Edge pair sum labeling of some cycle related graphs

P. Jeyanthi*1 and T. Saratha Devi†2

1 Govindammal Aditanar College for Women Tiruchendur-628 215, Tamil Nadu, India.
2 Department of Mathematics, G. Venkataswamy Naidu College, Kovilpatti-628502, Tamilnadu, India.

ABSTRACT

Let \( G \) be a \((p, q)\) graph. An injective map \( f : E(G) \to \{±1, ±2, \ldots, ±q\} \) is said to be an edge pair sum labeling if the induced vertex function \( f^* : V(G) \to \mathbb{Z} - \{0\} \) defined by \( f^*(v) = \sum_{e \in E_v} f(e) \) is one-one where \( E_v \) denotes the set of edges in \( G \) that are incident with a vertex \( v \) and \( f^*(V(G)) \) is either of the form \( \{±k_1, ±k_2, ±k_3, \ldots, ±k_p\} \) or \( \{±k_1, ±k_2, ±k_3, \ldots, ±k_p\} \bigcup \{±k_{p+1}\} \) according as \( p \) is even or odd. A graph with an edge pair sum labeling is called an edge pair sum graph. In this paper we prove that the graphs \( GL(n) \), double triangular snake \( D(T_n) \), \( W_n \), \( Fl_n \), \( ⟨C_m, K_{1,n}⟩ \) and \( ⟨C_m * K_{1,n}⟩ \) admit edge pair sum labeling.

Keywords: Edge pair sum labeling, edge pair sum graph, double triangular snake, wheel graph, flower graph

AMS subject Classification: 05C78.

1 Introduction

Throughout this paper we consider finite, simple and undirected graph \( G = (V(G), E(G)) \) with \( p \) vertices and \( q \) edges. \( G \) is also called a \((p, q)\) graph. We follow the basic notations

*Corresponding author: P. Jeyanthi. Email: jeyajeyanthi@rediffmail.com
†rajanvino03@gmail.com
and terminologies of graph theory as in [2]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. There are several types of labeling and for a dynamic survey of various graph labeling problems along with extensive bibliography we refer to Gallian [1]. Ponraj and Parthipan introduced the concept of pair sum labeling in [12]. An injective map $f : V(G) \rightarrow \{\pm 1, \pm 2, \ldots, \pm p\}$ is said to be a pair sum labeling of a graph $G(p, q)$ if the induced edge function $f_e : E(G) \rightarrow \mathbb{Z} - \{0\}$ defined by $f_e(uv) = f(u) + f(v)$ is one-one and $f_e(E(G))$ is either of the form $\{\pm k_1, \pm k_2, \ldots, \pm k_q\}$ or $\{\pm k_1, \pm k_2, \ldots, \pm k_{q-1}\} \cup \{\pm k_q + k\}$ according as $q$ is even or odd. A graph with a pair sum labeling, it is called a pair sum graph. Analogous to pair sum labeling we defined a new labeling called edge pair sum labeling [3] and further studied in [4-10]. In this paper we prove that the graphs $GL(n)$, double triangular snake $D(T_n)$, $W_n$, $Fl_n$, $\langle C_m, K_{1,n} \rangle$ and $\langle C_m * K_{1,n} \rangle$ admit edge pair sum labeling.

We use the following definitions in the subsequent sequel.

**Definition 1.** A complete bipartite graph is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. If partite sets $V_1$ and $V_2$ are having $m$ and $n$ vertices respectively then the related complete bipartite graph is denoted by $K_{m,n}$ and $V_1$ is called $m$-vertices part and $V_2$ is called $n$-vertices part of $K_{m,n}$.

**Definition 2.** The double triangular snake $D(T_n)$ is the graph obtained from the path $u_1, u_2, u_3, \ldots, u_n$ by joining $u_i$ and $u_{i+1}$ with two new vertices $v_i$ and $w_i$ for $1 \leq i \leq (n-1)$.

**Definition 3.** The wheel graph $W_n$ is the joining of the graphs $C_n$ and $K_1$ that is, $W_n = C_n + K_1$. Here the vertices corresponding to $C_n$ are called rim vertices and $C_n$ is called rim of $W_n$ while the vertex corresponds to $K_1$ is called apex vertex.

**Definition 4.** A helm $H_n$ is the graph obtained from the wheel $W_n$ by adding a pendant edge at each vertex on the wheel’s rim.

**Definition 5.** The flower graph $Fl_n$ is the graph obtained from a helm $H_n$ by joining each pendant vertex to the apex of the helm.

**Definition 6.** The graph $\langle C_m, K_{1,n} \rangle$ is the graph obtained from $C_m$ and $K_{1,n}$ by identifying any one of the vertices of $C_m$ with the central vertex of $K_{1,n}$ [11].

**Definition 7.** The graph $\langle C_m * K_{1,n} \rangle$ is the graph obtained from $C_m$ and $K_{1,n}$ by identifying any one of the vertices of $C_m$ with a pendant vertex of $K_{1,n}$ (that is a non-central vertex of $K_{1,n}$)[11].

## 2 Preliminary Results

The following results have been proved in [3].
- Every path $P_n$ is an edge pair sum graph for $n \geq 3$.
- Every cycle $C_n$ ($n \geq 3$) is an edge pair sum graph.
- The star graph $K_{1,n}$ is an edge pair sum graph if and only if $n$ is even.
- The complete graph $K_4$ is not an edge pair sum graph.

3 Main Results

Theorem 1. The complete bipartite graph $K_{2,n}$ is an edge pair sum graph.

Proof. Define $V(K_{2,n}) = \{u, v, u_i : 1 \leq i \leq 2n\}$ and $E(K_{2,n}) = \{e_i = uu_i, e'_i = vv_i : 1 \leq i \leq 2n\}$ are the vertices and edges of the graph $K_{2,n}$. Define the edge labeling $f : E(K_{2,n}) \rightarrow \{\pm 1, \pm 2, \pm 3, ..., \pm 4n\}$. For $1 \leq i \leq 2n$ $f(e_i) = i$ and $f(e'_i) = -(2n - i + 1)$. Then the induced vertex labeling is as follows: For $1 \leq i \leq n$ $f^*(u_i) = -(2n - 2i + 1)$ and $f^*(u_{n+i}) = 2i - 1$, $f^*(u) = n(2n + 1) = -f^*(v)$. Then $f^*(V(K_{2,n})) = \{\pm 1, \pm 3, \pm 5, ..., \pm (2n - 1), \pm (2n^2 + n)\}$. Hence $f$ is an edge pair sum labeling for all $n$. The example for the edge pair sum graph labeling of $K_{2,2}$ is shown in Figure 1.

Figure 1: Edge pair sum labeling for the graph $K_{2,2}$

Theorem 2. The double triangular snake $D(T_n)$ is an edge pair sum graph.

Proof. Let $G(V, E) = D(T_n)$. Then $V(G) = \{u_i : 1 \leq i \leq n, v_i, w_i : 1 \leq i \leq (n - 1)\}$ and $E(G) = \{e_{2i-1} = u_i v_i, e_{2i} = u_i + v_i, e'_{2i-1} = u_i w_i, e'_{2i} = u_i + w_i, e_i = u_i w_i : 1 \leq i \leq (n - 1)\}$ are the vertices and edges of the graph $G$. Define the edge labeling $f : E(G) \rightarrow \{\pm 1, \pm 2, \pm 3, ..., \pm (5n - 5)\}$ by considering the following two cases.

Case (i). $n$ is even.

Subcase (a). $n = 4$.

Define $f(e''_1) = -2$, $f(e''_2) = -1$, $f(e''_3) = 3$, for $1 \leq i \leq (n - 1)$ $f(e'_{2i-1}) = n - 2 + 2i = -f(e'_{2i-1})$ and $f(e_{2i}) = n - 1 + 2i = -f(e_{2i})$. The induced vertex labeling is as follows: $f^*(u_1) = -2 = -f^*(u_3)$, $f^*(u_2) = -3 = -f^*(u_4)$ and for $1 \leq i \leq (n - 1)$ $f^*(v_i) = 2n - 3 + 4i = -f^*(w_i)$. Then $f^*(V(G)) = \{\pm 2, \pm 3, \pm (2n + 1), \pm (2n + 5), \pm (2n + 9), ..., \pm (6n - 7)\}$. Hence $f$ is an edge pair sum labeling for $n = 4$.

Subcase (b). $n > 4$. 


For $1 \leq i \leq (n - 1)$ $f(e_{2i-1}) = n - 2 + 2i = -f(e'_{2i-1})$ and $f(e_{2i}) = n - 1 + 2i = -f(e'_{2i})$, for $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$ $f(e''_i) = n + 1 - 2i$, $f(e''_{2i-1}) = -2$, $f(e''_{2i}) = -1$, $f(e''_{2i+1}) = 3$ and for $\frac{n}{2} + 1 \leq i \leq (n - 1)$ $f(e''_i) = n - 1 - 2i$. The induced vertex labeling is as follows: for $1 \leq i \leq (n - 1)$ $f^*(v_i) = 2n - 3 + 4i = -f^*(w_i)$, $f^*(u_i) = n - 1 = -f^*(u_{n-i})$, for $2 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 2$ $f^*(u_i) = 2(n + 2 - 2i)$, $f^*(u_{2i-1}) = 3 = -f^*(u_{2i})$, $f^*(u_{2i+1}) = 2 = -f^*(u_{2i+2})$ and for $(\frac{n}{2} + 3) \leq i \leq (n - 1)$ $f^*(u_i) = 2(n - 2i)$. Then we get $f^*(V(G)) = \{\pm 2, \pm 3, \pm 12, \pm 16, \pm 20, ..., \pm (2n - 4), \pm (n - 1), \pm (2n + 1), \pm (2n + 5), \pm (2n + 9), ..., \pm (6n - 7)\}$. Hence $f$ is an edge pair sum labeling. The example for the edge pair sum graph labeling of $D(T_6)$ is shown in Figure 2.

![Figure 2: Edge pair sum labeling for the graph $D(T_6)$](image)

Case(ii). $n$ is odd.

Subcase (a). $n = 3$.

Define $f(e''_1) = -2$, $f(e''_2) = 1$, for $1 \leq i \leq (n - 1)$ $f(e_{2i-1}) = 2i + 1 = -f(e'_{2i-1})$ and $f(e_{2i}) = 2i + 2 = -f(e'_{2i})$. The induced vertex labeling is as follows: $f^*(v_1) = -2$, $f^*(u_2) = -1 = -f^*(u_3)$ and for $1 \leq i \leq (n - 1)$ $f^*(v_i) = 4i + 3 = -f^*(w_i)$. Then $f^*(V(G)) = \{\pm 1, \pm 7, \pm 11, \pm 15, ..., \pm (4n - 1)\} \cup \{-2\}$. Hence $f$ is an edge pair sum labeling for $n = 3$.

Subcase (b). $n > 3$.

For $1 \leq i \leq (n - 1)$ $f(e_{2i-1}) = n - 1 + 2i = -f(e'_{2i-1})$ and $f(e_{2i}) = n + 2i = -f(e'_{2i})$, for $1 \leq i \leq \left\lceil \frac{n - 3}{2} \right\rceil$ $f(e''_i) = -(n + 2 - 2i)$, $f(e''_{2i-1}) = 1$, $f(e''_{2i}) = 2$ and for $\frac{n + 3}{2} \leq i \leq (n - 1)$ $f(e''_i) = -n + 2 + 2i$. The induced vertex labeling is as follows: for $1 \leq i \leq (n - 1)$ $f^*(v_i) = 2n - 1 + 4i = -f^*(w_i)$, $f^*(u_1) = n = -f^*(u_{n-i})$, for $2 \leq i \leq \left\lceil \frac{n - 3}{2} \right\rceil$ $f^*(u_i) = 2(-n - 3 + 2i)$, $f^*(u_{2i-1}) = -3 = -f^*(u_{2i})$, $f^*(u_{2i+1}) = 6$ and for $\left\lceil \frac{n + 5}{2} \right\rceil \leq i \leq (n - 1)$ $f^*(u_i) = -2(n - 1 - 2i)$. Then we get $f^*(V(G)) = \{\pm 3, \pm 12, \pm 16, \pm 20, ..., \pm (2n - 2), \pm n, \pm (2n + 3), \pm (2n + 7), \pm (2n + 11), ..., \pm (6n - 5)\} \cup \{6\}$. Hence $f$ is an edge pair sum labeling.

The example for the edge pair sum graph labeling of $D(T_5)$ is shown in Figure 3.

**Theorem 3.** The wheel graph $W_n$ is an edge pair sum graph.

**Proof.** Let $V(W_n) = \{v, u_i : 1 \leq i \leq n\}$ and $E(W_n) = \{e'_i = vv_i : 1 \leq i \leq n, e_1 = u_nu_1, e_{1+i} = u_{i}u_{1+i} : 1 \leq i \leq (n - 1)\}$ are the vertices and edges of the graph $W_n$. Define
the edge labeling $f : E(W_n) \to \{\pm 1, \pm 2, \pm 3, ..., \pm 2n\}$ by considering the following two cases.

**Case(i).** $n$ is even.

For $1 \leq i \leq \frac{n}{2}$, $f(e_i) = i$ and $f(e'_{\frac{n}{2}+i}) = -(\frac{n}{2}+1-i)$, for $1 \leq i \leq (\frac{n}{2}-1)$, $f(e'_i) = n+2+2i$ and $f(e'_{\frac{n}{2}+i}) = -2(n+1-i)$, $f(e'_2) = -(n+2)$ and $f(e'_n) = (\frac{n}{2}+1)$. Then the induced vertex labeling is as follows: for $1 \leq i \leq (\frac{n}{2}-1)$, $f^*(u_i) = n+3+4i$ and $f^*(u_{\frac{n}{2}+i}) = -(3n+3-4i)$, $f^*(u_2) = -(n+2)$ and $f^*(u_n) = (\frac{n}{2}+1) = -f^*(v)$. Therefore we get $f^*(V(W_n)) = \{\pm (\frac{n}{2}+1), \pm (n+7), \pm (n+11), \pm (n+15), ..., \pm (3n-1)\} \cup \{-(n+2)\}$. Hence $f$ is an edge pair sum labeling. The examples for the edge pair sum graph labeling of $W_4$ and $W_8$ are shown in Figure 4.

**Case(ii).** $n$ is odd.
Subcase (a). $n = 1, 2 \pmod{3}$.
Define $f(e_1) = 1$, $f(e_2) = -2$, for $1 \leq i \leq \left( \frac{n-3}{2} \right)$, $f(e_{2i+1}) = 2 + i = -f(e_{n+3+i}), f(e_{n+4+i}) = -1$, $f(e'_{1}) = n - 1 = -f(e'_{2}),$ for $1 \leq i \leq \left( \frac{n-5}{2} \right)$, $f(e'_{2i+1}) = n - 1 + 2i = -f(e'_{n+3+i}),$ $f(e'_{n+4+i}) = 2n - 5 = -f(e'_{n})$ and $f(e'_{n+2}) = 2$. Then the induced vertex labeling is as follows: $f^*(u_1) = n - 2 = -f^*(u_2)$, for $1 \leq i \leq \left( \frac{n-5}{2} \right)$, $f^*(u_{2i+1}) = n + 4 + 4i = -f^*(u_{n+3+i}),$ $f^*(u_{n+4+i}) = \frac{5n-11}{2} = -f^*(u_n)$ and $f^*(u_{n+2}) = -2 = f^*(v)$. Then $f^*(V(W_n)) = \{ \pm 2, \pm \left( \frac{5n-11}{2} \right), \pm (n - 2), \pm (n + 8), \pm (n + 12), \pm (n + 16), \ldots, \pm (3n - 6) \}$. Hence $f$ is an edge pair sum labeling.

Subcase (b). $n = 0 \pmod{3}$.
Define $f(e_1) = 3$, $f(e_2) = 4$, $f(e_3) = -5$, for $1 \leq i \leq \left( \frac{n-5}{2} \right)$, $f(e_{3i+1}) = 5 + i = -f(e_{n+5+i}),$ $f(e_{n+4+i}) = -3$, $f(e_{n+5+i}) = -4$, $f(e'_1) = -(n + 1), f(e'_2) = -1 = -f(e'_3)$, for $1 \leq i \leq \left( \frac{n-3}{2} \right)$, $f(e'_{3i+1}) = \frac{n+5}{2} + i$, $f(e'_{n+5+i}) = 5$ and for $1 \leq i \leq \left( \frac{n-5}{2} \right)$, $f(e'_{n+5+i}) = -\left( \frac{n+5}{2} + i \right)$. Then the induced vertex labeling is as follows: $f^*(u_1) = -(n - 6) = -f^*(u_{n+3}), f^*(u_2) = -2 = -f^*(u_3)$, for $1 \leq i \leq \left( \frac{n-7}{2} \right)$, $f^*(u_{2i+1}) = \frac{n+5}{2} + 11 + 3i = -f^*(u_{n+5+i}), f^*(u_{n+4+i}) = \frac{3n-1}{2} = -f^*(u_n)$ and $f^*(u_{n+5+i}) = -5 = f^*(v)$. Then $f^*(V(W_n)) = \{ \pm 2, \pm 5, \pm (n - 6), \pm \left( \frac{3n-1}{2} \right), \pm \left( \frac{n+33}{2} \right), \pm \left( \frac{n+39}{2} \right), \pm \left( \frac{n+45}{2} \right), \ldots, \pm (2n + 3) \}$. Hence $f$ is an edge pair sum labeling. The examples for the edge pair sum graph labeling of $W_7$ and $W_9$ are shown in Figure 5.

\[ \text{Figure 5: Edge pair sum labeling for the graph } W_7 \text{ and } W_9 \]

**Theorem 4.** The flower graph $Fl_n$ is an edge pair sum graph.

**Proof.** Let $V(Fl_n) = \{ w, u_i, v_i : 1 \leq i \leq n \}$ and $E(Fl_n) = \{ e_i = u_n u_i, e_{1+i} = u_i u_{1+i} : 1 \leq i \leq (n - 1), e''_i = w u_i, e''_{1+i} = w v_i : 1 \leq i \leq n \}$ are the vertices and edges of the graph $Fl_n$. Define the edge labeling $f : E(Fl_n) \rightarrow \{ \pm 1, \pm 2, \pm 3, \ldots, \pm 4n \}$ by considering the following two cases.
Case(i). \( n \) is odd.
For \( 1 \leq i \leq \frac{n+1}{2} \) \( f(e_{2i-1}) = -(4i - 2) \), for \( 1 \leq i \leq \frac{n-1}{2} \) \( f(e_{2i}) = -(2n + 4i) \), for \( 1 \leq i \leq n \) \( f(e'_i) = -(2i - 1) \) and \( f(e''_i) = -(2n - 1 + 2i) \). Then the induced vertex labeling is as follows: for \( 1 \leq i \leq (n-1) \) \( f^*(u_i) = -(2n + 2 + 4i) \), for \( 1 \leq i \leq n \) \( f^*(v_i) = 2n - 2 + 4i \), \( f^*(u_n) = -(2n + 2) \) and \( f^*(w) = 2n^2 \). Then we get \( f^*(V(Fl_n)) = \{ \pm(2n+2), \pm(2n+6), \pm(2n+10), \pm(2n+14), \ldots, \pm(6n-2) \} \cup \{2n^2\} \). Hence \( f \) is an edge pair sum labeling for \( n \) is odd. The example for the edge pair sum graph labeling of \( Fl_5 \) is shown in Figure 6.

![Figure 6: Edge pair sum labeling for the graph Fl5](image)

Case(ii). \( n \) is even.
Subcase (a). \( n = 4 \).
For \( 1 \leq i \leq \frac{n}{2} \) \( f(e_i) = -f(e_{\frac{n}{2}+i}) \), \( f(e'_i) = 4n-i+1 = -f(e''_i) \), \( f(e'_{\frac{n}{2}+i}) = -(\frac{7n}{2}+i) = -f(e''_{\frac{n}{2}+i}) \), \( f(e''_i) = -(\frac{n}{2} - 1 + 2i) \) and \( f(e''_{\frac{n}{2}+i}) = n + 2i - 1 \). Then the induced vertex labeling is as follows: for \( 1 \leq i \leq \frac{n-2}{2} \) \( f^*(u_i) = 2i + 1 \) = \( -f^*(u_{\frac{n}{2}+i}) \), \( f^*(u_{\frac{n}{2}}) = 1 = -f^*(w) = \frac{n^2}{2} \). Then \( f^*(V(Fl_n)) = \{ \pm 1, \pm 3, \pm 5, \pm 7, \ldots, \pm (n-1), \pm (\frac{9n+2}{2}), \pm (\frac{9n+4}{2}), \pm (\frac{9n+6}{2}), \ldots, \pm 5n \} \cup \{\frac{n^2}{2}\} \). Hence \( f \) is an edge pair sum labeling.
Subcase (b). \( n = 2 \mod 4 \).
For \( 1 \leq i \leq \frac{n}{2} \) \( f(e_i) = -f(e_{\frac{n}{2}+i}) \), \( f(e'_i) = 4n-i+1 = -f(e''_i) \), \( f(e'_{\frac{n}{2}+i}) = -(\frac{7n}{2}+i) = -f(e''_{\frac{n}{2}+i}) \), \( f(e''_i) = -(\frac{n}{2} - 1 + 2i) \) and \( f(e''_{\frac{n}{2}+i}) = n + 2i - 1 \). Then the induced vertex labeling is as follows: for \( 1 \leq i \leq \frac{n-2}{2} \) \( f^*(u_i) = 2i + 1 = -f^*(u_{\frac{n}{2}+i}) \), \( f^*(u_{\frac{n}{2}}) = \frac{n^2}{2} = -f^*(u_n) \),
for \(1 \leq i \leq \frac{n}{2}\), \(f^*(v_i) = -\frac{1}{2}(9n + 2i) = -f^*(v_{n+i})\) and \(f^*(w) = \frac{n^2}{2}\). Then \(f^*(V(Fl_n)) = \{\pm(\frac{n-2}{2}), \pm 3, \pm 5, \pm 7, ..., \pm(n-1), \pm(\frac{3n+2}{2}), \pm(\frac{3n+4}{2}), \pm(\frac{3n+6}{2}), ..., \pm 5n\} \cup \{\pm \frac{n^2}{2}\}\). Hence \(f\) is an edge pair sum labeling.

Subcase (c). \(n = 0 (\text{mod} 4)\).

For \(1 \leq i \leq \frac{n}{2}\), \(f(e_i) = i, f(e_{\frac{n}{2}+1}) = -2, f(e_{\frac{n}{2}+2}) = -1, f(1 \leq i \leq \frac{n-4}{2}) \geq f(e_{\frac{n}{2}+2+i}) = f(e_{\frac{n}{2}+i}) - 2, f(1 \leq i \leq \frac{n}{2})\).

\(f(1 \leq i \leq \frac{n}{2})\)

\(f(e_i') = 4n - i + 1 = -f(e_i''), f(e_{\frac{n}{2}+i}) = -(\frac{7n+2}{2} - i) = -f(e_{2i}''),\)

\(f(e_i'') = n + 2i\) and \(f(v_{\frac{n}{2}+i}) = n + 2i\). Then the induced vertex labeling is as follows:

\(1 \leq i \leq \frac{n}{2}\) and \(f^*(v_{\frac{n}{2}+i}) = -f^*(v_{\frac{n}{2}+i})\) and \(f^*(w) = \frac{n^2}{2}\). Then \(f^*(V(Fl_n)) = \{\pm(\frac{n}{2} - 2), \pm 3, \pm 5, \pm 7, ..., \pm(n-1), \pm(\frac{3n+4}{2}), \pm(\frac{3n+6}{2}), \pm(\frac{3n+10}{2}), ..., \pm 5n\}\) \cup \{\pm \frac{n^2}{2}\}\). Hence \(f\) is an edge pair sum labeling. The example for the edge pair sum graph labeling of \(Fl_6\) is shown in Figure 7.

![Figure 7: Edge pair sum labeling for the graph Fl_6](image)

**Theorem 5.** The graph \(\langle C_m, K_{1,n} \rangle\) is an edge pair sum graph for \(m \geq 4\) and \(n\) is odd.

**Proof.** Let \(V(\langle C_m, K_{1,n} \rangle) = \{u_i: 1 \leq i \leq m, v_i: 1 \leq i \leq n\}\) and \(E(\langle C_m, K_{1,n} \rangle) = \{e_i = u_iu_{i+1}: 1 \leq i \leq (m - 1), e_m = u_mu_1, e_i = u_iv_i: 1 \leq i \leq n\}\) are the vertices and edges of the graph \(\langle C_m, K_{1,n} \rangle\). Define the edge labeling \(f: E(\langle C_m, K_{1,n} \rangle) \rightarrow \{-1, 2, -3, ..., \pm 1, 2, -3, ..., \pm (m + n)\}\) by considering the following four cases.

**Case(i).** \(m = 4\).

Define \(f(e_1) = 2 = -f(e_3), f(e_2) = -1 = -f(e_4), f(e'_1) = 3\) and for \(1 \leq i \leq \frac{n-1}{2}\)

\(f(e'_i + i) = 6 + i = -f(e_{2i + 1}).\)

The induced vertex labeling is as follows: \(f^*(u_1) = 6, f^*(u_2) = 1 = -f^*(u_4), f^*(u_3) = -3 = -f^*(v_1)\) and for \(1 \leq i \leq \frac{n-1}{2}\).

\(f^*(v_{i+1}) = 6 + i = -f^*(v_{n+i}).\)

Then \(f^*(V(\langle C_m, K_{1,n} \rangle)) = \{\pm 1, \pm 3, \pm 7, \pm 9, ..., \pm (\frac{n+1}{2})\} \cup \{6\}\). Hence \(f\) is an edge pair sum labeling.

**Case(ii).** \(m = 5\).
Define \( f(e_1) = -1 = -f(e_3) \), \( f(e_2) = -3 = -f(e_5) \), \( f(e_4) = -2 = f(e_1') \) and for \( 1 \leq i \leq \frac{n-1}{2} \), \( f(e_{i+1}) = 4+i = -f(e'_{\frac{n+1}{2}+i}) \). The induced vertex labeling is as follows: \( f^*(u_1) = 4 = -f^*(u_2), f^*(u_3) = -2 = -f^*(v_1), f^*(u_4) = -1 = -f^*(v_5) \) and for \( 1 \leq i \leq \frac{n-1}{2} ) \), \( f^*(v_{i+1}) = 4+i = -f^*(v_{\frac{n+1}{2}+i}) \). Then \( f^*(V((C_m,K_{1,n}))) = \{ \pm 1, \pm 2, \pm 4, \pm 5, \pm 6, \pm 7, ... , \pm (\frac{n+1}{2}) \} \). Hence \( f \) is an edge pair sum labeling. The examples for the edge pair sum graph labeling of \( < C_4, K_{1,3} > \) and \( < C_5, K_{1,5} > \) are shown in Figure 8.

![Figure 8: Edge pair sum labeling for the graph \( < C_4, K_{1,3} > \) and \( < C_5, K_{1,5} > \)](image-url)

**Case (iii).** \( m \) is even.

Subcase (a). \( m = 2 \pmod{4} \).

Define \( f(e_1) = \frac{m}{2} = -f(e_{m+2}) \), for \( 1 \leq i \leq \frac{m-2}{2} \), \( f(e_{i+1}) = -i = -f(e_{m+2+i}) \), \( f(e_1') = m-1 \) and for \( 1 \leq i \leq \frac{m-1}{2} \), \( f(e'_{i+1}) = m-1+2i = -f(e'_{\frac{m+1}{2}+i}) \). The induced vertex labeling is as follows: \( f^*(u_1) = 2(m-1), f^*(u_2) = \frac{m-2}{2} = -f^*(u_{m+4}) \), for \( 1 \leq i \leq \frac{m-4}{2} \), \( f^*(u_{2+i}) = -2i-1 = -f^*(u_{m+4+i}), f^*(u_{m+2}) = -m+1 = f^*(v_1) \) and for \( 1 \leq i \leq \frac{m-1}{2} \), \( f^*(v_{i+1}) = m-1+2i = f^*(v_{\frac{m+1}{2}+i}) \). Then \( f^*(V((C_m,K_{1,n}))) = \{ \pm (m-1), \pm (\frac{m-2}{2}), \pm 3, \pm 5, \pm 7, ... , \pm (m-3), \pm (m+1), \pm (m+3), \pm (m+5), ... , \pm (m+n-2) \} \). Hence \( f \) is an edge pair sum labeling.

Subcase (b). \( m = 0 \pmod{4} \).

Define \( f(e_1) = \frac{m}{2} = -f(e_{m+2}) \), \( f(e_2) = -2, f(e_3) = -1, \) for \( 1 \leq i \leq \frac{m-8}{2} \), \( f(e_{3+i}) = f(e_{i+1})-2, f(e_{m+2}) = -\left( \frac{m}{2} - 1 \right) = -f(e_m) \), for \( 1 \leq i \leq \frac{m-4}{2} \), \( f(e_{m+2+i}) = i, \) \( f(e_1') = m-1 \) and for \( 1 \leq i \leq \frac{m-1}{2} \), \( f(e'_{i+1}) = m-1+2i = -f(e'_{\frac{m+1}{2}+i}) \). The induced vertex labeling is as follows: \( f^*(u_1) = 2m-2, f^*(u_2) = \frac{m-4}{2} = f^*(u_{m+4}) \), for \( 1 \leq i \leq \frac{m-4}{2} \), \( f^*(u_{2+i}) = -(2i+1) = f^*(u_{m+4+i}), f^*(u_{m+2}) = -m+1 = f^*(v_1) \) and for \( 1 \leq i \leq \frac{m-1}{2} \), \( f^*(v_{i+1}) = m-1+2i = f^*(v_{\frac{m+1}{2}+i}) \). Then \( f^*(V((C_m,K_{1,n}))) = \{ \pm (m-1), \pm (\frac{m-4}{2}), \pm 3, \pm 5, \pm 7, ... , \pm (m-3), \pm (m+1), \pm (m+3), \pm (m+5), ... , \pm (m+n-2) \} \). Hence \( f \) is an edge pair sum labeling.

**Case (iv).** \( m \) is odd.

Subcase (a). \( m = 1, 3 \pmod{4} \).

For \( 1 \leq i \leq \frac{m-3}{2} \), \( f(e_i) = -2 - i = -f(e_{m+i}), f(e_{m-1}) = 1 = -f(e_m) \), \( f(e_{m+1}) = -2 = -f(e_{m+1}) \).

Figure 8: Edge pair sum labeling for the graph \( < C_4, K_{1,3} > \) and \( < C_5, K_{1,5} > \)
- \( f(e'_i) \) and for \( 1 \leq i \leq \frac{m-1}{2} \) \( f(e'_{i+1}) = m+2i = -f(e'_{m+1+i}) \). The induced vertex labeling is as follows: \( f^*(u_1) = -2 = -f^*(v_1) \), for \( 1 \leq i \leq \frac{m-5}{2} \) \( f^*(u_{i+1}) = -(5+2i) = -f^*(u_{m+2+i}) \), \( f^*(u_{m+1}) = -(\frac{m-1}{2}) = -f^*(u_m) \), \( f^*(u_{m+1}) = -1 = -f^*(u_{m+2+i}) \) and for \( 1 \leq i \leq \frac{m-1}{2} \) \( f^*(v_{i+1}) = m+2i = -f^*(v_{m+1+i}) \). Then \( f^*(V(C_m,K_{1,n})) = \{ \pm 1, \pm 2, \pm (\frac{m-1}{2}), \pm 7, \pm 9, \pm 11, ..., \pm m, \pm (m+2), \pm (m+4), \pm (m+6), ..., \pm (m+n-1) \} \).

Hence \( f \) is an edge pair sum labeling.

Subcase (b). \( m = 0 (\text{mod} 3) \).

Define \( f(e_1) = 2 = -f(e_{\frac{m+2}{2}}) \), \( f(e_2) = -3 = f(e'_1) \), for \( 1 \leq i \leq \frac{m-5}{2} \) \( f(e_{3+i}) = 3+i = -f(e'_{\frac{m+5}{2}}) \), \( f(e_{m+1}) = -1 = -f(e_m) \) and for \( 1 \leq i \leq \frac{m-1}{2} \) \( f(e'_{i+1}) = m+2i = -f(e'_{m+1+i}) \). The induced vertex labeling is as follows: \( f^*(u_1) = 6 = -f^*(u_{\frac{m+2}{2}}) \), \( f^*(u_2) = -1 = -f^*(u_3) \), for \( 1 \leq i \leq \frac{m-7}{2} \) \( f^*(u_{3+i}) = -(7+2i) = -f^*(u_{m+3+i}) \), \( f^*(u_{m+1}) = \frac{m-1}{2} = -f^*(u_m) \), \( f^*(u_{m+1}) = -3 = -f^*(v_1) \) and for \( 1 \leq i \leq \frac{m-1}{2} \) \( f^*(v_{i+1}) = m+2i = -f^*(v_{m+1+i}) \). Then \( f^*(V(C_m,K_{1,n})) = \{ \pm 1, \pm 3, \pm 6, \pm (\frac{m-1}{2}), \pm 9, \pm 11, \pm 13, ..., \pm m, \pm (m+2), \pm (m+4), \pm (m+6), ..., \pm (m+n-1) \} \). Hence \( f \) is an edge pair sum labeling. The example for the edge pair sum graph labeling of \( <C_9,K_{1,3}> \) is shown in Figure 10.

**Theorem 6.** The graph \( <C_m,K_{1,n}> \) is an edge pair sum graph for \( m \geq 4 \) and \( n \) is even.

**Proof.** In [3] we have proved that \( C_m \) is an edge pair sum graph for \( m \geq 3 \). Let \( f \) be an edge pair sum labeling of \( C_m \).

Then \( f^*(V(C_m)) = \{ \pm K_1, \pm K_2, ..., \pm K_p \} \) if \( p \) is even.

\( f^*(V(C_m)) = \{ \pm K_1, \pm K_2, ..., \pm K_{\frac{m-1}{2}} \} \cup \{ K_{\frac{m}{2}} \} \) if \( p \) is odd.

Let the vertex and edge sets are as follows: \( V(<C_m,K_{1,n})> = V(C_m) \cup \{ v_i : 1 \leq i \leq n \} \) and \( E(<C_m,K_{1,n})> = E(C_m) \cup \{ e'_i : 1 \leq i \leq n \} \).

Define the edge labeling \( h : E(<C_m,K_{1,n})> \rightarrow \{ \pm 1, \pm 2, \pm 3, ..., \pm (q+n) \} \).

\( h(e) = f(e) \) if \( e \in E(C_m) \)

\( h(e'_i) = q+2i : 1 \leq i \leq \frac{n}{2} \)
The induced vertex labeling is as follows:
$$h^*(v_i) = q + 2i : 1 \leq i \leq \frac{n}{2}$$
$$h^*(v_{\frac{n}{2}+i}) = -(q + 2i) : 1 \leq i \leq \frac{n}{2}$$
Then $h^*(V(\langle C_m, K_1, n \rangle)) = \{\pm K_1, \pm K_2, ..., \pm K_p \}$ if $p$ is even.
$$h^*(V(\langle C_m, K_1, n \rangle)) = \{\pm K_1, \pm K_2, ..., \pm K_{\frac{p-1}{2}} \} \bigcup \{K_{\frac{p}{2}} \}$$ if $p$ is odd.
Hence $h$ is an edge pair sum labeling.

**Corollary 7.** The graph $\langle C_m \ast K_1, n \rangle$ is an edge pair sum graph for $m \geq 4$ and $n$ is odd.

We use the previous edge labeling for this corollary. The example for the edge pair sum graph labeling of $\langle C_4 \ast K_1, 3 \rangle$ is shown in Figure 11.

**References**


