to \{0, 1, 2\}$ as follows:

\[
\begin{align*}
  f(u_i) &= 0, \quad 1 \leq i \leq 5t + 1 \\
  f(u_{5t+1+i}) &= 1, \quad 1 \leq i \leq t \\
  f(v_i) &= 2, \quad 1 \leq i \leq 5t - 1 \\
  f(v_{5t-1+i}) &= 1, \quad 1 \leq i \leq t + 1
\end{align*}
\]

and $f(u_{6t+1}) = 2$. Here, $ev_f(0) = ev_f(1) = ev_f(2) = 10t + 1$.

**Case 3.** $n \equiv 2 \pmod{6}$.
Let $n = 6t - 4$ and $t > 0$. Define a vertex labeling $f : V(L_n) \to \{0, 1, 2\}$ as follows:

\[
\begin{align*}
  f(u_i) &= 0, \quad 1 \leq i \leq 5t - 3 \\
  f(u_{5t-3+i}) &= 1, \quad 1 \leq i \leq t - 1 \\
  f(v_i) &= 2, \quad 1 \leq i \leq 5t - 4 \\
  f(v_{5t-4+i}) &= 1, \quad 1 \leq i \leq t.
\end{align*}
\]

In this case, $ev_f(0) = ev_f(1) = 10t - 7$, $ev_f(2) = 10t - 8$.

**Case 4.** $n \equiv 3 \pmod{6}$.
Let $n = 6t - 3$ and $t > 0$. Define a map $f : V(L_n) \to \{0, 1, 2\}$ by

\[
\begin{align*}
  f(u_i) &= 0, \quad 1 \leq i \leq 5t - 2 \\
  f(u_{5t-2+i}) &= 1, \quad 1 \leq i \leq t - 1 \\
  f(v_i) &= 2, \quad 1 \leq i \leq 5t - 3 \\
  f(v_{5t-3+i}) &= 1, \quad 1 \leq i \leq t.
\end{align*}
\]

In this case, $ev_f(0) = 10t - 5$, $ev_f(1) = ev_f(2) = 10t - 6$. 

Case 5. \( n \equiv 4 \pmod{6} \).

Let \( n = 6t - 2 \) and \( t > 0 \). When \( t = 1 \), the graph \( L_4 \) given in Figure 3 is total mean cordial.

\[
\begin{array}{c}
0 & 2 & 2 & 0 \\
2 & 6 & 0 & 6 \\
0 & 6 & 2 & 0 \\
\end{array}
\]

Figure 3:

Define a function \( f : V(L_n) \to \{0, 1, 2\} \) by

\[
\begin{align*}
f(u_i) &= 0, \quad 1 \leq i \leq 5t - 2 \\
f(u_{5t-2+i}) &= 1, \quad 1 \leq i \leq t \\
f(v_i) &= 2, \quad 1 \leq i \leq 5t - 3 \\
f(v_{5t-3+i}) &= 1, \quad 1 \leq i \leq t - 1 \\
\end{align*}
\]

\( f(v_{6t-3}) = 0 \) and \( f(v_{6t-2}) = 2 \). In this case, \( ev_f(0) = ev_f(1) = ev_f(2) = 10t - 4 \).

Case 6. \( n \equiv 5 \pmod{6} \).

Let \( n = 6t - 1 \) and \( t > 0 \). Define a map \( f : V(L_n) \to \{0, 1, 2\} \) by

\[
\begin{align*}
f(u_i) &= 0, \quad 1 \leq i \leq 5t - 1 \\
f(u_{5t-1+i}) &= 1, \quad 1 \leq i \leq t \\
f(v_i) &= 2, \quad 1 \leq i \leq 5t - 1 \\
f(v_{5t-1+i}) &= 1, \quad 1 \leq i \leq t. \\
\end{align*}
\]

Here, \( ev_f(0) = 10t - 3 \), \( ev_f(1) = ev_f(2) = 10t - 2 \). Hence \( L_n \) is total mean cordial.

Acknowledgement. The authors are sincerely thankful to the referees for their careful reading and constructive comments on earlier version of this manuscript, which resulted in better presentation of this article.

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