Journal of Algorithms and Computation

journal homepage: http://jac.ut.ac.ir

Theta Graph Designs

Anthony D. Forbes*[1], Terry S. Griggs†[2] and Tamsin J. Forbes‡[3]

[1,2]Department of Mathematics and Statistics, The Open University, Milton Keynes MK7 6AA, UK.
[3]22 St Albans Road, Kingston upon Thames KT2 5HQ, UK.

ABSTRACT

We solve the design spectrum problem for all theta graphs with 10, 11, 12, 13, 14 and 15 edges.

ARTICLE INFO

Article history:
Received 30, August 2016
Received in revised form 10, April 2017
Accepted 3 May 2017
Available online 01, June 2017

Keyword: Graph design; Graph decomposition; Theta graph.

AMS subject Classification: 05C51

1 Introduction

Let G be a simple graph. If the edge set of a simple graph K can be partitioned into edge sets of graphs each isomorphic to G, we say that there exists a *decomposition* of K into G. In the case where K is the complete graph K_n, we refer to the decomposition as a G *design* of order n. The *spectrum* of G is the set of non-negative integers n for which there exists a G design of order n. For completeness we remark that the empty set is a G design of order 0 as well as 1; these trivial cases will usually be omitted from discussion henceforth. A complete solution of the spectrum problem often seems to be very difficult. However it has been achieved in many cases, especially amongst the smaller graphs. We

*anthony.d.forbes@gmail.com
†terry.griggs@open.ac.uk
‡tamsin.forbes@gmail.com

Table 1: Design existence conditions for theta graphs

$a+b+c$	conditions
10	$n \equiv 0, 1, 5, 16 \pmod{20}$, $n \neq 5$
12	$n \equiv 0, 1, 9, 16 \pmod{24}$, $n \neq 9$
14	$n \equiv 0, 1, 8, 21 \pmod{28}$, $n \neq 8$
15	$n \equiv 0, 1, 6, 10 \pmod{15}$, $n \neq 6, 10$
odd prime power ≥ 5	$n \equiv 0, 1 \pmod{a+b+c}$

refer the reader to the survey article of Adams, Bryant and Buchanan, [2], and, for more up to date results, the Web site maintained by Bryant and McCourt, [5].
A *theta graph* $\Theta(a, b, c)$, $1 \leq a \leq b \leq c$, $b \geq 2$ is the graph with $a+b+c-1$ vertices and $a+b+c$ edges that consists of three paths of lengths a, b and c with common end points but internally disjoint. Each of the two common end points of the paths has degree 3 and all other vertices have degree 2. Observe that $\Theta(a, b, c)$ is bipartite if $a \equiv b \equiv c \pmod 2$, tripartite otherwise.
The aim of this paper is to establish the design spectrum for every theta graph with 10, 11, 12, 13, 14 or 15 edges. The design spectra for all theta graphs with 9 or fewer edges have been determined, [4]. Also there are some partial results for $\Theta(1, b, b)$, odd $b > 1$, [3], and it is known that there exists a $\Theta(a, b, c)$ design of order $2(a+b+c)+1$, [10]. See [2, Section 5.5], specifically Theorems 5.9 and 5.10, for details and further references. For convenience, we record the main result of [10] as follows.

Theorem 1.1. (Punnim & Pabhapote) *If* $1 \leq a \leq b \leq c$ *and* $b \geq 2$*, then there exists a* $\Theta(a, b, c)$ *design of order* $2(a+b+c)+1$*.*

It is clear that a $\Theta(a, b, c)$ design of order n can exist only if (i) $n \leq 1$, or $n \geq a+b+c-1$, and (ii) $n(n-1) \equiv 0 \pmod{2(a+b+c)}$. These necessary conditions are determined by elementary counting and given explicitly in Table 1 for some values of $a+b+c$. In this paper we show that the conditions are sufficient for $n =$ 10, 11, 12, 13, 14 and 15. We state our results formally.

Theorem 1.2. *Designs of order* n *exist for all theta graphs* $\Theta(a, b, c)$ *with* $a+b+c = 10$ *if and only if* $n \equiv 0, 1, 5$ *or* $16 \pmod{20}$ *and* $n \neq 5$*.*

Theorem 1.3. *Designs of order* n *exist for all theta graphs* $\Theta(a, b, c)$ *with* $a+b+c = 11$ *if and only if* $n \equiv 0$ *or* $1 \pmod{11}$*.*

Theorem 1.4. *Designs of order* n *exist for all theta graphs* $\Theta(a, b, c)$ *with* $a+b+c = 12$ *if and only if* $n \equiv 0, 1, 9$ *or* $16 \pmod{24}$ *and* $n \neq 9$*.*

Theorem 1.5. *Designs of order* n *exist for all theta graphs* $\Theta(a, b, c)$ *with* $a+b+c = 13$ *if and only if* $n \equiv 0$ *or* $1 \pmod{13}$*.*

Theorem 1.6. *Designs of order n exist for all theta graphs $\Theta(a,b,c)$ with $a+b+c=14$ if and only if $n \equiv 0, 1, 8$ or $21 \pmod{28}$ and $n \neq 8$.*

Theorem 1.7. *Designs of order n exist for all theta graphs $\Theta(a,b,c)$ with $a+b+c=15$ if and only if $n \equiv 0,\ 1,\ 6$ or $10 \pmod{15}$ and $n \neq 6,\ 10$.*

2 Constructions

We use Wilson's fundamental construction involving group divisible designs, [12]. Recall that a K-GDD of type $g_1^{t_1} \dots g_r^{t_r}$ is an ordered triple $(V, \mathcal{G}, \mathcal{B})$ where V is a set of cardinality $v = t_1 g_1 + \cdots + t_r g_r$, $\mathcal{G}$ is a partition of V into t_i subsets each of cardinality g_i, $i = 1, \dots, r$, called *groups* and $\mathcal{B}$ is a collection of subsets of cardinalities $k \in K$, called *blocks*, which collectively have the property that each pair of elements from different groups occurs in precisely one block but no pair of elements from the same group occurs at all. A $\{k\}$-GDD is also called a k-GDD. As is well known, whenever q is a prime power there exists a q-GDD of type q^q and a $(q+1)$-GDD of type q^{q+1} (arising from affine and projective planes of order q respectively). A *parallel class* in a group divisible design is a subset of the block set in which each element of the base set appears exactly once. A k-GDD is called *resolvable*, and denoted by k-RGDD, if the entire set of blocks can be partitioned into parallel classes.
Propositions 2.1 and 2.2 are for dealing with bipartite graphs, the latter being used only for 15-edge bipartite theta graphs, where a suitable decomposition of the form $K_{dr,ds}$ is not available. Propositions 2.3 to 2.8 deal with tripartite graphs.

Proposition 2.1. *Let d, f, r and s be positive integers, let g be a non-negative integer, and let $e = 0$ or 1. Suppose there exist G designs of orders $fdrs + e$ and $gds + e$, and suppose there exists a decomposition into G of the complete bipartite graph $K_{dr,ds}$. Then there exist G designs of order $tfdrs + gds + e$ for integers $t \geq 0$.*

Proof If $t = 0$, there is nothing to prove; so we may assume $t \geq 1$. Let $g = 0$ and use induction on t. Assume there exists a G design of order $tfdrs + e$ for some $t \geq 1$. Take a complete bipartite graph $K_{tfs,fr}$ and inflate the first part by a factor dr and the second part by a factor of ds, so that the edges become $K_{dr,ds}$ graphs. If $e = 1$, add an extra point, ∞. Overlay the inflated parts, together with ∞ when $e = 1$, with $K_{tfdrs+e}$ or K_{fdrs+e} as appropriate. The result is a complete graph $K_{(t+1)fdrs+e}$ which admits a decomposition into G since there exist decompositions into G of the components, $K_{tfdrs+e}$, K_{fdrs+e} and $K_{dr,ds}$.
For $g \geq 1$, the same construction but starting with a complete bipartite graph $K_{tfs,g}$ yields a design of order $K_{tfdrs+gds+e}$ since we may now assume that there exists a design of order $tfdrs + e$.
□

Proposition 2.2. *Let r and s be not necessarily distinct positive integers, and let $e = 0$ or 1. Suppose there exist G designs of orders $r + e$ and (if $s \neq r$) $s + e$, and suppose there exist decompositions into G of the complete bipartite graphs $K_{r,r}$ and (if $s \neq r$) $K_{r,s}$.*

Then there exist G designs of order $tr + s + e$ for integers $t \geq 0$.

Proof If $t = 0$, there is nothing to prove; so we may assume $t \geq 1$. Start with the graph K_{t+1}, inflate one point by a factor of s and all others by r so that the original edges become $K_{r,s}$ graphs and, when $t \geq 2$, $K_{r,r}$ graphs. Add a new point, ∞, if $e = 1$. Overlay the r-inflated parts together with ∞ if $e = 1$ by K_{r+e}. Overlay the s-inflated part together with ∞ if $e = 1$ by K_{s+e}. The result is a graph K_{tr+s+e} which admits a decomposition into G. □

Proposition 2.3. *Let p be a positive integer. Suppose there exist G designs of orders p, $p+1$, $2p$ and $2p+1$. Suppose also there exist decompositions into G of $K_{p,p,p}$, $K_{p,p,p,p}$ and $K_{p,p,p,p,p}$. Then there exist G designs of order n for $n \equiv 0, 1 \pmod{p}$.*

Proof It is known that there exists a $\{3, 4, 5\}$-GDD of type 1^t for $t \geq 3$, $t \neq 6, 8$; see [1]. Inflate each point of the GDD by a factor of p, thus expanding the blocks to complete multipartite graphs $K_{p,p,p}$, $K_{p,p,p,p}$ and $K_{p,p,p,p,p}$. Let $e = 0$ or 1. If $e = 1$, add an extra point, ∞. Overlay each inflated group, together with ∞ if $e = 1$, with K_{p+e}. Since a design of order $p + e$ and decompositions into G of $K_{p,p,p}$, $K_{p,p,p,p}$ and $K_{p,p,p,p,p}$ exist, this construction creates a design of order $pt + e$ for $t \geq 3$, $t \neq 6, 8$.
For order $6p + e$, use a p-inflated 3-GDD of type 2^3, plus an extra point if $e = 1$, with decompositions into G of K_{2p+e} and $K_{p,p,p}$. Similarly, for order $8p + e$, use a 3-GDD of type 2^4 instead. □

Proposition 2.4. *Let p be a positive integer, let $f = 0$ or 1 and write f' for $1 - f$. Suppose there exist G designs of orders $3p + f'$, $4p$, $4p + 1$, $5p + f$, $7p + f'$, $8p$, $8p + 1$, $9p+f$, $11p+f'$, $13p+f$ and suppose there exist decompositions into G of $K_{p,p,p,p}$, $K_{p,p,p,p,p}$, $K_{p,p,p,p,3p}$ and $K_{p,p,p,p,4p}$, $K_{2p,2p,2p}$ and $K_{4p,4p,4p,5p}$. Then there exist G designs of order n for $n \equiv 0, 1, p + f, 3p + f' \pmod{4p}$, $n \neq p + f$.*

Proof Start with a 4-RGDD of type 4^{3t+1}, $t \geq 1$, [8]; see also [9]. There are $4t$ parallel classes. Let $x, y, z \geq 0$ and $w = x + y + z \leq 4$. If $w > 0$, add a new group of size w and adjoin each point of this new group to all blocks of a parallel class to create a {4,5}-GDD of type $4^{3t+1}w^1$, $t \geq 1$, which degenerates to a 5-GDD of type 4^5 if $t = 1$ and $w = 4$. In the new group inflate x points by a factor of p, y points by a factor of $3p$ and z points by a factor of $4p$. Inflate all points in the other groups by a factor of p. So the original blocks become $K_{p,p,p,p}$ graphs, and new ones $K_{p,p,p,p,p}$ or $K_{p,p,p,p,3p}$ or $K_{p,p,p,p,4p}$ graphs. Either overlay the groups with K_{4p} and, if $w > 0$, $K_{px+3py+4pz}$, or add a new point and overlay with K_{4p+1} and, if $w > 0$, $K_{px+3py+4pz+1}$. Using decompositions into G of K_{4p}, K_{4p+1}, $K_{p,p,p,p}$, $K_{p,p,p,p,p}$, $K_{p,p,p,p,3p}$ and $K_{p,p,p,p,4p}$ this construction yields a design of order $12pt + 4p + px + 3py + 4pz + e$ if $t \geq 1$, where $e = 0$ or 1, whenever a design of order $px + 3py + 4pz + e$ exists.
As illustrated in Table 2, all design orders $n \equiv 0, 1, p+f, 3p+f' \pmod{4p}$, $n \neq p+f$, are covered except for those values indicated as missing. The missing values not assumed as

Table 2: The construction for Proposition 2.4

$12pt+4p$,	$x=0,\ y=0,\ z=0,$	$e=0,$	missing $4p$
$12pt+4p+4p$,	$x=0,\ y=0,\ z=1,$	$e=0,$	missing $8p$
$12pt+4p+8p$,	$x=0,\ y=0,\ z=2,$	$e=0,$	missing $12p$
$12pt+4p+1$,	$x=0,\ y=0,\ z=0,$	$e=1,$	missing $4p+1$
$12pt+4p+4p+1$,	$x=0,\ y=0,\ z=1,$	$e=1,$	missing $8p+1$
$12pt+4p+8p+1$,	$x=0,\ y=0,\ z=2,$	$e=1,$	missing $12p+1$
$12pt+4p+5p+f$,	$x=1,\ y=0,\ z=1,$	$e=f,$	missing $9p+f$
$12pt+4p+9p+f$,	$x=1,\ y=0,\ z=2,$	$e=f,$	missing $13p+f$
$12pt+4p+13p+f$,	$x=1,\ y=0,\ z=3,$	$e=f,$	missing $5p+f$, $17p+f$
$12pt+4p+3p+f'$,	$x=0,\ y=1,\ z=0,$	$e=f',$	missing $7p+f'$
$12pt+4p+7p+f'$,	$x=0,\ y=1,\ z=1,$	$e=f',$	missing $11p+f'$
$12pt+4p+11p+f'$,	$x=0,\ y=1,\ z=2,$	$e=f',$	missing $3p+f'$, $15p+f'$

given are handled independently as follows. See [7, Tables 4.3 and 4.10] for lists of small 3- and 4-GDDs.
For $12p+e$, use decompositions of K_{4p+e} and $K_{2p,2p,2p}$ with a $2p$-inflated 3-GDD of type 2^3 plus an extra point if $e=1$.
For $15p+f'$, use decompositions of $K_{3p+f'}$ and $K_{p,p,p,p}$ with a p-inflated 4-GDD of type 3^5 plus an extra point if $f'=1$.
For $17p+f$, use decompositions of K_{4p+f}, K_{5p+f}, $K_{4p,4p,4p,5p}$ with the trivial 4-GDD of type 1^4 plus an extra point if $f=1$. □

Proposition 2.5. *Suppose there exist G designs of orders* 16, 20, 21, 25, 36, 40, 41, 45, 56, 65 *and suppose there exist decompositions into G of $K_{10,10,10}$, $K_{5,5,5,5}$, $K_{20,20,20,25}$, $K_{5,5,5,5,5}$, $K_{5,5,5,5,15}$ and $K_{5,5,5,5,20}$. Then there exist G designs of order n for $n \equiv 0, 1, 5, 16 \pmod{20}$, $n \neq 5$.*

Proof Use Proposition 2.4 with $p=5$ and $f=0$. □

Proposition 2.6. *Suppose there exist G designs of orders* 16, 24, 25, 33, 40, 49, 57, 81 *and suppose there exist decompositions into G of $K_{8,8,8}$, $K_{8,8,8,8}$ and $K_{8,8,8,24}$. Then there exist G designs of order n for $n \equiv 0, 1, 9, 16 \pmod{24}$, $n \neq 9$.*

Proof Start with a 3-RGDD of type 3^{2t+1}, $t \geq 1$, [11] (see also [9]), which has $3t$ parallel classes. Let $x, y \geq 0$, $x+y \leq 3$ and if $x+y > 0$, add a new group of size $x+y$ and adjoin each point of this new group to all blocks of a parallel class to create a {3,4}-GDD of type $3^{2t+1}(x+y)^1$. This degenerates to a 4-GDD of type 3^4 if $t=1$ and $x+y=3$. In the new group inflate x points by a factor of 8 and y points by a factor of 24. Inflate all other points by a factor of 8. The original blocks become $K_{8,8,8}$ graphs and new ones $K_{8,8,8,8}$ or $K_{8,8,8,24}$ graphs. Either overlay the groups with K_{24} and possibly K_{8x+24y}, or add a

Table 3: The construction for Proposition 2.6

$48t+24$,	$x=0$, $y=0$,	$e=0$,	missing 24
$48t+24+24$,	$x=0$, $y=1$,	$e=0$,	missing 48
$48t+24+1$,	$x=0$, $y=0$,	$e=1$,	missing 25
$48t+24+25$,	$x=0$, $y=1$,	$e=1$,	missing 49
$48t+24+33$,	$x=1$, $y=1$,	$e=1$,	missing 57
$48t+24+57$,	$x=1$, $y=2$,	$e=1$,	missing 33, 81
$48t+24+16$,	$x=2$, $y=0$,	$e=0$,	missing 40
$48t+24+40$,	$x=2$, $y=1$,	$e=0$,	missing 16, 64

new point and overlay the groups with K_{25} and possibly $K_{8x+24y+1}$. Using decompositions into G of K_{24}, K_{25}, $K_{8,8,8}$, $K_{8,8,8,8}$ and $K_{8,8,8,24}$ this construction yields a design of order $48t+24+8x+24y+e$, $t \geq 1$, $e=0$ or 1, whenever a design of order $8x+24y+e$ exists. As illustrated in Table 3, all design orders $n \equiv 0,1,9,16 \pmod{24}$, $n \neq 9$, are covered except for those values indicated as missing. The missing values not assumed as given in the statement of the proposition, namely 48 and 64, are constructed using decompositions of K_{16} and $K_{8,8,8}$ with an 8-inflated 3-GDD of type 2^3 for 48, or type 2^4 for 64. □

Proposition 2.7. *Suppose there exist G designs of orders 21, 28, 29, 36, 49, 56, 57, 64, 77, 92 and suppose there exist decompositions into G of $K_{14,14,14}$, $K_{7,7,7,7}$, $K_{28,28,28,35}$, $K_{7,7,7,7,7}$, $K_{7,7,7,7,21}$ and $K_{7,7,7,7,28}$. Then there exist G designs of order n for $n \equiv 0,1,8,21 \pmod{28}$, $n \neq 8$.*

Proof Use Proposition 2.4 with $p=7$ and $f=1$. □

Proposition 2.8. *Suppose there exist G designs of orders 15, 16, 21, 25, 30, 31, 36, 40, 51, 55, 66 and 70, and suppose there exist decompositions into G of $K_{5,5,5}$ and $K_{5,5,5,5}$. Then there exist G designs of order n for $n \equiv 0,1,6,10 \pmod{15}$, $n \neq 6,10$.*

Proof Start with a 3-RGDD of type 3^{2t+1}, $t \geq 1$, [11] (see also [9]), which has $3t$ parallel classes. Let $w \geq 0$, $w \leq 3t$, and if $w > 0$, add a new group of size w and adjoin each point of this new group to all blocks of a parallel class to create a {3,4}-GDD of type $3^{2t+1}w^1$. This degenerates to a 4-GDD of type 3^4 if $t=1$ and $w=3$. Inflate all points by a factor of 5, so that the original blocks become $K_{5,5,5}$ graphs and new ones $K_{5,5,5,5}$ graphs. Let $e=0$ or 1, and if $e=1$, add a new point, ∞. Overlay each group, including ∞ if $e=1$, with K_{15+e} or K_{5w+e}, as appropriate. Using decompositions of K_{15+e}, $K_{5,5,5}$, and $K_{5,5,5,5}$ this construction yields a design of order $30t+15+5w+e$, $t \geq w/3$, $e=0$ or 1, whenever a design of order $5w+e$ exists.
As illustrated in Table 4, all design orders $n \equiv 0,1,6,10 \pmod{15}$, $n \neq 6,10$, are covered except for those values indicated as missing. The missing values not assumed as given are constructed as follows. See [7, Table 4.3] for a list of small 3-GDDs.

Table 4: The construction for Proposition 2.8

$30t + 15$,	$w = 0$, $e = 0$,	$t \geq 1$,	missing 15
$30t + 15 + 15$,	$w = 3$, $e = 0$,	$t \geq 1$,	missing 30
$30t + 15 + 1$,	$w = 0$, $e = 1$,	$t \geq 1$,	missing 16
$30t + 15 + 16$,	$w = 3$, $e = 1$,	$t \geq 1$,	missing 31
$30t + 15 + 21$,	$w = 4$, $e = 1$,	$t \geq 2$,	missing 36, 66
$30t + 15 + 36$,	$w = 7$, $e = 1$,	$t \geq 3$,	missing 21, 51, 81, 111
$30t + 15 + 25$,	$w = 5$, $e = 0$,	$t \geq 2$,	missing 40, 70
$30t + 15 + 40$,	$w = 8$, $e = 0$,	$t \geq 3$,	missing 25, 55, 85, 115

For 81, use decompositons of K_{21} and $K_{5,5,5}$ with a 5-inflated 3-GDD of type 4^4 and an extra point.
For 85, use decompositons of K_{15}, K_{25} and $K_{5,5,5}$ with a 5-inflated 3-GDD of type $3^4 5^1$.
For 111, use decompositons of K_{21}, K_{31} and $K_{5,5,5}$ with a 5-inflated 3-GDD of type $4^4 6^1$ and an extra point.
For 115, use decompositons of K_{15}, K_{25} and $K_{5,5,5}$ with a 5-inflated 3-GDD of type $3^1 5^4$. $\square$

3 Proofs of the theorems

We deal with each of the theta graphs stated in the theorems. The lemmas in this section assert the existence of specific graph designs and decompositions of multipartite graphs. These are used as ingredients for the propositions of Section 2 to construct the decompositions of complete graphs required to prove the theorems. With a few exceptions, the details of the decompositions that constitute the proofs of the lemmas have been deferred to sections in the rather lengthy Appendix to this paper. If absent, the Appendix may be obtained from the ArXiv (identifier 1703.01483), or by request from the first author. In the presentation of our results we represent $\Theta(a, b, c)$ by a subscripted ordered $(a + b + c - 1)$-tuple $(v_1, v_2, \ldots, v_{a+b+c-1})_{\Theta(a,b,c)}$, where v_1 and v_2 are the vertices of degree 3 and the three paths are

$$\begin{aligned} &\text{(i)} \quad (v_1,\ v_2) \ \text{ if } a = 1, \quad (v_1,\ v_3,\ \ldots,\ v_{a+1},\ v_2) \ \text{ if } a \geq 2, \\ &\text{(ii)} \quad (v_1,\ v_{a+2},\ \ldots,\ v_{a+b},\ v_2), \\ &\text{(iii)} \quad (v_1,\ v_{a+b+1},\ \ldots,\ v_{a+b+c-1},\ v_2). \end{aligned}$$

If a graph G has e edges, the numbers of occurrences of G in a decomposition into G of the complete graph K_n, the complete r-partite graph K_{n^r} and the complete $(r+1)$-partite graph $K_{n^r m^1}$ are respectively

$$\frac{n(n-1)}{2e}, \quad \frac{n^2 r(r-1)}{2e} \quad \text{and} \quad \frac{nr(n(r-1)+2m)}{2e}.$$

The number of theta graphs with e edges is $\lfloor e^2/12-1/2 \rfloor$ of which $\lfloor e^2/48+(e \bmod 2)(e-8)/8+1/2 \rfloor$ are bipartite; see [6] for example.
All decompositions were created by a computer program written in the C language, and a MATHEMATICA program provided final assurance regarding their correctness. Most of the decompositions were obtained without difficulty. However, there were a few that seemed to present a challenge, notably K_{20} for theta graphs of 10 edges, K_{24} for theta graphs of 12 edges and K_{30} for theta graphs of 15 edges. So, for these three cases, it is appropriate to include the decomposition details in the main body of the paper.

Lemma 3.1. *There exist* $\Theta(a,b,c)$ *designs of orders* 16*,* 20 *and* 25 *for* $a+b+c=10$*. There exist* $\Theta(a,b,c)$ *designs of orders* 36*,* 40*,* 41*,* 45*,* 56*,* 65 *for* $a+b+c=10$ *with* a*,* b*,* c *not all even.*

Proof There are seven theta graphs of which two are bipartite. We present designs of order 20 here; the remaining decompositions are presented in the Appendix.

$\boldsymbol{K_{20}}$ Let the vertex set be Z_{20}. The decompositions consist of the graphs

$(4,17,0,9,7,1,12,15,13)_{\Theta(1,2,7)}$, $(11,17,18,10,1,13,3,7,16)_{\Theta(1,2,7)}$,
$(1,6,18,14,11,12,2,0,15)_{\Theta(1,2,7)}$,
$(12,4,10,18,0,8,15,3,11)_{\Theta(1,2,7)}$, $(6,2,8,11,19,7,0,14,16)_{\Theta(1,2,7)}$,
$(2,4,18,7,15,10,14,8,16)_{\Theta(1,2,7)}$, $(6,12,0,10,16,3,18,14,19)_{\Theta(1,2,7)}$,

$(0,1,9,18,12,15,4,3,5)_{\Theta(1,3,6)}$, $(0,7,17,5,10,14,12,8,2)_{\Theta(1,3,6)}$,
$(8,6,14,1,13,3,17,4,19)_{\Theta(1,3,6)}$,
$(3,15,2,1,11,17,18,7,6)_{\Theta(1,3,6)}$, $(3,7,14,11,19,5,6,18,10)_{\Theta(1,3,6)}$,
$(7,19,15,18,13,14,2,10,11)_{\Theta(1,3,6)}$, $(15,19,11,2,14,6,3,9,10)_{\Theta(1,3,6)}$,

$(0,2,3,6,9,11,10,14,8)_{\Theta(1,4,5)}$, $(17,2,7,12,13,9,0,18,11)_{\Theta(1,4,5)}$,
$(6,11,5,3,17,18,8,0,7)_{\Theta(1,4,5)}$,
$(8,12,13,15,16,4,3,11,9)_{\Theta(1,4,5)}$, $(1,8,3,15,7,16,13,0,5)_{\Theta(1,4,5)}$,
$(1,17,5,7,19,4,9,16,0)_{\Theta(1,4,5)}$, $(12,17,5,9,13,11,19,0,4)_{\Theta(1,4,5)}$,

$(2,1,8,10,18,7,12,0,6)_{\Theta(2,2,6)}$, $(10,15,7,8,12,11,6,13,5)_{\Theta(2,2,6)}$,
$(10,8,11,9,0,16,7,14,17)_{\Theta(2,2,6)}$,
$(5,1,7,3,8,13,11,17,15)_{\Theta(2,2,6)}$, $(1,9,4,5,19,3,7,13,15)_{\Theta(2,2,6)}$,
$(5,17,0,19,11,9,13,16,1)_{\Theta(2,2,6)}$, $(9,17,3,12,7,11,15,19,13)_{\Theta(2,2,6)}$,

$(0,5,18,2,10,4,9,8,17)_{\Theta(2,3,5)}$, $(13,5,2,11,9,3,7,6,12)_{\Theta(2,3,5)}$,
$(7,6,0,1,15,2,12,9,10)_{\Theta(2,3,5)}$,
$(11,3,12,16,15,0,8,7,4)_{\Theta(2,3,5)}$, $(3,7,10,0,15,8,16,4,19)_{\Theta(2,3,5)}$,
$(7,19,16,14,11,12,4,15,2)_{\Theta(2,3,5)}$, $(11,19,8,3,6,18,15,12,0)_{\Theta(2,3,5)}$,

$(8,2,7,5,13,9,4,18,17)_{\Theta(2,4,4)}$, $(18,6,10,9,8,0,7,17,4)_{\Theta(2,4,4)}$,
$(13,11,15,8,18,0,10,3,5)_{\Theta(2,4,4)}$,
$(7,3,6,13,4,11,15,0,9)_{\Theta(2,4,4)}$, $(3,7,0,2,19,4,8,11,10)_{\Theta(2,4,4)}$,
$(15,19,12,3,16,5,8,17,11)_{\Theta(2,4,4)}$, $(15,19,18,1,12,7,14,11,16)_{\Theta(2,4,4)}$,

$(15, 10, 18, 13, 6, 12, 11, 0, 8)_{\Theta(3,3,4)}$, $(0, 11, 16, 3, 17, 8, 10, 1, 5)_{\Theta(3,3,4)}$,
$(5, 7, 0, 13, 3, 8, 17, 19, 14)_{\Theta(3,3,4)}$,
$(6, 18, 7, 17, 13, 12, 0, 1, 2)_{\Theta(3,3,4)}$, $(6, 10, 5, 4, 14, 18, 2, 16, 17)_{\Theta(3,3,4)}$,
$(10, 14, 9, 8, 11, 1, 2, 3, 13)_{\Theta(3,3,4)}$, $(14, 18, 10, 6, 15, 5, 2, 9, 19)_{\Theta(3,3,4)}$
under the action of the mapping $x \mapsto x + 4 \pmod{20}$ for only the first three graphs in each design. □

Lemma 3.2. *There exist decompositions into $\Theta(a, b, c)$ of the complete bipartite graph $K_{10,5}$ for $a + b + c = 10$ with a, b, c all even. There exist decompositions into $\Theta(a, b, c)$ of the complete multipartite graphs $K_{10,10,10}$, $K_{5,5,5,5}$, $K_{20,20,20,25}$, $K_{5,5,5,5,5}$, $K_{5,5,5,5,15}$ and $K_{5,5,5,5,20}$ for $a + b + c = 10$ with a, b, c not all even.*

Proof The decompositions are presented in the Appendix. □

Theorem 1.2 for $\Theta(a, b, c)$ with $a + b + c = 10$ follows from Theorem 1.1, Lemmas 3.1 and 3.2, and Proposition 2.1 (with $d = 5$, $r = 2$, $s = 1$, $f = 2$ and $g = 0$, 3 or 5) if a, b and c are all even, Proposition 2.5 otherwise. □

Lemma 3.3. *There exist $\Theta(a, b, c)$ designs of orders 11 and 12 for $a + b + c = 11$. There exists a $\Theta(a, b, c)$ design of order 22 for $a + b + c = 11$ with a, b, c not all odd.*

Proof There are nine theta graphs of which three are bipartite. The decompositions are presented in the Appendix. □

Lemma 3.4. *There exist decompositions into $\Theta(a, b, c)$ of the complete bipartite graph $K_{11,11}$ for $a + b + c = 11$ with a, b, c all odd. There exist decompositions into $\Theta(a, b, c)$ of the complete multipartite graphs $K_{11,11,11}$, $K_{11,11,11,11}$ and $K_{11,11,11,11,11}$ for $a + b + c = 11$ with a, b, c not all odd.*

Proof The decompositions are presented in the Appendix. □

Theorem 1.3 for $\Theta(a, b, c)$ with $a + b + c = 11$ follows from Theorem 1.1, Lemmas 3.3 and 3.4, and Proposition 2.1 (with $d = 11$, $r = s = f = 1$ and $g = 0$) if a, b and c are all odd, Proposition 2.3 otherwise. □

Lemma 3.5. *There exist $\Theta(a, b, c)$ designs of orders 16, 24 and 33 for $a + b + c = 12$. There exist $\Theta(a, b, c)$ designs of orders 40, 49, 57 and 81 for $a + b + c = 12$ with a, b, c not all even.*

Proof There are eleven theta graphs of which three are bipartite. We present designs of order 24 here; the remaining decompositions are presented in the Appendix.

$\boldsymbol{K_{24}}$ Let the vertex set be Z_{24}. The decompositions consist of
$(11, 9, 16, 19, 10, 12, 6, 15, 7, 2, 8)_{\Theta(1,2,9)}$, $(8, 0, 10, 7, 6, 19, 18, 1, 20, 22, 12)_{\Theta(1,2,9)}$,

$(2,6,13,15,1,17,18,14,22,19,21)_{\Theta(1,2,9)}$, $(10,7,1,3,21,8,14,9,19,16,13)_{\Theta(1,2,9)}$, $(16,21,1,7,9,22,15,19,5,17,4)_{\Theta(1,2,9)}$, $(18,4,10,21,17,14,16,6,11,12,19)_{\Theta(1,2,9)}$, $(21,20,7,2,14,13,18,6,5,4,12)_{\Theta(1,2,9)}$, $(21,12,13,22,10,5,15,20,16,3,9)_{\Theta(1,2,9)}$, $(4,20,0,1,19,23,11,15,3,7,5)_{\Theta(1,2,9)}$, $(4,23,13,8,15,12,7,0,11,17,20)_{\Theta(1,2,9)}$, $(12,23,16,8,19,7,4,15,13,5,21)_{\Theta(1,2,9)}$,

$(22,18,1,11,8,16,17,13,9,15,2)_{\Theta(1,3,8)}$, $(9,20,10,22,1,3,21,4,19,12,2)_{\Theta(1,3,8)}$, $(7,4,17,5,10,12,21,6,19,16,20)_{\Theta(1,3,8)}$, $(16,2,3,9,21,22,0,18,13,15,11)_{\Theta(1,3,8)}$, $(18,19,14,7,9,12,11,13,0,6,22)_{\Theta(1,3,8)}$, $(12,7,16,11,17,22,4,15,10,13,6)_{\Theta(1,3,8)}$, $(8,15,17,0,10,21,11,19,5,18,16)_{\Theta(1,3,8)}$, $(5,21,13,7,15,6,11,3,22,9,0)_{\Theta(1,3,8)}$, $(7,23,8,5,22,12,0,2,13,16,1)_{\Theta(1,3,8)}$, $(8,23,1,14,20,6,17,15,7,9,16)_{\Theta(1,3,8)}$, $(13,23,21,15,3,19,14,4,16,7,0)_{\Theta(1,3,8)}$,

$(22,19,10,9,20,11,16,8,2,21,4)_{\Theta(1,4,7)}$, $(19,9,3,21,18,14,1,7,2,20,4)_{\Theta(1,4,7)}$, $(20,18,15,0,3,6,2,16,21,10,11)_{\Theta(1,4,7)}$, $(21,0,7,20,22,14,18,16,15,3,13)_{\Theta(1,4,7)}$, $(4,11,7,18,15,8,12,0,9,17,13)_{\Theta(1,4,7)}$, $(4,1,22,13,15,18,2,9,11,7,0)_{\Theta(1,4,7)}$, $(6,22,1,8,14,0,11,17,5,20,21)_{\Theta(1,4,7)}$, $(7,9,15,14,6,22,8,19,1,13,21)_{\Theta(1,4,7)}$, $(1,22,5,21,15,23,7,14,13,9,16)_{\Theta(1,4,7)}$, $(6,23,7,13,5,16,3,9,14,17,22)_{\Theta(1,4,7)}$, $(15,17,23,14,0,6,5,4,13,12,21)_{\Theta(1,4,7)}$,

$(1,6,22,19,9,8,5,13,12,11,17)_{\Theta(1,5,6)}$, $(0,7,9,17,19,4,8,3,16,2,21)_{\Theta(1,5,6)}$, $(5,17,19,16,9,15,6,4,22,12,18)_{\Theta(1,5,6)}$, $(3,15,22,16,12,10,21,8,20,9,4)_{\Theta(1,5,6)}$, $(13,4,22,15,7,12,16,20,3,5,18)_{\Theta(1,5,6)}$, $(15,1,16,6,22,10,0,2,19,18,21)_{\Theta(1,5,6)}$, $(18,2,10,19,3,11,1,4,21,14,15)_{\Theta(1,5,6)}$, $(5,7,0,18,15,11,12,9,2,6,10)_{\Theta(1,5,6)}$, $(7,13,22,11,19,6,3,14,23,2,8)_{\Theta(1,5,6)}$, $(10,23,14,18,22,5,17,20,13,15,19)_{\Theta(1,5,6)}$, $(21,23,16,10,2,22,15,6,7,18,3)_{\Theta(1,5,6)}$,

$(11,15,2,13,19,1,0,4,12,3,20)_{\Theta(2,2,8)}$, $(16,13,18,1,21,17,4,14,15,0,10)_{\Theta(2,2,8)}$, $(12,11,18,0,14,19,20,5,21,7,15)_{\Theta(2,2,8)}$, $(13,3,17,14,20,18,15,19,22,0,8)_{\Theta(2,2,8)}$, $(12,14,2,17,13,19,18,9,22,7,8)_{\Theta(2,2,8)}$, $(13,20,7,0,3,1,15,8,5,22,17)_{\Theta(2,2,8)}$, $(2,17,9,18,22,4,7,19,16,6,10)_{\Theta(2,2,8)}$, $(14,2,6,10,18,22,15,3,0,17,1)_{\Theta(2,2,8)}$, $(1,2,7,8,23,17,15,12,6,22,13)_{\Theta(2,2,8)}$, $(9,10,15,16,7,14,22,8,11,23,18)_{\Theta(2,2,8)}$, $(14,18,0,5,20,23,6,21,10,9,1)_{\Theta(2,2,8)}$,

$(7,11,0,1,14,3,15,10,22,17,16)_{\Theta(2,3,7)}$, $(1,20,15,17,3,8,18,5,22,14,4)_{\Theta(2,3,7)}$, $(7,0,13,6,8,16,4,9,2,12,14)_{\Theta(2,3,7)}$, $(10,14,13,1,5,2,3,21,19,17,18)_{\Theta(2,3,7)}$, $(9,7,21,20,18,6,19,15,22,4,5)_{\Theta(2,3,7)}$, $(1,3,19,4,10,16,13,21,2,6,0)_{\Theta(2,3,7)}$, $(3,11,12,13,20,18,16,15,4,8,2)_{\Theta(2,3,7)}$, $(13,15,17,4,7,8,10,19,20,0,18)_{\Theta(2,3,7)}$, $(5,23,12,0,2,9,7,10,16,21,1)_{\Theta(2,3,7)}$, $(15,21,12,6,11,23,20,7,22,3,4)_{\Theta(2,3,7)}$, $(19,23,14,4,0,5,20,16,12,8,7)_{\Theta(2,3,7)}$,

$(4,5,18,0,12,3,7,16,6,8,15)_{\Theta(2,4,6)}$, $(1,16,13,10,6,22,5,11,2,19,8)_{\Theta(2,4,6)}$, $(22,21,4,12,11,16,5,14,3,17,15)_{\Theta(2,4,6)}$, $(18,9,17,8,4,5,0,13,21,12,14)_{\Theta(2,4,6)}$, $(15,17,10,19,12,7,4,1,20,18,6)_{\Theta(2,4,6)}$, $(3,5,19,15,9,6,1,12,4,10,7)_{\Theta(2,4,6)}$, $(10,3,2,8,17,0,11,6,15,22,7)_{\Theta(2,4,6)}$, $(3,18,22,9,0,2,6,7,15,14,19)_{\Theta(2,4,6)}$, $(10,18,13,5,2,21,23,0,1,8,7)_{\Theta(2,4,6)}$, $(16,23,15,1,19,22,9,8,11,14,7)_{\Theta(2,4,6)}$,

$(16, 23, 19, 18, 10, 14, 17, 11, 15, 2, 6)_{\Theta(2,4,6)}$,

$(15, 14, 21, 3, 0, 4, 11, 19, 20, 1, 13)_{\Theta(2,5,5)}$, $(14, 0, 18, 15, 2, 4, 19, 8, 16, 1, 5)_{\Theta(2,5,5)}$, $(18, 1, 11, 9, 8, 6, 21, 19, 17, 0, 15)_{\Theta(2,5,5)}$, $(10, 21, 2, 13, 5, 19, 8, 7, 14, 0, 20)_{\Theta(2,5,5)}$, $(21, 13, 4, 3, 11, 17, 15, 0, 7, 1, 2)_{\Theta(2,5,5)}$, $(19, 22, 10, 21, 7, 15, 11, 14, 12, 18, 1)_{\Theta(2,5,5)}$, $(2, 7, 12, 0, 10, 20, 8, 6, 17, 1, 4)_{\Theta(2,5,5)}$, $(10, 18, 8, 15, 6, 12, 4, 14, 9, 20, 23)_{\Theta(2,5,5)}$, $(4, 22, 17, 14, 1, 12, 9, 20, 15, 16, 18)_{\Theta(2,5,5)}$, $(7, 12, 22, 2, 16, 4, 15, 20, 6, 14, 23)_{\Theta(2,5,5)}$, $(20, 22, 14, 12, 0, 23, 4, 17, 9, 1, 6)_{\Theta(2,5,5)}$,

$(14, 13, 20, 7, 16, 9, 22, 17, 18, 21, 0)_{\Theta(3,3,6)}$, $(21, 9, 3, 5, 22, 18, 2, 16, 1, 17, 19)_{\Theta(3,3,6)}$, $(21, 12, 4, 13, 16, 18, 14, 15, 20, 17, 11)_{\Theta(3,3,6)}$, $(19, 2, 22, 8, 15, 9, 12, 1, 13, 21, 11)_{\Theta(3,3,6)}$, $(7, 2, 21, 10, 9, 4, 14, 17, 16, 22, 12)_{\Theta(3,3,6)}$, $(18, 14, 7, 2, 15, 1, 19, 4, 16, 8, 23)_{\Theta(3,3,6)}$, $(8, 19, 7, 0, 11, 6, 12, 20, 16, 3, 14)_{\Theta(3,3,6)}$, $(14, 3, 5, 7, 12, 15, 18, 11, 16, 23, 0)_{\Theta(3,3,6)}$, $(0, 3, 11, 19, 20, 22, 4, 7, 23, 15, 8)_{\Theta(3,3,6)}$, $(6, 19, 4, 8, 21, 23, 10, 3, 11, 15, 7)_{\Theta(3,3,6)}$, $(16, 22, 15, 13, 19, 2, 12, 4, 20, 23, 11)_{\Theta(3,3,6)}$,

$(16, 9, 17, 13, 2, 11, 4, 8, 22, 14, 19)_{\Theta(3,4,5)}$, $(21, 19, 6, 7, 22, 4, 15, 2, 20, 8, 11)_{\Theta(3,4,5)}$, $(3, 5, 8, 7, 4, 1, 19, 6, 9, 14, 18)_{\Theta(3,4,5)}$, $(22, 4, 5, 17, 10, 11, 0, 20, 6, 2, 7)_{\Theta(3,4,5)}$, $(14, 11, 16, 9, 8, 10, 13, 7, 1, 17, 15)_{\Theta(3,4,5)}$, $(22, 21, 11, 5, 7, 10, 20, 9, 2, 1, 16)_{\Theta(3,4,5)}$, $(10, 20, 1, 15, 3, 12, 4, 18, 7, 21, 0)_{\Theta(3,4,5)}$, $(16, 20, 10, 12, 7, 0, 13, 5, 23, 9, 18)_{\Theta(3,4,5)}$, $(2, 16, 10, 23, 15, 7, 13, 4, 8, 21, 12)_{\Theta(3,4,5)}$, $(7, 15, 12, 5, 17, 2, 8, 23, 13, 4, 21)_{\Theta(3,4,5)}$, $(18, 23, 0, 15, 2, 19, 4, 11, 20, 5, 8)_{\Theta(3,4,5)}$,

$(20, 6, 21, 4, 11, 14, 18, 17, 2, 10, 5)_{\Theta(4,4,4)}$, $(13, 12, 10, 14, 22, 17, 16, 0, 2, 8, 11)_{\Theta(4,4,4)}$, $(5, 19, 7, 4, 2, 8, 22, 0, 17, 15, 18)_{\Theta(4,4,4)}$, $(8, 1, 14, 15, 22, 13, 21, 3, 7, 2, 16)_{\Theta(4,4,4)}$, $(5, 21, 14, 12, 8, 19, 3, 15, 20, 0, 7)_{\Theta(4,4,4)}$, $(21, 7, 14, 9, 3, 19, 8, 1, 17, 20, 15)_{\Theta(4,4,4)}$, $(20, 11, 10, 22, 7, 4, 17, 1, 9, 2, 14)_{\Theta(4,4,4)}$, $(11, 18, 2, 15, 1, 20, 7, 16, 22, 19, 6)_{\Theta(4,4,4)}$, $(2, 4, 0, 15, 19, 12, 23, 9, 17, 7, 18)_{\Theta(4,4,4)}$, $(3, 10, 12, 20, 1, 14, 23, 8, 17, 9, 19)_{\Theta(4,4,4)}$, $(3, 12, 6, 15, 4, 18, 9, 1, 23, 10, 17)_{\Theta(4,4,4)}$

under the action of the mapping $x \mapsto x + 8 \pmod{24}$ for only the first six graphs in each design. □

Lemma 3.6. *There exist decompositions into $\Theta(a, b, c)$ of the complete bipartite graph $K_{12,8}$ for $a + b + c = 12$ with a, b, c all even. There exist decompositions into $\Theta(a, b, c)$ of the complete multipartite graphs $K_{8,8,8}$, $K_{8,8,8,8}$ and $K_{8,8,8,24}$ for $a + b + c = 12$ with a, b, c not all even.*

Proof The decompositions are presented in the Appendix. □

Theorem 1.4 for $\Theta(a, b, c)$ with $a + b + c = 12$ follows from Theorem 1.1, Lemmas 3.5 and 3.6, and Proposition 2.1 (with $d = 4$, $r = 3$, $s = 2$, $f = 1$ and $g = 0$, 2 or 4) if a, b and c are all even, Proposition 2.6 otherwise. □

Lemma 3.7. *There exist $\Theta(a, b, c)$ designs of orders 13 and 14 for $a + b + c = 13$. There exists a $\Theta(a, b, c)$ design of order 26 for $a + b + c = 13$ with a, b, c not all odd.*

Proof There are thirteen theta graphs of which four are bipartite. The decompositions are presented in the Appendix. □

Lemma 3.8. *There exist decompositions into $\Theta(a,b,c)$ of the complete bipartite graph $K_{13,13}$ for $a+b+c=13$ with a, b, c all odd. There exist decompositions into $\Theta(a,b,c)$ of the complete multipartite graphs $K_{13,13,13}$, $K_{13,13,13,13}$ and $K_{13,13,13,13,13}$ for $a+b+c=13$ with a, b, c not all odd.*

Proof The decompositions are presented in the Appendix. □

Theorem 1.5 for $\Theta(a,b,c)$ with $a+b+c=13$ follows from Theorem 1.1, Lemmas 3.7 and 3.8, and Proposition 2.1 (with $d=13$, $r=s=f=1$ and $g=0$) if a, b and c are all odd, Proposition 2.3 otherwise. □

Lemma 3.9. *There exist $\Theta(a,b,c)$ designs of order 21, 28 and 36 for $a+b+c=14$. There exist $\Theta(a,b,c)$ designs of order 49, 56, 57, 64, 77 and 92 for $a+b+c=14$ with a, b, c not all even.*

Proof There are fifteen theta graphs of which four are bipartite. The decompositions are presented in the Appendix. □

Lemma 3.10. *There exist decompositions into $\Theta(a,b,c)$ of the complete bipartite graph $K_{14,7}$ for $a+b+c=14$ with a, b, c all even. There exist decompositions into $\Theta(a,b,c)$ of the complete multipartite graphs $K_{14,14,14}$, $K_{7,7,7,7}$, $K_{28,28,28,35}$, $K_{7,7,7,7,7}$, $K_{7,7,7,7,21}$ and $K_{7,7,7,7,28}$ for $a+b+c=14$ with a, b, c not all even.*

Proof The decompositions are presented in the Appendix. □

Theorem 1.6 for $\Theta(a,b,c)$ with $a+b+c=14$ follows from Theorem 1.1, Lemmas 3.9 and 3.10, and Proposition 2.1 (with $d=7$, $r=2$, $s=1$, $f=2$ and $g=0$, 3 or 5) if a, b and c are all even, Proposition 2.7 otherwise. □

Lemma 3.11. *There exist $\Theta(a,b,c)$ designs of orders 15, 16, 21, and 25 for $a+b+c=15$. There exist $\Theta(a,b,c)$ designs of orders 30, 36, 40, 51, 55, 66 and 70 for $a+b+c=15$ with a, b, c not all odd.*

Proof There are eighteen 15-edge theta graphs of which six are bipartite. We present designs of order 30 here; the remaining decompositions are presented in the Appendix.

$\boldsymbol{K_{30}}$ Let the vertex set be Z_{30}. The decompositions consist of the graphs

$(6,2,0,7,22,5,19,15,18,25,23,1,21,12)_{\Theta(1,2,12)}$,
$(7,20,15,2,10,9,8,5,6,17,3,18,28,22)_{\Theta(1,2,12)}$,
$(20,4,14,19,1,3,5,18,11,16,24,13,2,21)_{\Theta(1,2,12)}$,
$(7,13,10,14,6,11,8,4,25,9,18,21,27,23)_{\Theta(1,2,12)}$,

$(27, 5, 16, 22, 6, 18, 13, 0, 14, 2, 9, 21, 11, 4)_{\Theta(1,2,12)}$,
$(5, 17, 26, 11, 23, 2, 15, 22, 4, 6, 10, 28, 21, 8)_{\Theta(1,2,12)}$,
$(23, 5, 14, 17, 2, 11, 7, 28, 24, 22, 18, 16, 25, 29)_{\Theta(1,2,12)}$,
$(17, 29, 11, 13, 4, 0, 28, 16, 12, 10, 22, 1, 5, 20)_{\Theta(1,2,12)}$,
$(23, 29, 8, 19, 10, 3, 20, 11, 26, 9, 16, 4, 27, 14)_{\Theta(1,2,12)}$,

$(19, 1, 6, 11, 23, 0, 27, 21, 5, 18, 14, 7, 15, 17)_{\Theta(1,4,10)}$,
$(15, 19, 0, 28, 20, 1, 6, 9, 18, 22, 11, 16, 25, 8)_{\Theta(1,4,10)}$,
$(17, 26, 16, 15, 8, 23, 10, 28, 5, 9, 20, 12, 19, 21)_{\Theta(1,4,10)}$,
$(0, 9, 1, 10, 21, 14, 12, 23, 15, 25, 5, 7, 2, 26)_{\Theta(1,4,10)}$,
$(10, 15, 4, 2, 28, 25, 19, 16, 13, 17, 27, 26, 6, 22)_{\Theta(1,4,10)}$,
$(12, 0, 5, 20, 23, 24, 4, 14, 11, 2, 16, 6, 28, 18)_{\Theta(1,4,10)}$,
$(12, 18, 4, 20, 10, 11, 26, 23, 14, 29, 6, 5, 8, 17)_{\Theta(1,4,10)}$,
$(0, 24, 10, 26, 16, 29, 2, 17, 14, 28, 8, 22, 12, 6)_{\Theta(1,4,10)}$,
$(18, 24, 11, 8, 23, 6, 0, 22, 2, 5, 26, 29, 20, 17)_{\Theta(1,4,10)}$,

$(5, 2, 26, 10, 18, 8, 4, 16, 25, 14, 20, 17, 21, 6)_{\Theta(1,6,8)}$,
$(9, 21, 18, 15, 4, 17, 22, 11, 12, 24, 13, 10, 0, 14)_{\Theta(1,6,8)}$,
$(10, 6, 15, 2, 3, 27, 13, 11, 17, 0, 8, 23, 12, 7)_{\Theta(1,6,8)}$,
$(24, 0, 22, 19, 15, 20, 9, 26, 14, 13, 8, 28, 12, 5)_{\Theta(1,6,8)}$,
$(21, 13, 19, 28, 4, 26, 17, 18, 1, 14, 7, 5, 15, 23)_{\Theta(1,6,8)}$,
$(19, 7, 5, 17, 3, 10, 25, 11, 4, 22, 15, 28, 13, 1)_{\Theta(1,6,8)}$,
$(7, 23, 13, 3, 16, 28, 5, 27, 10, 22, 29, 15, 25, 11)_{\Theta(1,6,8)}$,
$(1, 19, 17, 10, 28, 21, 4, 16, 9, 22, 7, 29, 13, 25)_{\Theta(1,6,8)}$,
$(1, 23, 21, 5, 13, 19, 9, 25, 17, 29, 11, 27, 4, 16)_{\Theta(1,6,8)}$,

$(12, 18, 17, 28, 24, 15, 7, 19, 22, 3, 23, 9, 27, 2)_{\Theta(2,2,11)}$,
$(18, 8, 15, 7, 5, 2, 24, 9, 22, 1, 27, 28, 11, 4)_{\Theta(2,2,11)}$,
$(15, 22, 17, 10, 11, 19, 23, 8, 6, 12, 13, 26, 20, 2)_{\Theta(2,2,11)}$,
$(15, 13, 23, 22, 25, 14, 28, 6, 9, 0, 4, 26, 17, 20)_{\Theta(2,2,11)}$,
$(5, 21, 7, 19, 23, 22, 20, 0, 25, 12, 8, 9, 15, 2)_{\Theta(2,2,11)}$,
$(13, 16, 19, 10, 6, 17, 24, 5, 26, 1, 7, 0, 28, 22)_{\Theta(2,2,11)}$,
$(4, 25, 10, 19, 23, 17, 28, 1, 24, 22, 11, 18, 29, 20)_{\Theta(2,2,11)}$,
$(4, 13, 7, 28, 6, 29, 23, 14, 19, 12, 5, 11, 17, 8)_{\Theta(2,2,11)}$,
$(16, 25, 1, 18, 5, 29, 10, 12, 23, 0, 11, 2, 7, 22)_{\Theta(2,2,11)}$,

$(12, 19, 17, 15, 23, 14, 4, 21, 1, 25, 5, 2, 6, 3)_{\Theta(2,3,10)}$,
$(22, 2, 1, 6, 17, 25, 28, 11, 4, 0, 18, 10, 16, 27)_{\Theta(2,3,10)}$,
$(6, 0, 21, 20, 28, 12, 2, 25, 11, 15, 10, 14, 26, 17)_{\Theta(2,3,10)}$,
$(11, 16, 17, 2, 9, 12, 5, 23, 28, 26, 15, 6, 13, 1)_{\Theta(2,3,10)}$,
$(22, 9, 10, 12, 1, 11, 21, 23, 26, 20, 4, 25, 0, 8)_{\Theta(2,3,10)}$,
$(1, 13, 0, 27, 9, 18, 19, 2, 7, 20, 25, 8, 21, 15)_{\Theta(2,3,10)}$,
$(3, 1, 20, 7, 26, 15, 19, 8, 13, 5, 21, 25, 27, 14)_{\Theta(2,3,10)}$,
$(3, 9, 21, 1, 23, 17, 25, 14, 19, 11, 27, 15, 29, 7)_{\Theta(2,3,10)}$,
$(9, 13, 26, 15, 2, 3, 27, 21, 19, 6, 7, 24, 25, 12)_{\Theta(2,3,10)}$,

$(19, 3, 9, 0, 22, 7, 6, 20, 26, 10, 28, 17, 4, 18)_{\Theta(2,4,9)}$,
$(14, 6, 4, 26, 19, 13, 9, 20, 7, 8, 16, 12, 11, 24)_{\Theta(2,4,9)}$,
$(8, 28, 15, 18, 26, 5, 21, 17, 19, 14, 25, 4, 7, 23)_{\Theta(2,4,9)}$,
$(11, 20, 23, 7, 28, 21, 17, 27, 6, 15, 16, 22, 3, 5)_{\Theta(2,4,9)}$,
$(10, 4, 8, 11, 3, 1, 15, 12, 23, 9, 21, 7, 29, 24)_{\Theta(2,4,9)}$,
$(1, 25, 0, 11, 20, 24, 19, 7, 17, 26, 23, 2, 6, 3)_{\Theta(2,4,9)}$,
$(25, 18, 13, 5, 14, 11, 7, 12, 6, 29, 2, 0, 24, 19)_{\Theta(2,4,9)}$,
$(1, 12, 9, 6, 8, 5, 13, 23, 0, 26, 24, 17, 20, 18)_{\Theta(2,4,9)}$,
$(12, 18, 14, 13, 21, 24, 8, 29, 19, 27, 0, 6, 7, 15)_{\Theta(2,4,9)}$
$(5, 15, 17, 7, 4, 20, 28, 16, 26, 25, 9, 27, 18, 2)_{\Theta(2,5,8)}$,
$(25, 27, 24, 6, 26, 5, 1, 3, 2, 8, 11, 18, 10, 7)_{\Theta(2,5,8)}$,
$(11, 24, 4, 0, 12, 6, 22, 19, 14, 18, 17, 26, 8, 1)_{\Theta(2,5,8)}$,
$(26, 28, 21, 7, 9, 15, 24, 3, 6, 14, 12, 25, 13, 19)_{\Theta(2,5,8)}$,
$(5, 3, 18, 22, 27, 28, 23, 0, 25, 10, 19, 2, 17, 11)_{\Theta(2,5,8)}$,
$(5, 11, 25, 9, 28, 22, 10, 21, 2, 4, 8, 27, 16, 14)_{\Theta(2,5,8)}$,
$(10, 17, 21, 14, 3, 22, 16, 4, 28, 29, 15, 26, 23, 7)_{\Theta(2,5,8)}$,
$(5, 10, 8, 4, 22, 26, 28, 19, 29, 13, 23, 9, 20, 16)_{\Theta(2,5,8)}$,
$(11, 17, 1, 27, 23, 22, 20, 15, 4, 16, 28, 2, 29, 3)_{\Theta(2,5,8)}$,
$(4, 7, 25, 26, 27, 15, 23, 10, 9, 12, 28, 14, 16, 22)_{\Theta(2,6,7)}$,
$(26, 12, 16, 9, 19, 11, 8, 1, 5, 6, 24, 18, 27, 13)_{\Theta(2,6,7)}$,
$(3, 5, 14, 22, 1, 20, 26, 11, 7, 17, 12, 2, 9, 10)_{\Theta(2,6,7)}$,
$(13, 0, 11, 8, 5, 23, 27, 2, 18, 14, 26, 22, 4, 7)_{\Theta(2,6,7)}$,
$(7, 18, 1, 8, 24, 2, 19, 5, 11, 10, 17, 27, 25, 3)_{\Theta(2,6,7)}$,
$(21, 22, 0, 15, 12, 4, 27, 24, 28, 17, 3, 5, 16, 9)_{\Theta(2,6,7)}$,
$(22, 27, 11, 12, 3, 16, 18, 10, 15, 28, 6, 4, 23, 21)_{\Theta(2,6,7)}$,
$(9, 21, 18, 6, 29, 15, 24, 4, 23, 0, 3, 10, 12, 5)_{\Theta(2,6,7)}$,
$(9, 27, 3, 15, 17, 24, 16, 6, 11, 18, 28, 0, 10, 29)_{\Theta(2,6,7)}$,
$(19, 17, 3, 15, 13, 16, 9, 23, 7, 24, 12, 14, 21, 2)_{\Theta(3,4,8)}$,
$(11, 25, 16, 17, 18, 24, 2, 1, 19, 28, 8, 3, 6, 10)_{\Theta(3,4,8)}$,
$(4, 14, 9, 10, 15, 11, 20, 2, 5, 7, 26, 1, 6, 27)_{\Theta(3,4,8)}$,
$(6, 15, 26, 14, 17, 0, 9, 2, 19, 18, 7, 4, 12, 28)_{\Theta(3,4,8)}$,
$(11, 19, 22, 12, 21, 6, 9, 27, 1, 3, 25, 16, 29, 20)_{\Theta(3,4,8)}$,
$(22, 11, 16, 4, 28, 20, 17, 10, 12, 26, 23, 24, 8, 5)_{\Theta(3,4,8)}$,
$(10, 22, 16, 8, 17, 23, 29, 26, 4, 6, 5, 0, 28, 14)_{\Theta(3,4,8)}$,
$(10, 29, 28, 5, 4, 22, 24, 2, 16, 23, 18, 17, 12, 11)_{\Theta(3,4,8)}$,
$(11, 29, 14, 0, 23, 5, 17, 6, 20, 4, 28, 16, 18, 2)_{\Theta(3,4,8)}$,
$(21, 6, 10, 11, 5, 2, 16, 3, 8, 24, 17, 27, 23, 4)_{\Theta(3,6,6)}$,
$(2, 6, 18, 25, 7, 27, 8, 14, 26, 19, 4, 28, 24, 15)_{\Theta(3,6,6)}$,
$(11, 16, 4, 8, 0, 17, 25, 22, 6, 23, 15, 10, 7, 26)_{\Theta(3,6,6)}$,
$(4, 22, 17, 13, 16, 14, 27, 24, 0, 3, 5, 10, 19, 15)_{\Theta(3,6,6)}$,
$(7, 24, 8, 9, 19, 18, 0, 13, 20, 12, 11, 26, 23, 2)_{\Theta(3,6,6)}$,

$(1, 9, 17, 7, 3, 15, 27, 19, 25, 11, 13, 23, 14, 21)_{\Theta(3,6,6)}$,
$(25, 1, 5, 29, 11, 17, 8, 15, 7, 27, 13, 21, 3, 9)_{\Theta(3,6,6)}$,
$(9, 19, 27, 21, 2, 11, 5, 26, 3, 15, 13, 7, 23, 29)_{\Theta(3,6,6)}$,
$(19, 25, 17, 23, 5, 7, 21, 15, 1, 13, 29, 20, 27, 3)_{\Theta(3,6,6)}$,
$(19, 9, 5, 1, 12, 26, 15, 22, 16, 10, 23, 21, 7, 28)_{\Theta(4,4,7)}$,
$(13, 3, 2, 28, 18, 19, 24, 7, 14, 5, 10, 25, 23, 15)_{\Theta(4,4,7)}$,
$(10, 20, 12, 3, 25, 22, 11, 15, 9, 4, 0, 1, 21, 5)_{\Theta(4,4,7)}$,
$(9, 23, 7, 20, 18, 26, 4, 12, 0, 24, 6, 22, 2, 3)_{\Theta(4,4,7)}$,
$(6, 1, 20, 17, 23, 13, 22, 19, 14, 8, 15, 21, 18, 11)_{\Theta(4,4,7)}$,
$(11, 14, 24, 23, 16, 8, 10, 26, 12, 2, 6, 5, 18, 17)_{\Theta(4,4,7)}$,
$(14, 20, 23, 22, 29, 28, 26, 0, 18, 8, 17, 5, 2, 4)_{\Theta(4,4,7)}$,
$(11, 29, 10, 17, 0, 23, 5, 28, 2, 14, 24, 20, 8, 26)_{\Theta(4,4,7)}$,
$(20, 29, 2, 16, 17, 22, 8, 12, 23, 6, 26, 5, 4, 11)_{\Theta(4,4,7)}$,
$(11, 7, 6, 8, 15, 18, 4, 9, 19, 28, 27, 0, 10, 26)_{\Theta(4,5,6)}$,
$(24, 28, 2, 17, 19, 5, 14, 13, 10, 25, 16, 3, 8, 21)_{\Theta(4,5,6)}$,
$(26, 9, 28, 5, 21, 8, 16, 27, 13, 22, 17, 18, 2, 11)_{\Theta(4,5,6)}$,
$(1, 22, 24, 27, 18, 5, 9, 7, 23, 25, 12, 4, 6, 2)_{\Theta(4,5,6)}$,
$(28, 13, 25, 20, 26, 22, 7, 14, 24, 17, 9, 8, 11, 23)_{\Theta(4,5,6)}$,
$(23, 9, 1, 6, 24, 17, 11, 14, 3, 26, 15, 0, 12, 18)_{\Theta(4,5,6)}$,
$(0, 9, 25, 17, 20, 6, 27, 12, 29, 18, 24, 11, 21, 15)_{\Theta(4,5,6)}$,
$(0, 5, 24, 19, 11, 17, 27, 3, 18, 21, 6, 12, 7, 29)_{\Theta(4,5,6)}$,
$(5, 23, 13, 18, 6, 15, 24, 12, 3, 8, 27, 21, 2, 29)_{\Theta(4,5,6)}$
under the action of the mapping $x \mapsto x + 6 \pmod{30}$ for only the first five graphs in each design. □

Lemma 3.12. *There exist decompositions into $\Theta(a, b, c)$ of the complete bipartite graphs $K_{15,15}$, $K_{15,20}$ and $K_{15,25}$ for $a+b+c = 15$ with a, b, c all odd. There exist decompositions into $\Theta(a, b, c)$ of the complete multipartite graphs $K_{5,5,5}$ and $K_{5,5,5,5}$ for $a+b+c = 15$ with a, b, c not all odd.*

Proof The decompositions are presented in the Appendix. □

Theorem 1.7 follows from Theorem 1.1, Lemmas 3.11 and 3.12, and Proposition 2.2 (with $(r, s, e) = (15, 15, 0)$, $(15, 15, 1)$, $(15, 20, 1)$ or $(15, 25, 0)$) if a, b and c are all odd, Proposition 2.8 otherwise. □

References

[1] Abel. R., F. E. Bennett and M. Greig, PBD-Closure, *Handbook of Combinatorial Designs*, second edition (ed. C. J. Colbourn and J. H. Dinitz), Chapman & Hall/CRC Press (2007), 247–255.

[2] Adams. P, D. E. Bryant and M. Buchanan, A survey on the existence of G-designs, *J. Combin. Des.* **16** (2008), 373–410.

[3] Blinco. A. , On diagonal cycle systems, *Australas. J. Combin.* **24** (2001), 221–230.

[4] Blinco. A. B, Decompositions of complete graphs into theta graphs with fewer than ten edges, *Utilitas Mathematica* **64** (2003), 197–212.

[5] Bryant. D. E. and T. A. McCourt, Existence results for G–designs, http://wiki.smp.uq.edu.au/G-designs/.

[6] Forbes. A. D. , Solution 265.6 – Triples, *M500* **276** (June 2017).

[7] Ge. G. , Group divisible designs, *Handbook of Combinatorial Designs* second edition (ed. C. J. Colbourn and J. H. Dinitz), Chapman & Hall/CRC Press (2007), 255–260.

[8] Ge. G. and A. C. H. Ling, Asymptotic results on the existence of 4-RGDDs and uniform 5-GDDs, *J. Combin. Des.* **13** (2005) 222–237.

[9] Ge. G. and Y. Miao, PBDs, Frames and Resolvability, *Handbook of Combinatorial Designs*, second edition (ed. C. J. Colbourn and J. H. Dinitz), Chapman & Hall/CRC Press (2007), 261–265.

[10] Punnim. N. and N. Pabhapote, On graceful graphs: cycles with a P_k chord, $k \geq 4$, *Ars Combin.* **23A** (1987), 225–228.

[11] Rees. R. S. , Two New Direct Product-Type Constructions for Resolvable Group-Divisible Designs, *J. Combin. Designs* **1** (1993), 15–26.

[12] Wilson. R. M. , An existence theory for pairwise balanced designs I: Composition theorems and morphisms, *J. Combin. Theory Ser. A*, **13** (1971), 220–245.