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A note on 3-Prime cordial graphs

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ABSTRACT

Let G be a (p, q) graph. Let $f : V(G) \to \{1, 2, \ldots, k\}$ be a map. For each edge uv, assign the label $\gcd(f(u), f(v))$. f is called k-prime cordial labeling of G if $|v_f(i) - v_f(j)| \leq 1$, $i, j \in \{1, 2, \ldots, k\}$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(x)$ denotes the number of vertices labeled with x, $e_f(1)$ and $e_f(0)$ respectively denote the number of edges labeled with 1 and not labeled with 1. A graph with a k-prime cordial labeling is called a k-prime cordial graph. In this paper we investigate 3-prime cordial labeling behavior of union of a 3-prime cordial graph and a path P_n.

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1 Introduction

All graphs in this paper are finite, simple and undirected. Let G be a (p, q) graph where p refers the number of vertices of G and q refers the number of edge of G. For a graph

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G, the splitting graph of G, $S'(G)$, is obtained from G by adding for each vertex v of G a new vertex v' so that v' is adjacent to every vertex that is adjacent to v, see Gallian survey [2]. Note that if G is a (p, q) graph then $S'(G)$ is a $(2p, 3q)$ graph. All graphs considered here are finite simple and undirected. The number of vertices of a graph G is called order of G, and the number of edges is called size of G. In 1987, Cahit introduced the concept of cordial labeling of graphs [1]. Sundaram, Ponraj, Somasundaram [6] have introduced the notion of prime cordial labeling. A prime cordial labeling of a graph G with vertex set V is a bijection $f : V \to \{1, 2, \ldots, |V|\}$ such that if each edge uv is assigned the label 1 if $\gcd(f(u), f(v)) = 1$ and 0 if $\gcd(f(u), f(v)) > 1$, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. Also they discussed the prime cordial labeling behavior of various graphs. Recently Ponraj et al. [8], introduced k-prime cordial labeling of graphs. In this paper we investigate the 3-prime cordial labeling behavior of union of a 3-prime cordial graph and a path P_n. Let x be any real number. Then $\lfloor x \rfloor$ stands for the largest integer less than or equal to x and $\lceil x \rceil$ stands for smallest integer greater than or equal to x. Terms not defined here follow from Harary [3].

2 Preliminaries

Definition 2.1. *Let G be a (p, q) graph and $2 \leq p \leq k$. Let $f : V(G) \to \{1, 2, \ldots, k\}$ be a function. For each edge uv, assign the label* $\gcd(f(u), f(v))$. *f is called a k-prime cordial labeling of G if $|v_f(i) - v_f(j)| \leq 1$, $i, j \in \{1, 2, \ldots, k\}$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(x)$ denotes the number of vertices labeled with x, $e_f(1)$ and $e_f(0)$ respectively denote the number of edges labeled with* 1 *and not labeled with* 1. *A graph with a k-prime cordial labeling is called a k-prime cordial graph.*

Theorem 2.1. *[8] The path P_n is* 3*-prime cordial if and only if $n \neq 3$.*

Proof. For $n = 3$, it is trivial that, for any labeling g, $v_g(1) = v_g(2) = v_g(3) = 1$. But $e_g(0) = 0$. This implies $|e_g(0) - e_g(1)| > 1$. Assume $n \neq 3$. Let P_n be the path $v_1 v_2 \ldots v_n$.
Case 1. $n \equiv 0, 1 \pmod 3$.
Assign the label 2 to the vertices $v_1, v_2, \ldots, v_{\lceil \frac{n}{3} \rceil}$. Then assign the label 3 consecutively to the vertices $v_{\lceil \frac{n}{3} \rceil + 1}, v_{\lceil \frac{n}{3} \rceil + 2}, \ldots$ until we have received the $\lceil \frac{n}{2} \rceil$ edges with the label 0. If all the $\lfloor \frac{n}{3} \rfloor$ 3's are exhausted then assign the label 1 to the remaining vertices; otherwise consider the non labeled vertex v_i such that v_{i-1} is labeled and assign the labels 1, 3 to the vertices $v_i, v_{i+1}, v_{i+2}, \ldots$ alternatively until $\lfloor \frac{n}{3} \rfloor$ 3's are exhausted. Finally assign the label 1 to the remaining vertices.
Case 2. $n \equiv 2 \pmod 3$.
As in case 1, assign the labels to the vertices $v_1, v_2, \ldots, v_n$. Now, let i be the least positive integer such that the label of v_{i-1} = the label of $v_{i+1} = 3$, and the label of $v_i = 1$. Finally

interchange the labels of v_i and v_{i+1}. Clearly this vertex labeling satisfies both vertex and edge conditions. □

3 Main Results

First we prove that union of any 3-prime cordial with P_n is also a 3-prime cordial if $n > 12$.

Theorem 3.1. *If G is a (p,q) 3-prime cordial graph then $G \cup P_n$ also a 3-prime cordial graph if $n > 12$.*

Proof. Let f be a 3-prime cordial labeling of G and let g be a 3-prime cordial labeling of P_n defined in theorem 2.1. Let $v_1, v_2, \ldots, v_n$ be the vertices of P_n. We define a map $h : V(G \cup P_n) \to \{1,2,3\}$ by $h(v_i) = g(v_i)$ where $1 \le i \le n$ and $h(u_j) = f(u_j)$ for $1 \le j \le p$. Then we have the following cases.

Case 1. $p \equiv 0 \pmod 3$ and $q \equiv 0 \pmod 2$.

Let $p = 3t_1$ and $q = 2r_1$. In this case $v_f(1) = v_f(2) = v_f(3) = t_1$ and $e_f(0) = e_f(1) = r_1$.

Subcase 1a. $n \equiv 0 \pmod 3$.

Let $n = 3t_2$. Here $v_g(1) = v_g(2) = v_g(3) = t_2$. This implies $v_h(1) = v_h(2) = v_h(3) = t_1 + t_2$. If $n - 1 \equiv 0 \pmod 2$ then $n - 1 = 2r_2$. Here $e_g(0) = e_g(1) = r_2$. Therefore $e_h(0) = e_h(1) = r_1 + r_2$.

If $n-1 \equiv 1 \pmod 2$ then put $n-1 = 2r_2+1$. Here $e_g(0) = r_2+1$ and $e_g(1) = r_2$. Hence $e_h(0) = r_1 + r_2 + 1$ and $e_h(1) = r_1 + r_2$.

Subcase 1b. $n \equiv 1 \pmod 3$.

Let $n = 3t_2 + 1$. Here $v_g(1) = v_g(3) = t_2$, $v_g(2) = t_2 + 1$. This implies $v_h(1) = v_h(3) = t_1+t_2$, $v_h(2) = t_1+t_2+1$. If $n-1 \equiv 0 \pmod 2$ then $n-1 = 2r_2$. Here $e_g(0) = e_g(1) = r_2$. Therefore $e_h(0) = e_h(1) = r_1 + r_2$.

If $n-1 \equiv 1 \pmod 2$ then put $n-1 = 2r_2+1$. Here $e_g(0) = r_2+1$ and $e_g(1) = r_2$. Hence $e_h(0) = r_1 + r_2 + 1$ and $e_h(1) = r_1 + r_2$.

Subcase 1c. $n \equiv 2 \pmod 3$.

Let $n = 3t_2 + 2$. Here $v_g(2) = v_g(3) = t_2 + 1$, $v_g(1) = t_2$. This implies $v_h(2) = v_h(3) = t_1+t_2+1$, $v_h(1) = t_1+t_2$. If $n-1 \equiv 0 \pmod 2$ then $n-1 = 2r_2$. Here $e_g(0) = e_g(1) = r_2$. Therefore $e_h(0) = e_h(1) = r_1 + r_2$.

If $n-1 \equiv 1 \pmod 2$ then put $n-1 = 2r_2+1$. Here $e_g(0) = r_2+1$ and $e_g(1) = r_2$. Hence $e_h(0) = r_1 + r_2 + 1$ and $e_h(1) = r_1 + r_2$.

Case 2. $p \equiv 0 \pmod 3$ and $q \equiv 1 \pmod 2$.

Let $p = 3t_1$ and $q = 2r_1 + 1$. In this case $v_f(1) = v_f(2) = v_f(3) = t_1$ and $e_f(0) = r_1 + 1$, $e_f(1) = r_1$ or $e_f(0) = r_1$, $e_f(1) = r_1 + 1$.

Subcase 2a. $n \equiv 0 \pmod 3$.

Let $n = 3t_2$. Here $v_g(1) = v_g(2) = v_g(3) = t_2$. This implies $v_h(1) = v_h(2) = v_h(3) = t_1 + t_2$. If $n - 1 \equiv 0 \pmod 2$ then $n - 1 = 2r_2$. Here $e_g(0) = e_g(1) = r_2$. Therefore $e_h(0) = r_1 + r_2 + 1$, $e_h(1) = r_1 + r_2$ or $e_h(0) = r_1 + r_2$, $e_h(1) = r_1 + r_2 + 1$.
If $n - 1 \equiv 1 \pmod 2$ then put $n - 1 = 2r_2 + 1$. Here $e_g(0) = r_2 + 1$ and $e_g(1) = r_2$. If $e_f(0) = r_1 + 1$, $e_f(1) = r_1$ then consider the vertex v_i such that $g(v_{i-1}) = g(v_i) = 3$ and $g(v_{i+1}) = 1$. Relabel the vertex v_i and v_n by 1 and 3 respectively. Then $e_g(0) = r_2$ and $e_g(1) = r_2 + 1$. Now $e_h(0) = e_h(1) = r_1 + r_2 + 1$. If $e_f(0) = r_1$ and $e_f(1) = r_1 + 1$ then $e_h(0) = e_h(1) = r_1 + r_2 + 1$.
Subcase 2b. $n \equiv 1 \pmod 3$.
Let $n = 3t_2 + 1$. Here $v_g(2) = t_2 + 1$, $v_g(1) = v_g(3) = t_2$. This implies $v_h(1) = v_h(3) = t_1 + t_2$, $v_h(2) = t_1 + t_2 + 1$. If $n - 1 \equiv 0 \pmod 2$ then $n - 1 = 2r_2$. Here $e_g(0) = e_g(1) = r_2$. Therefore $e_h(0) = r_1 + r_2 + 1$, $e_h(1) = r_1 + r_2$ or $e_h(0) = r_1 + r_2$, $e_h(1) = r_1 + r_2 + 1$.
If $n - 1 \equiv 1 \pmod 2$ then put $n - 1 = 2r_2 + 1$. Here $e_g(0) = r_2 + 1$ and $e_g(1) = r_2$. If $e_f(0) = r_1 + 1$, $e_f(1) = r_1$ then consider the vertex v_i such that $g(v_{i-1}) = g(v_i) = 3$ and $g(v_{i+1}) = 1$. Relabel the vertex v_i and v_n by 1 and 3 respectively. Then $e_g(0) = r_2$ and $e_g(1) = r_2 + 1$. Now $e_h(0) = e_h(1) = r_1 + r_2 + 1$. If $e_f(0) = r_1$ and $e_f(1) = r_1 + 1$ then $e_h(0) = e_h(1) = r_1 + r_2 + 1$.
Subcase 2c. $n \equiv 2 \pmod 3$.
Let $n = 3t_2 + 2$. Here $v_g(2) = v_g(3) = t_2 + 1$, $v_g(1) = t_2$. This implies $v_h(2) = v_h(3) = t_1 + t_2 + 1$, $v_h(1) = t_1 + t_2$. If $n - 1 \equiv 0 \pmod 2$ then $n - 1 = 2r_2$. Here $e_g(0) = e_g(1) = r_2$. Therefore $e_h(0) = r_1 + r_2 + 1$, $e_h(1) = r_1 + r_2$ or $e_h(0) = r_1 + r_2$, $e_h(1) = r_1 + r_2 + 1$.
If $n - 1 \equiv 1 \pmod 2$ then put $n - 1 = 2r_2 + 1$. Here $e_g(0) = r_2 + 1$ and $e_g(1) = r_2$. If $e_f(0) = r_1 + 1$, $e_f(1) = r_1$ then consider the vertex v_i such that $g(v_{i-1}) = g(v_i) = 3$ and $g(v_{i+1}) = 1$. Relabel the vertex v_i and v_n by 1 and 3 respectively. Then $e_g(0) = r_2$ and $e_g(1) = r_2 + 1$. Now $e_h(0) = e_h(1) = r_1 + r_2 + 1$. If $e_f(0) = r_1$ and $e_f(1) = r_1 + 1$ then $e_h(0) = e_h(1) = r_1 + r_2 + 1$.
Case 3. $p \equiv 1 \pmod 3$ and $q \equiv 0 \pmod 2$.
Let $p = 3t_1 + 1$ and $q = 2r_1$. In this case $v_f(1) = v_f(2) = t_1$, $v_f(3) = t_1 + 1$ or $v_f(1) = v_f(3) = t_1$, $v_f(2) = t_1 + 1$ or $v_f(2) = v_f(3) = t_1$, $v_f(1) = t_1 + 1$ and $e_f(0) = e_f(1) = r_1$.
Subcase 3a. $n \equiv 0 \pmod 3$.
Let $n = 3t_2$. Here $v_g(1) = v_g(2) = v_g(3) = t_2$. This implies $v_h(1) = v_h(2) = t_1 + t_2$, $v_h(3) = t_1 + t_2 + 1$ or $v_h(1) = v_h(3) = t_1 + t_2$, $v_h(2) = t_1 + t_2 + 1$ or $v_h(2) = v_h(3) = t_1 + t_2$, $v_h(1) = t_1 + t_2 + 1$. If $n - 1 \equiv 0 \pmod 2$ then $n - 1 = 2r_2$. Here $e_g(0) = e_g(1) = r_2$. Therefore $e_h(0) = e_h(1) = r_1 + r_2$.
If $n - 1 \equiv 1 \pmod 2$ then put $n - 1 = 2r_2 + 1$. Here $e_g(0) = r_2 + 1$ and $e_g(1) = r_2$. So $e_h(0) = r_1 + r_2 + 1$, $e_h(1) = r_1 + r_2$.
Subcase 3b. $n \equiv 1 \pmod 3$.
Let $n = 3t_2 + 1$. Here $v_g(2) = t_2 + 1$, $v_g(1) = v_g(3) = t_2$. Then i) $v_h(1) = t_1 + t_2$, $v_h(2) = v_h(3) = t_1 + t_2 + 1$ or ii) $v_h(2) = t_1 + t_2 + 2$, $v_h(1) = v_h(3) = t_1 + t_2$ or iii)

$v_h(3) = t_1 + t_2$, $v_h(1) = v_h(2) = t_1 + t_2 + 1$.
If $n-1 \equiv 0 \pmod 2$ then $n-1 = 2r_2$. Here $e_g(0) = e_g(1) = r_2$. Therefore $e_h(0) = e_h(1) = r_1 + r_2$.
If $n-1 \equiv 1 \pmod 2$ then put $n-1 = 2r_2+1$. Here $e_g(0) = r_2+1$ and $e_g(1) = r_2$. So $e_h(0) = r_1 + r_2 + 1$, $e_h(1) = r_1 + r_2$.
For the case (ii), we consider the vertex v_i such that $g(v_{i-1}) = 2$ and $g(v_i) = 3$. Now, relabel the vertex v_{i-1} by 3. Then $v_g(1) = v_g(2) = t_2$ and $v_g(3) = t_2 + 1$. Hence $v_h(1) = t_1 + t_2$, $v_h(2) = v_h(3) = t_1 + t_2 + 1$ and $e_h(0) = r_1 + r_2 + 1$, $e_h(1) = r_1 + r_2$.
Subcase 3c. $n \equiv 2 \pmod 3$.
Let $n = 3t_2 + 2$. Here $v_g(1) = t_2$, $v_g(2) = v_g(3) = t_2 + 1$. Then i) $v_h(1) = t_1 + t_2$, $v_h(2) = t_1 + t_2 + 1$, $v_h(3) = t_1 + t_2 + 2$ or ii) $v_h(1) = t_1 + t_2$, $v_h(2) = t_1 + t_2 + 2$, $v_h(3) = t_1 + t_2 + 1$ or iii) $v_h(1) = v_h(2) = v_h(3) = t_1 + t_2 + 1$.
If $n-1 \equiv 0 \pmod 2$ then $n-1 = 2r_2$. Here $e_g(0) = e_g(1) = r_2$. Therefore $e_h(0) = e_h(1) = r_1 + r_2$.
If $n-1 \equiv 1 \pmod 2$ then put $n-1 = 2r_2+1$. Here $e_g(0) = r_2+1$ and $e_g(1) = r_2$. So $e_h(0) = r_1 + r_2 + 1$, $e_h(1) = r_1 + r_2$.
For the case (i), we consider the vertex v_i such that $g(v_{i-1}) = g(v_{i+1}) = 1$ and $g(v_i) = 3$. Now, relabel the vertex v_i by 1. Then $v_h(1) = v_h(2) = v_h(3) = t_1 + t_2 + 1$ and the edge condition is not affected.
Now we consider the case (ii). Here we interchange the labels of the vertices with labels 2 and 3 in P_n then proceed as above, we have the same case.
Case 4. $p \equiv 1 \pmod 3$ and $q \equiv 1 \pmod 2$.
Let $p = 3t_1 + 1$ and $q = 2r_1 + 1$. In this case $v_f(2) = v_f(3) = t_1$, $v_f(1) = t_1 + 1$ or $v_f(1) = v_f(3) = t_1$, $v_f(2) = t_1 + 1$ or $v_f(1) = v_f(2) = t_1$, $v_f(3) = t_1 + 1$ and $e_f(0) = r_1 + 1$, $e_f(1) = r_1$ or $e_f(0) = r_1$, $e_f(1) = r_1 + 1$.
Subcase 4a. $n \equiv 0 \pmod 3$.
Let $n = 3t_2$. Here $v_g(1) = v_g(2) = v_g(3) = t_2$. This implies $v_h(2) = v_h(3) = t_1 + t_2$, $v_h(1) = t_1 + t_2 + 1$ or $v_h(1) = v_h(3) = t_1 + t_2$, $v_h(2) = t_1 + t_2 + 1$ or $v_h(1) = v_h(2) = t_1 + t_2$, $v_h(3) = t_1 + t_2 + 1$. If $n-1 \equiv 0 \pmod 2$ then $n-1 = 2r_2$. Here $e_g(0) = e_g(1) = r_2$. Therefore $e_h(0) = r_1 + r_2 + 1$, $e_h(1) = r_1 + r_2$ or $e_h(0) = r_1 + r_2$, $e_h(1) = r_1 + r_2 + 1$.
If $n-1 \equiv 1 \pmod 2$ then put $n-1 = 2r_2+1$. Here $e_g(0) = r_2+1$ and $e_g(1) = r_2$. If $e_f(0) = r_1$, $e_f(1) = r_1 + 1$ then $e_h(0) = e_h(1) = r_1 + r_2 + 1$. If $e_f(0) = r_1 + 1$, $e_f(1) = r_1$ then consider the vertex v_i such that $g(v_{i-1}) = g(v_i) = 3$, $g(v_{i+1}) = 1$. Note that v_n is labeled by 1. Now interchange the labels of v_i and v_n. Then $e_h(0) = e_h(1) = r_1 + r_2 + 1$.
Subcase 4b. $n \equiv 1 \pmod 3$.
Let $n = 3t_2 + 1$. Here $v_g(2) = t_2 + 1$, $v_g(1) = v_g(3) = t_2$. Then i) $v_h(3) = t_1 + t_2$, $v_h(2) = v_h(3) = t_1 + t_2 + 1$ or ii) $v_h(1) = v_h(3) = t_1 + t_2$, $v_h(2) = t_1 + t_2 + 2$ or iii) $v_h(1) = t_1 + t_2$, $v_h(2) = v_h(3) = t_1 + t_2 + 1$.
Consider the case (ii). In this case, we find a vertex v_i such that $g(v_{i-1}) = g(v_i) = 2$,

$g(v_{i+1}) = g(v_{i+2}) = 3$. Now assign the label 3 to the vertex v_i. Now $v_h(1) = t_1 + t_2$, $v_h(2) = v_h(3) = t_1 + t_2 + 1$.
If $n-1 \equiv 0 \pmod 2$ then $n-1 = 2r_2$. Here $e_g(0) = e_g(1) = r_2$. Therefore $e_h(0) = r_1 + r_2 + 1$, $e_h(1) = r_1 + r_2$ or $e_h(0) = r_1 + r_2$, $e_h(1) = r_1 + r_2 + 1$.
If $n-1 \equiv 1 \pmod 2$ then put $n-1 = 2r_2+1$. Here $e_g(0) = r_2 + 1$, $e_g(1) = r_2$. Suppose $e_f(0) = r_1$, $e_f(1) = r_1 + 1$ then $e_h(0) = e_h(1) = r_1 + r_2 + 1$. If $e_f(0) = r_1 + 1$, $e_f(1) = r_1$ then consider the vertex v_i such that $f(v_i) = f(v_{i-1}) = 3$, $f(v_{i+1}) = 1$. Note that v_n is labeled by 1. Now interchange the labels of v_i and v_n then $e_h(0) = e_h(1) = r_1 + r_2 + 1$.
Subcase 4c. $n \equiv 2 \pmod 3$.
Let $n = 3t_2 + 2$. Here $v_g(1) = t_2$, $v_g(2) = v_g(3) = t_2 + 1$. Then i) $v_h(1) = v_h(2) = v_h(3) = t_1 + t_2 + 1$ or ii) $v_h(1) = t_1 + t_2$, $v_h(2) = t_1 + t_2 + 2$, $v_h(3) = t_1 + t_2 + 1$ or iii) $v_h(1) = t_1 + t_2$, $v_h(2) = t_1 + t_2 + 1$, $v_h(3) = t_1 + t_2 + 3$.
If $n-1 \equiv 0 \pmod 2$ then $n-1 = 2r_2$. Here $e_g(0) = e_g(1) = r_2$. Therefore $e_h(0) = r_1 + r_2 + 1$, $e_h(1) = r_1 + r_2$ or $e_h(0) = r_1 + r_2$, $e_h(1) = r_1 + r_2 + 1$.
If $n-1 \equiv 1 \pmod 2$ then put $n-1 = 2r_2+1$. Here $e_g(0) = r_2 + 1$ and $e_g(1) = r_2$. For the edge conditions of the labeling h we proceed as in subcase 4b.
For the case (ii), relabel the vertices with the label 3 by 2 and vice versa. Let the new labeling be h'. Then Consider the vertex v_i such that $h'(v_{i-1}) = h'(v_i) = 2$, $h'(v_{i+1}) = h'(v_{i+2}) = 2$. Now, relabel the vertex v_{i+2} by 1. Clearly this vertex labeling h' satisfy the vertex and edge conditions.
The same case may be arised for the case (iii) when without interchanging the labels 3 and 2.
Case 5. $p \equiv 2 \pmod 3$ and $q \equiv 0 \pmod 2$.
Let $p = 3t_1 + 2$ and $q = 2r_1$. In this case $v_f(1) = v_f(2) = t_1 + 1$, $v_f(3) = t_1$ or $v_f(1) = v_f(3) = t_1 + 1$, $v_f(2) = t_1$ or $v_f(2) = v_f(3) = t_1 + 1$, $v_f(1) = t_1$ and $e_f(0) = e_f(1) = r_1$.
Subcase 5a. $n \equiv 0 \pmod 3$.
Let $n = 3t_2$. Here $v_g(1) = v_g(2) = v_g(3) = t_2$. This implies $v_h(1) = v_h(2) = t_1 + t_2 + 1$, $v_h(3) = t_1 + t_2$ or $v_h(1) = v_h(3) = t_1 + t_2 + 1$, $v_h(2) = t_1 + t_2$ or $v_h(2) = v_h(3) = t_1 + t_2 + 1$, $v_h(1) = t_1 + t_2$. If $n-1 \equiv 0 \pmod 2$ then $n-1 = 2r_2$. Here $e_g(0) = e_g(1) = r_2$. Therefore $e_h(0) = e_h(1) = r_1 + r_2$.
If $n-1 \equiv 1 \pmod 2$ then put $n-1 = 2r_2+1$. Here $e_g(0) = r_2 + 1$ and $e_g(1) = r_2$. So $e_h(0) = r_1 + r_2 + 1$, $e_h(1) = r_1 + r_2$.
Subcase 5b. $n \equiv 1 \pmod 3$.
Let $n = 3t_2 + 1$. Here $v_g(2) = t_2 + 1$, $v_g(1) = v_g(3) = t_2$. Then i) $v_h(1) = t_1 + t_2 + 1$, $v_h(2) = t_1 + t_2 + 2$, $v_h(3) = t_1 + t_2$ or ii) $v_h(1) = v_h(2) = v_h(3) = t_1 + t_2 + 1$ or iii) $v_h(1) = t_1 + t_2$, $v_h(2) = t_1 + t_2 + 2$, $v_h(3) = t_1 + t_2 + 1$.
For the case (i), we consider the vertex v_i such that $g(v_{i-1}) = g(v_i) = 2$ and $g(v_{i+1}) = g(v_{i+2}) = 3$. Now, relabel the vertex v_i by 3. Then $v_h(1) = v_h(2) = v_h(3) = t_1 + t_2 + 1$. Consider the case (iii). Here relabel the vertices with the label 3 by 2 and vice versa.

Let the new labeling be h'. Then consider the vertex v_i such that $h'(v_{i-1}) = h'(v_i) = 2$ and $h'(v_{i+1}) = 1$, $h'(v_{i+2}) = 2$. Now relabel the vertex v_{i+2} by 1. Then $v_{h'}(1) = v_{h'}(2) = v_{h'}(3) = t_1 + t_2 + 1$.
If $n-1 \equiv 0 \pmod 2$ then $n-1 = 2r_2$. Here $e_g(0) = e_g(1) = r_2$. Therefore $e_h(0) = e_h(1) = r_1 + r_2$.
If $n-1 \equiv 1 \pmod 2$ then put $n-1 = 2r_2+1$. Here $e_g(0) = r_2+1$ and $e_g(1) = r_2$. So $e_h(0) = r_1 + r_2 + 1$, $e_h(1) = r_1 + r_2$.
Subcase 5c. $n \equiv 2 \pmod 3$.
Let $n = 3t_2 + 2$. Here $v_g(1) = t_2$, $v_g(2) = v_g(3) = t_2 + 1$. Then i) $v_h(2) = t_1 + t_2 + 2$, $v_h(1) = v_h(3) = t_1 + t_2 + 1$ or ii) $v_h(1) = v_h(2) = t_1 + t_2 + 1$, $v_h(3) = t_1 + t_2 + 2$ or iii) $v_h(1) = t_1 + t_2$, $v_h(2) = v_h(3) = t_1 + t_2 + 2$.
For the case (iii), we consider the vertex v_i such that $g(v_{i-1}) = g(v_i) = 3$, $g(v_{i+1}) = 1$ and $g(v_{i+2}) = 3$. Now, relabel the vertex v_{i+2} by 1. Then $v_h(1) = v_h(3) = t_1 + t_2 + 1$, $v_h(2) = t_1 + t_2 + 2$.
If $n-1 \equiv 0 \pmod 2$ then $n-1 = 2r_2$. Here $e_g(0) = e_g(1) = r_2$. Therefore $e_h(0) = e_h(1) = r_1 + r_2$.
If $n-1 \equiv 1 \pmod 2$ then put $n-1 = 2r_2+1$. Here $e_g(0) = r_2+1$ and $e_g(1) = r_2$. So $e_h(0) = r_1 + r_2 + 1$, $e_h(1) = r_1 + r_2$.
Case 6. $p \equiv 2 \pmod 3$ and $q \equiv 1 \pmod 2$.
Let $p = 3t_1 + 2$ and $q = 2r_1 + 1$. In this case $v_f(1) = v_f(2) = t_1 + 1$, $v_f(3) = t_1$ or $v_f(1) = v_f(3) = t_1 + 1$, $v_f(2) = t_1$ or $v_f(2) = v_f(3) = t_1 + 1$, $v_f(1) = t_1$. Also $e_f(0) = r_1 + 1$, $e_f(1) = r_1$ or $e_f(0) = r_1$, $e_f(1) = r_1 + 1$.
Subcase 6a. $n \equiv 0 \pmod 3$.
Let $n = 3t_2$. Here $v_g(1) = v_g(2) = v_g(3) = t_2$. This implies $v_h(1) = v_h(2) = t_1 + t_2 + 1$, $v_h(3) = t_1 + t_2$ or $v_h(1) = v_h(3) = t_1 + t_2 + 1$, $v_h(2) = t_1 + t_2$ or $v_h(2) = v_h(3) = t_1 + t_2 + 1$, $v_h(1) = t_1 + t_2$. If $n-1 \equiv 0 \pmod 2$ then $n-1 = 2r_2$. Here $e_g(0) = e_g(1) = r_2$. Therefore $e_h(0) = r_1 + r_2 + 1$, $e_h(1) = r_1 + r_2$ or $e_h(0) = r_1 + r_2$, $e_h(1) = r_1 + r_2 + 1$.
If $n-1 \equiv 1 \pmod 2$ then put $n-1 = 2r_2+1$. Here $e_g(0) = r_2+1$ and $e_g(1) = r_2$. So $e_h(0) = e_h(1) = r_1 + r_2 + 1$, $e_h(1) = r_1 + r_2 + 2$, $e_h(1) = r_1 + r_2$. Now interchange the labels of v_1 and v_n. Then $e_h(0) = e_h(1) = r_1 + r_2 + 1$.
Subcase 6b. $n \equiv 1 \pmod 3$.
Let $n = 3t_2 + 1$. Here $v_g(2) = t_2 + 1$, $v_g(1) = v_g(3) = t_2$. Then i) $v_h(1) = t_1 + t_2 + 1$, $v_h(2) = t_1 + t_2 + 2$, $v_h(3) = t_1 + t_2$ or ii) $v_h(1) = v_h(2) = v_h(3) = t_1 + t_2 + 1$ or iii) $v_h(1) = t_1 + t_2$, $v_h(2) = t_1 + t_2 + 2$, $v_h(3) = t_1 + t_2 + 1$.
Consider the case (i). In this case, we consider the vertex v_i such that $g(v_{i-1}) = g(v_i) = 2$, $g(v_{i+1}) = g(v_{i+2}) = 3$. Now assign the label 3 to the vertex v_i. Now $v_h(1) = v_h(2) = v_h(3) = t_1 + t_2 + 1$. Consider the case (iii). Here interchange the labels 3 and 2. Now consider the vertex v_i such that $g(v_{i-1}) = g(v_i) = 2$, $g(v_{i+1}) = 1$, $g(v_{i+2}) = 2$. Now relabel the vertex v_{i+2} by 1. Then $v_h(1) = v_h(2) = v_h(3) = t_1 + t_2 + 1$.

If $n-1 \equiv 0 \pmod 2$ then $n-1 = 2r_2$. Here $e_g(0) = e_g(1) = r_2$. Therefore $e_h(0) = r_1 + r_2 + 1$, $e_h(1) = r_1 + r_2$ or $e_h(0) = r_1 + r_2$, $e_h(1) = r_1 + r_2 + 1$.
If $n-1 \equiv 1 \pmod 2$ then put $n-1 = 2r_2 + 1$. Here $e_g(0) = r_2 + 1$, $e_g(1) = r_2$. Then $e_h(0) = r_1 + r_2 + 2$, $e_h(1) = r_1 + r_2$ or $e_h(0) = e_h(1) = r_1 + r_2 + 1$. If $e_h(0) = r_1 + r_2 + 2$, $e_h(1) = r_1 + r_2$ then interchange the labels of v_1 and v_n then $e_h(0) = e_h(1) = r_1 + r_2 + 1$.
Subcase 6c. $n \equiv 2 \pmod 3$.
Let $n = 3t_2+2$. Here $v_g(1) = t_2$, $v_g(2) = v_g(3) = t_2+1$. Then i) $v_h(1) = v_h(3) = t_1+t_2+1$, $v_h(2) = t_1+t_2+2$ or ii) $v_h(1) = v_h(2) = t_1+t_2+1$, $v_h(3) = t_1+t_2+2$ or iii) $v_h(1) = t_1+t_2$, $v_h(2) = v_h(3) = t_1 + t_2 + 2$.
For the case (iii), we consider the vertex v_i such that $g(v_{i-1}) = g(v_i) = 3$, $g(v_{i+1}) = 1$, $g(v_{i+2}) = 3$. Now, relabel the vertex v_{i+2} by 1. Then $v_h(1) = v_h(3) = t_1 + t_2 + 1$, $v_h(2) = t_1 + t_2 + 2$.
If $n-1 \equiv 0 \pmod 2$ then $n-1 = 2r_2$. Here $e_g(0) = e_g(1) = r_2$. Therefore $e_h(0) = r_1 + r_2 + 1$, $e_h(1) = r_1 + r_2$ or $e_h(0) = r_1 + r_2$, $e_h(1) = r_1 + r_2 + 1$.
If $n-1 \equiv 1 \pmod 2$ then put $n-1 = 2r_2+1$. Here $e_g(0) = r_2+1$ and $e_g(1) = r_2$. Then $e_h(0) = e_h(1) = r_1 + r_2 + 1$ or $e_h(0) = r_1 + r_2 + 2$, $e_h(1) = r_1 + r_2$. Now interchange the labels of the vertices v_1 and v_n. Then $e_h(0) = e_h(1) = r_1 + r_2 + 1$.
Thus $G \cup P_n$ is 3-prime cordial if $n > 12$. ☐

Next we show that the splitting graph of a star is not a 3-prime cordial graph. Let $V(S'(K_{1,n})) = \{u, v, u_i, v_i : 1 \leq i \leq n\}$ and $E(S'(K_{1,n})) = \{uu_i, vv_i, uv_i : 1 \leq i \leq n\}$.

Theorem 3.2. *$S'(K_{1,n})$ is not 3-prime cordial.*

Proof. Suppose there exists a 3-prime cordial labeling f, then we have the following possible cases.
Case 1. $f(u) = f(v) = 2$.
Subcase 1a. $n \equiv 0 \pmod 3$.
Let $n = 3t$. Then $p = 6t + 2$ and $q = 9t$. Here we have the following three cases: (a) $v_f(1) = v_f(2) = 2t + 1$, $v_f(3) = 2t$. (b) $v_f(1) = v_f(3) = 2t + 1$, $v_f(2) = 2t$. (c) $v_f(2) = v_f(3) = 2t + 1$, $v_f(1) = 2t$. Consider the case (a) and (c). In this case $e_f(0) \leq 4t-2$, a contradiction. Consider the case (b). Here $e_f(0) \leq 4t-4$, a contradiction.
Subcase 1b. $n \equiv 1 \pmod 3$.
Let $n = 3t + 1$. Then $p = 6t + 4$ and $q = 9t + 3$. In this case we have the following three cases: (a) $v_f(1) = v_f(2) = 2t + 1$, $v_f(3) = 2t + 2$. (b) $v_f(1) = v_f(3) = 2t + 1$, $v_f(2) = 2t + 2$. (c) $v_f(2) = v_f(3) = 2t + 1$, $v_f(1) = 2t + 2$. Consider the case (a) and (c). In this case $e_f(0) \leq 4t - 2$, a contradiction. Consider the case (b). Here $e_f(0) \leq 4t$, a contradiction.
Subcase 1c. $n \equiv 2 \pmod 3$.
Let $n = 3t + 2$. Then $p = 6t + 6$ and $q = 9t + 6$. Here $v_f(1) = v_f(2) = v_f(3) = 2t + 2$. But $e_f(0) \leq 4t$, a contradiction.

Case 2. $f(u) = f(v) = 3$.
Similar to case 1.
Case 3. $f(u) = f(v) = 1$.
Similar to case 1.
Case 4. $f(u) = 2$, $f(v) = 1$ or $f(u) = 1$, $f(v) = 2$.
Subcase 4a. $n \equiv 0 \pmod 3$.
We consider the three cases as given in subcase 1a. For the cases (a) and (c), we have $e_f(0) \leq 2t - 1$, a contradiction. If we consider the case (b), we get $e_f(0) \leq 2t - 2$, a contradiction.
Subcase 4b. $n \equiv 1 \pmod 3$.
Her also we have three cases as in subcase 1b. First we consider the cases (a) and (c). Here $e_f(0) \leq 2t-1$, a contradiction. For the case (b), we get $e_f(0) \leq 2t$, a contradiction.
Subcase 4c. $n \equiv 2 \pmod 3$.
As in subcase 1c, we have $v_f(1) = v_f(2) = v_f(3) = 2t+2$. But $e_f(0) \leq 2t$, a contradiction.
Case 5. $f(u) = 3$, $f(v) = 1$ or $f(u) = 1$, $f(v) = 3$.
Similar to case 4.
Case 6. $f(u) = 2$, $f(v) = 3$ or $f(u) = 3$, $f(v) = 2$.
Subcase 6a. $n \equiv 0 \pmod 3$.
Consider the three cases given in subcase 1a. For the cases (a) and (b), we have $e_f(0) \leq 4t - 1$, and for the case (b), we get $e_f(0) \leq 4t$, both gives a contradiction to a 3-prime cordial labeling.
Subcase 6b. $n \equiv 1 \pmod 3$.
In this case we consider the three cases given in subcase 1b. If we consider the cases (a) and (b), we have $e_f(0) \leq 4t + 1$, and for the case (b), we get $e_f(0) \leq 4t$, a contradiction.
Subcase 6c. $n \equiv 2 \pmod 3$.
As in subcase 1c, we have $v_f(1) = v_f(2) = v_f(3) = 2t + 2$. Here $e_f(0) \leq 4t + 2$, a contradiction.
Hence the splitting graph of a star is not a 3-prime cordial graph. □

Theorem 3.3. *Let G be a graph obtained from the star $K_{1,n}$ by identifying each pendent vertex to the central vertex of the star $K_{1,m}$. Then G is 3-prime cordial.*

Proof. Let $V(K_{1,n}) = \{u, u_i : 1 \leq i \leq n\}$ and $E(K_{1,n}) = \{uu_i : 1 \leq i \leq n\}$. Let the vertex set of the i^{th} $K_{1,m}$ be $\{v_i, v_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$ and the edge set be $\{v_i v_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$. Identify u_i with v_i. It is obvious that the graph G has $mn + n + 1$ vertices and $mn + n$ edges. Assign the label 2 to the vertex u. We now move to the first star $K_{1,m}$. Assign the label 2 to the vertex v_1. Then assign the label 2 to the vertices $v_1^1, v_2^1, \ldots$ etc, until we have used $\lceil \frac{mn+n+1}{3} \rceil$ 2's as the vertex labels. If the used 2's is less than $\lceil \frac{mn+n+1}{3} \rceil$ then we move to the next star $K_{1,n}$. Assign the label 1 to the vertex v_2. Then assign the label 2 to the vertices $v_1^2, v_2^2, \ldots$ etc. If the number of

vertices labeled with 2 is $\lceil \frac{mn+n+1}{3} \rceil$ then stop. Otherwise, we move to the next step and countinuing as in above. Suppose the process is stop in the i^{th} star, then assign 3 to the unlabeled pendent vertices of the i^{th} star. We now move to $(i+1)^{th}$ star. Assign 3 to the central vertex. Next assign 3 to the pendent pendent vertices of the $(i+1)^{th}$ star. Each time count the value of $e_f(0)$. If it is $\lceil \frac{mn+n}{2} \rceil$ then assign 1 to the remaining vertices of the i^{th} star. Otherwise, assign 1 to the pendent vertices. This process is repeated until we have $\lceil \frac{mn+n}{2} \rceil$ edges with label 0.
Assign the label 1 to the central vertex of the non-labeled stars then move to its pendent veritces corresponding to it and 1 to the pendent vertices. Count the value of $v_f(1)$. If it is $\lfloor \frac{mn+n+1}{3} \rfloor$ then stop. Finally assign 3 to the non-labeled vertices.
It is easy to verify that this vertex labeling is a 3-prime cordial labeling. □

Jelly fish graphs $J(m,n)$ obtained from a cycle $C_4 : uvxyu$ by joining x and y with an edge and appending m pendent edges to u and n pendent edges to v.

Theorem 3.4. *The jelly fish $J(m,n)$ is 3-prime cordial if $10m \geq n+2$.*

Proof. Let the vertex set of $J(m,n)$ be $\{u, v, x, y, u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and the edge set be $\{uu_i, vv_j, ux, xv, uy, yv, xy : 1 \leq i \leq m, 1 \leq j \leq n\}$. We give the labeling f to the vertices of $J(m,n)$ as follows: Assign the label 2 to the vertices u, x, y. Then assign the label 2 to all the vertices $u_i \,(1 \leq i \leq n)$. Then assign the label 3 to the vertex v. Next assign the label 3 to the vertices $v_i \left(1 \leq i \leq \lceil \frac{m+n+4}{3} \rceil - 1\right)$. Next assign 1 to the vertices $v_{n-i} \left(0 \leq i \leq \lfloor \frac{m+n+4}{3} \rfloor\right)$. Finally, assign the label 2 to the non-labeled vertices v_i. We now count the edges with label 0 and 1. If the number of edges with label 0 is 2 more than the number of edges labeled with 1, then we relabel the vertices u_1 and v_1. Clearly, thus the relabeled graph $J(m,n)$ is 3-prime cordial; otherwise f is automatically a 3-prime cordial labeling. □

References

[1] I.Cahit, Cordial Graphs: A weaker version of Graceful and Harmonious graphs, *Ars combin.,* **23** (1987) 201-207.

[2] J.A.Gallian, A Dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, **17** (2015) #Ds6.

[3] F.Harary, Graph theory, *Addision wesley*, New Delhi (1969).

[4] M. Hovey, A-cordial graphs, *Discrete Math.*, **93** (1991), 183194.

[5] E. Salehi, PC-labelings of a graphs and its PC-sets, *Bull. Inst. Combin. Appl.,* **58** (2010) 112-121.

[6] M.Sundaram, R.Ponraj and S.Somasundaram, Prime cordial labeling of graphs, *J. Indian Acad. Math.,* **27**(2005) 373-390.

[7] M. Sundaram, R. Ponraj and S. Somasundaram, Product cordial labeling of graphs, *Bull. Pure and Appl. Sci. (Math. & Stat.)*, **23E** (2004) 155-163.

[8] R.Ponraj, Rajpal singh, R.Kala and S. Sathish Narayanan, k-prime cordial labeling of graphs, *J. Appl. Math. & Informatics*,**34**(2016), No. 3- 4, pp. 227-237.