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Total vertex irregularity strength of corona product of some graphs

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ABSTRACT

A vertex irregular total k-labeling of a graph G with vertex set V and edge set E is an assignment of positive integer labels $\{1, 2, ..., k\}$ to both vertices and edges so that the weights calculated at vertices are distinct. The total vertex irregularity strength of G, denoted by $tvs(G)$is the minimum value of the largest label k over all such irregular assignment. In this paper, we study the total vertex irregularity strength for $n \geq 3, m \geq 2, P_n \odot K_1, P_n \odot K_2, C_n \odot K_2, L_n \odot K_1, CL_n \odot K_1, P_2 \odot C_n, P_n \odot \overline{K_m}, C_n \odot \overline{K_m}$

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1 Introduction

As a standard notation, assume that $G = (V, E)$ is a finite ,simple and undirected graph with p vertices and q edges. A labeling of a graph is any mapping that sends some set of graph elements to a set of numbers(usually positive integers). If the domain is the vertex -set (or) the edge- set ,the labeling are called respectively vertex labeling (or) edge labeling. If the domain is V∪E then we call the labeling a total labeling.
In many cases it is interesting to consider the sum of all labels associated with a graph element. This will be called the weight of element. The graph labeling has caught the attention of many authors and many new labeling results appear every year. This popularity is not only due to the mathematical challenges of graph labeling, but also for the wide range of its application for instance X-ray, crystallography, coding theory, radar, astronomy, circuit, design, network design and communication design.
Chartrand et al. [7] introduced labelings of the edges of a graph G with positive integers such that the sum of the labels of edges incident with a vertex is different for all the vertices. Such labelings were called *irregular assignments* and *the irregularity strength s(G)* of a graph G is known as the minimum k for which G has an irregular assignment using labels at most k.
The irregularity strength $s(G)$ can be interpreted as the smallest integer k for which G can be turned into a multigraph G' by replacing each edge by a set of at most k parallel edges, such that the degrees of the vertices in G' are all different.Finding the irregularity strength of a graph seems to be hard even for graphs with simple structure in [6, 16]. Karonski et al. [9] conjectured that the edges of every connected graph of order at least 3 can be assigned labels from $\{1, 2, 3\}$such that for all pairs of adjacent vertices the sums of the labels of the incident edges are different.
Motivated by irregular assignments Bača et al. [5] defined a labeling $f : V(G) \cup E(G) \to \{1, 2, ..., k\}$ to be a vertex irregular total k-labeling if for every two different vertices x and y the vertex-weights $wt_f(x) \neq wt_f(y)$ where the vertex-weight$wt_f(x) = f(x) + \sum_{xy \in E} f(xy)$. A minimum k for which G has a vertex irregular total k-labeling is defined as the total vertex irregularity strength of G and denoted by $tvs(G)$. It is easy to see that irregularity strength $s(G)$ of a graph G is defined only for graphs containing at most one isolated vertex and no connected component of order 2. On the other hand, the total vertex irregularity strength $tvs(G)$ is defined for every graph G. If an edge labeling $f : E \to \{1, 2, ..., \delta(G)\}$ provides the irregularity strength $s(G)$, then we extend this labeling total labeling ϕ in such a way

$$\phi(xy) = f(xy) \text{ for every} xy \in E(G),$$
$$\phi(x) = 1 \text{ for every } x \in V(G).$$

Thus,the total labeling ϕ is a vertex irregular total labeling and for graphs with no component of order ≤ 2 has $tvs(G) \leq s(G)$. Nierhoff [10] proved that for all (p, q)-graphs G with no component of order at most 2 and $G \neq K_3$the irregularity strength $s(G) \leq p - 1$. From this result it follows that

$$tvs(G) \leq p - 1. \tag{1}$$

In [5] several bounds and exact values of $tvs(G)$ were determined for different types of graphs (in particular for stars, cliques and prisms). Among others, the authors proved that for every (p, q)- graph G with minimum degree $\delta = \delta(G)$and maximum degree $\Delta = \Delta(G)$,

$$\left\lceil \frac{p + \delta(G)}{\Delta(G) + 1} \right\rceil \leq tvs(G) \leq p + \Delta(G) - 2\delta(G) + 1. \tag{2}$$

In the case of r-regular graphs (2) gives

$$\left\lceil \frac{p + r}{r + 1} \right\rceil \leq tvs(G) \leq p - r + 1. \tag{3}$$

For graphs with no component of order≤ 2, Bača et al. [5] strengthened also these upper bounds, proving that $tvs(G) \leq p - 1 - \left\lceil \frac{p-2}{\Delta(G)+1} \right\rceil$. These results were then improved by Przybylo in [14] for sparse graphs and for graphs with large minimum degree. Ahmad et al [1,3] determined an exact value of the total vertex irregularity strength for wheel related graphs and cubic graphs. Wijaya et al. [18] determined an exact value of the total vertex irregularity strength for complete bipartite graphs. Wijaya et al. [17] found the exact values of tvs for wheels, fans, suns and friendship graphs. Nurdin et al. [11] proved the following lower bound of tvs for any graph G.

Theorem:1 [11] Let G be a connected graph having n_i vertices of degree $i(i = \delta, \delta + 1, \delta + 2, ..., \Delta)$ whereδ and Δ are the minimum and the maximum degree of G, respectively. Then

$$tvs(G) \geq max \left\{ \left\lceil \frac{\delta + n_\delta}{\delta + 1} \right\rceil, \left\lceil \frac{\delta + n_\delta + n_{\delta+1}}{\delta + 2} \right\rceil, ..., \left\lceil \frac{\delta + \sum_{i=\delta}^{\Delta}(n_i)}{\Delta + 1} \right\rceil \right\}. \tag{4}$$

Also [11] Nurdin et al. posed the following conjecture.

Conjecture:1.2 [11] Let G be a connected graph having n_i vertices of a degree $i(i = \delta, \delta + 1, \delta + 2, ..., \Delta)$ whereδ and Δ are the minimum and the maximum degree of G, respectively.Then

$$tvs(G) = max \left\{ \left\lceil \frac{\delta + n_\delta}{\delta + 1} \right\rceil, \left\lceil \frac{\delta + n_\delta + n_{\delta+1}}{\delta + 2} \right\rceil, ..., \left\lceil \frac{\delta + \sum_{i=\delta}^{\Delta}(n_i)}{\Delta + 1} \right\rceil \right\}. \tag{5}$$

Conjecture 1.2 has been verified by several authors for several families of graphs. For a regular Hamiltonian (p, q) graph G, it was showed in [5] that $tvs(G) \leq \left\lceil \frac{p+2}{3} \right\rceil$. Thus for cycle C_p we have that $tvs(G) = \left\lceil \frac{p+2}{3} \right\rceil$. In [11,12,13], Nurdin et al. found the exact values of total vertex irregularity strength of trees, several types of trees and disjoint union of t copies of path. Slamin et al. [15] determined the total vertex irregularity strength of disjoint union of sun graphs. In [2] Ahmad, Bača and Numan determined the total vertex irregularity strength of disjoint union of friendship graphs. In [4] Ashfaq Ahmad, Syed Ahtsham ul Haq Bokhary, Roslan Hasni and Slamin found the exact value of the total vertex irregularity strength of ladder related graphs.

Definition 1.1. *A corona product of two graphs G and H, denoted by $G \odot H$, is the graph that is obtained by placing a copy of G and $|V(G)|$ copies of H so that all vertices in the same copy of H are joined with exactly one vertex of G, while each vertex of G is joined to exactly one copy of H.*

Definition 1.2. *The ladder L_n is the planar grid P_n X P_2.*

2 Main Results

In this paper we determine exact values of the total vertex irregularity strength for $n \geq 3, m \geq 2, P_n \odot K_1, P_n \odot K_2, C_n \odot K_2, L_n \odot K_1, CL_n \odot K_1, P_2 \odot C_n, P_n \odot \overline{K_m}, C_n \odot \overline{K_m}$.

Theorem 2.1. $tvs(P_n \odot K_1) = \left\lceil \frac{n+1}{2} \right\rceil, n \geq 3$.

Proof. Let $V(P_n \odot K_1) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(P_n \odot K_1) = \{u_iv_i, v_iv_{i+1} : 1 \leq i \leq n\}$. Let $k = \left\lceil \frac{n+1}{2} \right\rceil$, then from (4) it follows that,$tvs(P_n \odot K_1) \geq max \left\{ \left\lceil \frac{n+1}{2} \right\rceil, \left\lceil \frac{n+3}{3} \right\rceil, \left\lceil \frac{2n+1}{4} \right\rceil \right\}$ $= \left\lceil \frac{n+1}{2} \right\rceil$. That is $tvs(P_n \odot K_1) \geq k$. To prove the reverse inequality,we define a function f from $V \cup E$ to $\{1, 2, 3, ..., k\}$ in the following way.

$$f(u_i) = \begin{cases} 1, & if \quad 1 \leq i \leq k \\ 1 + i - k, & if \quad k + 1 \leq i \leq n; \end{cases}$$

$$f(v_i) = \begin{cases} k, & if \quad i = 1 \\ 2k - n, & if \quad 2 \leq i \leq k \\ k - n + i, & if \quad k + 1 \leq i \leq n - 2 \\ n - k, & if \quad i = n - 1 \\ n + 3 - 2k, & if \quad i = n; \end{cases}$$

$$f(u_iv_i) = \begin{cases} i, & if \quad 1 \leq i \leq k \\ k, & if \quad k + 1 \leq i \leq n; \end{cases}$$

$$f(v_iv_{i+1}) = \begin{cases} n - k + 1, & if \quad 1 \leq i \leq n - 2 \\ k, & if \quad i = n - 1. \end{cases}$$

We observe that for $1 \leq i \leq n$,

$$wt(u_i) = i + 1,$$

$$wt(v_i) = \begin{cases} n+2, & if \;\; i = 1 \\ n+2+i, & if \;\; 2 \le i \le k \\ n+2+i, & if \;\; k+1 \le i \le n-2 \\ 2n+1, & if \;\; i = n-1 \\ n+3, & if \;\; i = n. \end{cases}$$

It is easy to check that the weights of the vertices are distinct.This labeling construction shows that $tvs(P_n \odot K_1) \le k$. Combining this with the lower bound, we conclude that $tvs(P_n \odot K_1) = k$. Figure 1 shows the vertex irregular total labeling of $P_8 \odot K_1$.

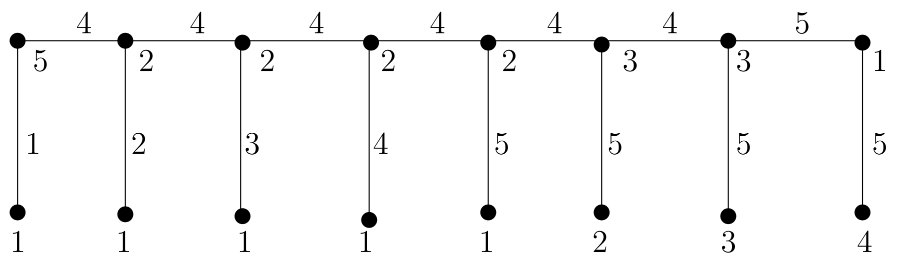

Figure 1: $tvs(P_8 \odot K_1) = 5$

□

Theorem 2.2. $tvs(P_n \odot K_2) = \left\lceil \frac{2n+2}{3} \right\rceil, n \ge 3.$

Proof. Let $V(P_n \odot K_2) = \{v_i, a^i, b^i : 1 \le i \le n\}$ and $E(P_n \odot K_2) = \{v_iv_{i+1}, v_ia^i, v_ib^i, a^ib^i : 1 \le i \le n\}$. Let $k = \left\lceil \frac{2n+2}{3} \right\rceil$, then from (4) it follows that, $tvs(P_n \odot K_2) \ge max \left\{ \left\lceil \frac{2n+2}{3} \right\rceil, \left\lceil \frac{2n+4}{4} \right\rceil, \left\lceil \frac{3n+2}{5} \right\rceil \right\} = \left\lceil \frac{2n+2}{3} \right\rceil$. That is $tvs(P_n \odot K_2) \ge \left\lceil \frac{2n+2}{3} \right\rceil = k$. To prove the reverse inequality, we define a function f from $V \cup E \to \{1, 2, 3, ..., k\}$ in the following way.

$f(a^i) = 1, 1 \le i \le n;$
$f(v_ib^i) = k, 1 \le i \le n;$

$$f(v_ia^i) = \begin{cases} 1, & if \;\; 1 \le i \le k \\ 1+i-k, & if \;\; k+1 \le i \le n; \end{cases}$$

$$f(b^i) = \begin{cases} n+2-k, & if \;\; 1 \le i \le k \\ n+2-2k+i, & if \;\; k+1 \le i \le n; \end{cases}$$

$$f(a^ib^i) = \begin{cases} i, & if \;\; 1 \le i \le k \\ k, & if \;\; k+1 \le i \le n; \end{cases}$$

$$f(v_i) = \begin{cases} 2n+2-k-\left\lfloor \frac{2n+2}{3} \right\rfloor, & if \;\; i=1 \\ 2k+4-2k-\left\lfloor \frac{2n+2}{3} \right\rfloor, & if \;\; i=2 \\ 2n+2-3k+i, & if \;\; 3 \le i \le k \\ 2n+2-2k, & if \;\; k+1 \le i \le n-1 \\ n+3-k, & if \;\; i=n; \end{cases}$$

$$f(v_i v_{i+1}) = \begin{cases} \left\lfloor \frac{2n+2}{3} \right\rfloor, & if \;\; i=1 \\ k, & if \;\; 2 \le i \le n. \end{cases}$$

We observe that,

$wt(a^i) = 2+i, 1 \le i \le n;$
$wt(b_i) = n+2+i, 1 \le i \le n;$

$$wt(v_i) = \begin{cases} 2n+3, & if \;\; i=1 \\ 2n+5, & if \;\; i=2 \\ 2n+3+i, & if \;\; 3 \le i \le k \\ 2n+3+i, & if \;\; k+1 \le i \le n-1 \\ 2n+4, & if \;\; i=n. \end{cases}$$

It is easy to check that the weights of the vertices are distinct. This labeling construction shows that $tvs(P_n \odot K_2) \le k$. Combining this with the lower bound, we conclude that $tvs(P_n \odot K_2) = k$. Figure 2 shows the vertex irregular total labeling of $P_6 \odot K_2$. $\square$

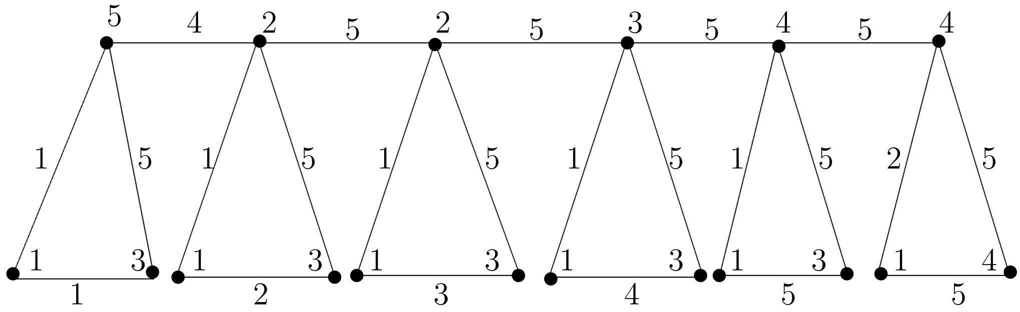

Figure 2: $tvs(P_6 \odot K_2) = 5$

Theorem 2.3. $tvs(C_n \odot K_2) = \left\lceil \frac{2n+2}{3} \right\rceil, \; n \ge 3.$

Proof. Let $V(C_n \odot K_2) = \{v_i, a^i, b^i : 1 \le i \le n\}$ and $E(C_n \odot K_2) = \{v_i v_{i+1}, v_i a^i, v_i b^i, a^i b^i : 1 \le i \le n\}$ with indices taken modulo n. Let $k = \left\lceil \frac{2n+2}{3} \right\rceil$, then from (4) it follows that, $tvs(C_n \odot K_2) \ge max \left\{ \left\lceil \frac{2n+2}{3} \right\rceil, \left\lceil \frac{2n+4}{4} \right\rceil, \left\lceil \frac{3n+2}{5} \right\rceil \right\}$

$= \lceil \frac{2n+2}{3} \rceil$. That is $tvs(C_n \odot K_2) \geq \lceil \frac{2n+2}{3} \rceil = k$. To prove the reverse inequality, we define a function f from $V \cup E$ to $\{1, 2, 3, ..., k\}$ in the following way.

$$f(a^i) = \begin{cases} 1, & if \;\; 1 \leq i \leq k \\ 2i - 2k + 1, & if \;\; k+1 \leq i \leq n; \end{cases}$$

$$f(b^i) = \begin{cases} 1, & if \;\; 1 \leq i \leq k-1 \\ 2, & if \;\; i = k \\ 2i - 2k + 2, & if \;\; k+1 \leq i \leq n; \end{cases}$$

$$f(v_i a^i) = \begin{cases} i, & if \;\; 1 \leq i \leq k \\ k, & if \;\; k+1 \leq i \leq n; \end{cases}$$

$$f(v_i b^i) = \begin{cases} 1+i, & if \;\; 1 \leq i \leq k \\ k, & if \;\; k+1 \leq i \leq n; \end{cases}$$

$$f(a^i b^i) = \begin{cases} i, & if \;\; 1 \leq i \leq k \\ k, & if \;\; k+1 \leq i \leq n; \end{cases}$$

$$f(v_i) = \begin{cases} 1+k-i, & if \;\; 1 \leq i \leq k-1 \\ 2, & if \;\; i = k \\ 2+i-k, & if \;\; k+1 \leq i \leq n; \end{cases}$$

$f(v_i v_{i+1}) = k, 1 \leq i \leq n.$

We observe that,
$wt(a^i) = 2i + 1, 1 \leq i \leq n;$
$wt(b^i) = 2i + 2, 1 \leq i \leq n;$
$wt(v_i) = 3k + 2 + i, 1 \leq i \leq n.$

It is easy to check that the weights of the vertices are distinct. This labeling construction shows that $tvs(C_n \odot K_2) \leq k$. Combining this with the lower bound, we conclude that $tvs(C_n \odot K_2) = k$. Figure 3 shows the vertex irregular total labeling of $C_6 \odot K_2$.

□

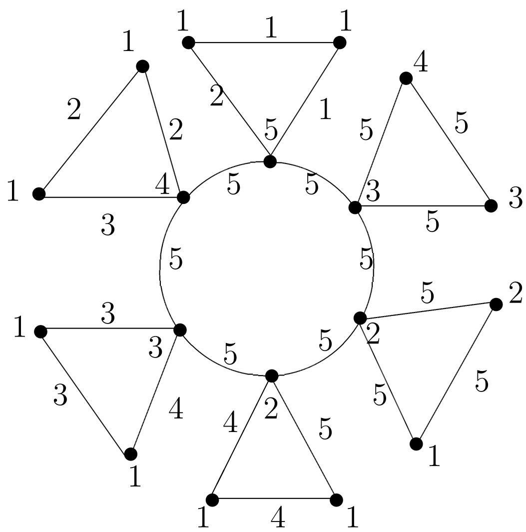

Figure 3: $tvs(C_6 \odot K_2) = 5$

Theorem 2.4. $tvs(L_n \odot K_1) = n + 1, n \geq 3$.

Proof. Let $V(L_n \odot K_1) = \{v_i, u_i, a^i, b^i : 1 \leq i \leq n\}$ and $E(L_n \odot K_1) = \{v_iv_{i+1}, u_iu_{i+1}, v_iu_i, v_ia^i, v_ib^i : 1 \leq i \leq n\}$, then from (4) it follows that, $tvs(L_n \odot K_1) \geq max\left\{\left\lceil\frac{2n+1}{2}\right\rceil, \left\lceil\frac{2n+3}{4}\right\rceil, \left\lceil\frac{3n+1}{5}\right\rceil\right\} = \left\lceil\frac{2n+1}{2}\right\rceil = n + 1$. That is $tvs(L_n \odot K_1) \geq n + 1$. To prove the reverse inequality, we define a function f from $V \cup E$ to $\{1, 2, 3, ..., n + 1\}$ in the following way.

$$f(v_i) = \begin{cases} n, & if \;\; 1 \leq i \leq n-1 \\ 2, & if \;\; i = n; \end{cases}$$

$$f(u_i) = \begin{cases} 1+n, & if \;\; i = 1, n \\ 1, & if \;\; 2 \leq i \leq n-1; \end{cases}$$

$f(v_iv_{i+1}) = 1, 1 \leq i \leq n;$
$f(u_iu_{i+1}) = n, 1 \leq i \leq n;$
$f(a^i) = 1, 1 \leq i \leq n;$
$f(b^i) = n + 1, 1 \leq i \leq n;$
$f(v_ia^i) = f(u_ib^i) = i, 1 \leq i \leq n;$
$f(v_iu_i) = n, 1 \leq i \leq n.$

We observe that,
$wt(a^i) = 1 + i, 1 \leq i \leq n;$
$wt(b_i) = n + 1 + i, 1 \leq i \leq n;$

$$wt(u_i) = \begin{cases} 3n+2, & if \ \ i=1 \\ 3n+1+i, & if \ \ 2 \leq i \leq n-1 \\ 4n+1, & if \ \ i=n; \end{cases}$$

$$wt(v_i) = \begin{cases} 2n+2, & if \ \ i=1 \\ 2n+2+i, & if \ \ 2 \leq i \leq n-1 \\ 2n+3, & if \ \ i=n. \end{cases}$$

It is easy to check that the weights of the vertices are distinct. This labeling construction shows that $tvs(L_n \odot K_1) \leq n+1$. Combining this with the lower bound, we conclude that $tvs(L_n \odot K_1) = n+1$. Figure 4 shows the vertex irregular total labeling of $L_5 \odot K_1$. $\square$

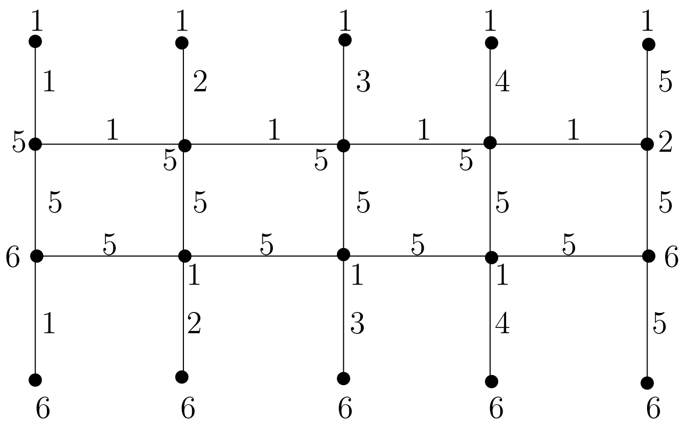

Figure 4: $tvs(L_5 \odot K_1) = 6$

Theorem 2.5. $tvs(CL_n \odot K_1) = n+1, n \geq 3$.

Proof. Let $V(CL_n \odot K_1) = \{v_i, u_i, a^i, b^i : 1 \leq i \leq n\}$ and
$E(CL_n \odot K_1) = \{v_iv_{i+1}, u_iu_{i+1}, v_iu_i, v_ia^i, v_ib^i : 1 \leq i \leq n\}$, with indices taken modulo n. Then from (4) it follows that, $tvs(CL_n \odot K_1) \geq max\left\{\left\lceil\frac{2n+1}{2}\right\rceil, \left\lceil\frac{4n+1}{5}\right\rceil\right\} = \left\lceil\frac{2n+1}{2}\right\rceil = n+1$. That is $tvs(CL_n \odot K_1) \geq n+1$. To prove the reverse inequality, we define a function f from $V \cup E$ to $\{1, 2, 3, ..., n+1\}$ in the following way.

$f(v_i) = n-1, 1 \leq i \leq n;$
$f(u_i) = 1, 1 \leq i \leq n;$
$f(v_iv_{i+1}) = 1, 1 \leq i \leq n;$
$f(u_iu_{i+1}) = n, 1 \leq i \leq n;$
$f(a^i) = 1, 1 \leq i \leq n;$
$f(b^i) = n+1, 1 \leq i \leq n;$
$f(v_ia^i) = f(u_ib^i) = i, 1 \leq i \leq n;$
$f(v_iu_i) = n, 1 \leq i \leq n.$

We observe that,
$wt(a^i) = 1 + i, 1 \leq i \leq n;$
$wt(b^i) = n + 1 + i, 1 \leq i \leq n;$
$wt(v_i) = 2n + 1 + i, 1 \leq i \leq n;$
$wt(u_i) = 3n + 1 + i, 1 \leq i \leq n.$

It is easy to check that the weights of the vertices are distinct. This labeling construction shows that $tvs(CL_n \odot K_1) \leq n + 1$. Combining this with the lower bound, we conclude that $tvs(CL_n \odot K_1) = n+1$. Figure 5 shows the vertex irregular total labeling of $CL_5 \odot K_1$. □

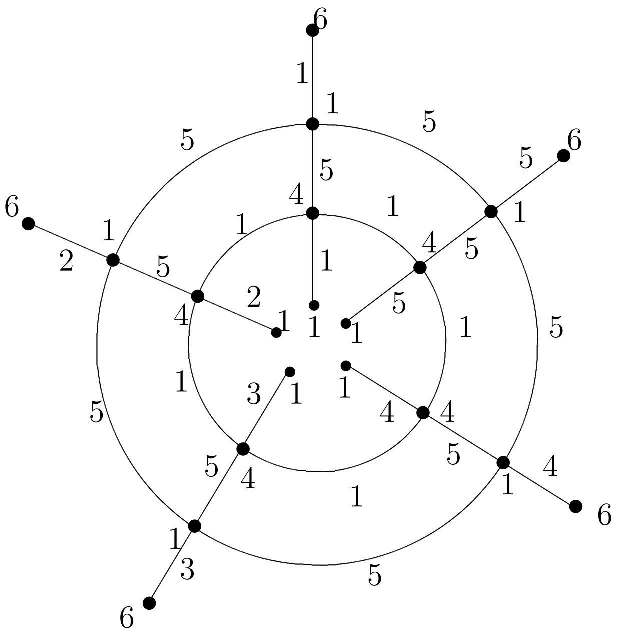

Figure 5: $tvs(CL_5 \odot K_1) = 6$

Theorem 2.6. $tvs(P_2 \odot C_n) = \left\lceil \frac{2n+3}{4} \right\rceil, n \geq 3$.

Proof. Let$V(P_2 \odot C_n) = \{c_j, v_i^j : 1 \leq i \leq n, 1 \leq j \leq 2\}$ and $E(P_2 \odot C_n) = \{v_i^j, v_{i+1}^j, c_j v_i^j, : 1 \leq i \leq n, 1 \leq j \leq 2\}$. Let $k = \left\lceil \frac{2n+3}{3} \right\rceil$, then from (4) it follows that,$tvs(P_2 \odot C_n) \geq max\left\{\left\lceil \frac{2n+3}{4} \right\rceil, \left\lceil \frac{3n+4}{n+2} \right\rceil\right\} = \left\lceil \frac{2n+3}{4} \right\rceil = k$. That is $tvs(P_2 \odot C_n) \geq \left\lceil \frac{2n+3}{4} \right\rceil = k$. To prove the reverse inequality, we define a function f from $V \cup E$ to $\{1, 2, 3, ..., k\}$ in the following way.
For $1 \leq i \leq n, 1 \leq j \leq 2$,

$$f(v_i^j) = \begin{cases} 1, & if \;\; 1 \leq i \leq k \\ 1 + i - k, & if \;\; k + 1 \leq i \leq n; \end{cases}$$

$$f(c_jv_i^j) = \begin{cases} i, & if \;\; 1 \leq i \leq k \\ k, & if \;\; k+1 \leq i \leq n; \end{cases}$$

$f(v_i^1v_{i+1}^1) = 1, 1 \leq i \leq n;$
$f(v_i^2v_{i+1}^2) = k, 1 \leq i \leq n;$
$f(c_1) = k - 1,$
$f(c_2) = k,$
$f(c_1c_2) = 1.$

We observe that,
$wt(v_i^1) = 3 + i, 1 \leq i \leq n;$
$wt(v_i^2) = 2k + 1 + i, 1 \leq i \leq n;$
$wt(c_1) = k(m + 1 - k) + \sum_{i=1}^{k}(i),$
$wt(c_2) = k(m + 1 - k) + \sum_{i=1}^{k}(i + 1).$

It is easy to check that the weights of the vertices are distinct. This labeling construction shows that $tvs(P_2 \odot C_n) \leq k$. Combining this with the lower bound, we conclude that $tvs(P_2 \odot C_n) = k$. Figure 6 shows the vertex irregular total labeling of $P_2 \odot C_8$. □

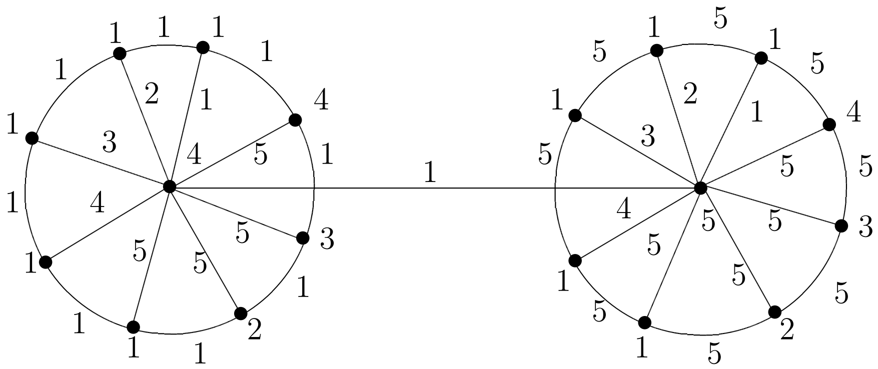

Figure 6: $tvs(P_2 \odot C_8) = 5$

Theorem 2.7. $tvs(C_n \odot \overline{K_m}) = \left\lceil \frac{1+nm}{2} \right\rceil, n \geq 3, m \geq 2.$

Proof. Let $V(C_n \odot \overline{K_m}) = \{v_i, a_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(C_n \odot \overline{K_m}) = \{v_iv_{i+1}, v_ia_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$ with indices taken modulo n. Let $k = \left\lceil \frac{1+nm}{2} \right\rceil$, then from (4) it follows that, $tvs(C_n \odot \overline{K_m}) \geq max\left\{\left\lceil \frac{1+nm}{2} \right\rceil, \left\lceil \frac{1+nm+n}{n+3} \right\rceil\right\} = \left\lceil \frac{1+nm}{2} \right\rceil = k$. That is $tvs(C_n \odot \overline{K_m}) \geq k$. To prove the reverse inequality, we define a function f from $V \cup E$ to $\{1, 2, 3, ..., k\}$ in the following way.
$f(v_i) = i, 1 \leq i \leq n;$
$f(v_iv_{i+1}) = k, 1 \leq i \leq n;$
$f(a_i^1) = i, 1 \leq i \leq n;$
$f(v_ia_i^j) = min\{1 + (n - 1)(j - 1), k\}; 1 \leq i \leq n, 1 \leq j \leq m.$

- For $1 \le i \le n, 2 \le j \le m, 1 + (n-1)(j-1) < k$
  $f(a_i^j) = i + j - 1.$

- For$1 \le i \le n, 2 \le j \le m, 1 + (n-1)(j-1) \ge k$
  $f(a_i^j) = (j-1)n + 1 - k + i.$

We observe that, $wt(a_i^j) = (j-1)n + 1 + i, 1 \le i \le n, 1 \le j \le m.$

Also the weights of v_i's are distinct.It is easy to check that the weights of the vertices are distinct.This labeling construction shows that $tvs(C_n \odot \overline{K_m}) \le k$. Combining this with the lower bound, we conclude that $tvs(C_n \odot \overline{K_m}) = k$. Figure 7 shows the vertex irregular total labeling of $C_3 \odot \overline{K_3}$. □

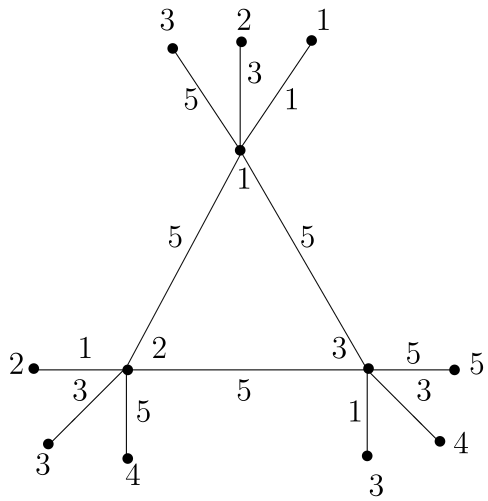

Figure 7: $tvs(C_3 \odot \overline{K_3}) = 5$

Theorem 2.8. $tvs(P_n \odot \overline{K_m}) = \left\lceil \frac{1+nm}{2} \right\rceil, n \ge 3, m \ge 2.$

Proof. Let $V(P_n \odot \overline{K_m}) = \{v_i, a_i^j : 1 \le i \le n, 1 \le j \le m\}$ and $E(P_n \odot \overline{K_m}) = \{v_i v_{i+1}, v_i a_i^j : 1 \le i \le n, 1 \le j \le m\}$.Let $k = \left\lceil \frac{1+nm}{2} \right\rceil$, then from (4) it follows that,$tvs(P_n \odot \overline{K_m}) \ge max\left\{\left\lceil \frac{1+nm}{2} \right\rceil, \left\lceil \frac{1+nm+n}{n+3} \right\rceil\right\} = \left\lceil \frac{1+nm}{2} \right\rceil = k$. That is $tvs(P_n \odot \overline{K_m}) \ge k$. To prove the reverse inequality, we define a function f from $V \cup E$ to $\{1, 2, 3, ..., k\}$ in the following way

$$f(v_i) = \begin{cases} k-1, & if \;\; i = 1 \\ i-1, & if \;\; 2 \le i \le n-1 \\ k, & if \;\; i = n; \end{cases}$$

$f(v_i v_{i+1}) = k, 1 \le i \le n;$
$f(a_i^1) = i, i \le i \le n;$

$f(v_i a_i^j) = min\{1 + (n-1)(j-1), k\}; 1 \leq i \leq n, 1 \leq j \leq m.$
For $1 \leq i \leq n, 2 \leq j \leq m, 1 + (n-1)(j-1) < k$
$f(a_i^j) = i + j - 1.$
For$1 \leq i \leq n, 2 \leq j \leq m, 1 + (n-1)(j-1) \geq k$
$f(a_i^j) = (j-1)n + 1 - k + i.$
We observe that for $1 \leq i \leq n, 1 \leq j \leq m,$
$wt(a_i^j) = (j-1)n + 1 + i.$
Also the weights of v_i's are distinct.It is easy to check that the weights of the vertices are distinct.This labeling construction shows that $tvs(P_n \odot \overline{K_m}) \leq k$. Combining this with the lower bound, we conclude that $tvs(P_n \odot \overline{K_m}) = k$. Figure 8 shows the vertex irregular total labeling of $P_5 \odot \overline{K_4}$. □

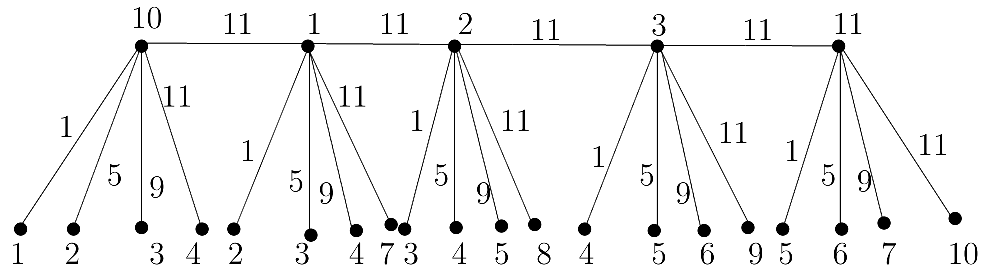

Figure 8: $tvs(P_5 \odot \overline{K_4}) = 11$

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