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Asteroidal number for some product graphs

S. ALAGU*[1] and R. KALA†[2]

[1,2]Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli - 627 012, Tamilnadu, India.

ABSTRACT

The notion of Asteroidal triples was introduced by Lekkerkerker and Boland [6]. D.G.Corneil and others [2], Ekkehard Kohler [3] further investigated asteroidal triples. Walter generalized the concept of asteroidal triples to asteroidal sets [8]. Further study was carried out by Haiko Muller [4]. In this paper we find asteroidal numbers for Direct product of cycles, Direct product of path and cycle, Strong product of paths and cycles and some more graphs.

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*alagu391@gmail.com

†karthipyi91@yahoo.co.in

1 Introduction

We collect all the relevant definitions needed in the sequel.

Definition 1.1. *The* ***neighbourhood*** *of a vertex u in a graph G is the set* $N(u)$ *consisting of all vertices v which are adjacent with u. The closed neighbourhood is* $N[u] = N(u) \cup \{u\}$ *.*

Definition 1.2. *Let* $G = (V, E)$ *be a graph. A subset A of vertices is called an* ***asteroidal set*** *of the graph G if for each vertex* $a \in A$ *all elements of* $A \setminus \{a\}$ *are contained in the same connected component of G-*$N[a]$ *.*

Definition 1.3. *The* ***asteroidal number*** *of* G *denoted by* $an(G)$ *is defined as the maximum cardinality of an asteroidal set in* G *.*

Definition 1.4. *Let* $G = (V_G, E_G)$ *and* $H = (V_H, E_H)$ *be two graphs. The* ***Cartesian product*** *of* G *and* H *, denoted by* $G \circ H$ *, has* $V(G \circ H) = \{(g, h) / g \in G; h \in H\}$ *and* $E(G \circ H) = \{(g_1, h_1)(g_2, h_2) / g_1 = g_2, h_1h_2 \in E(H)$ *or* $g_1g_2 \in E(G), h_1 = h_2\}$. [7]

Definition 1.5. *Let* $G = (V_G, E_G)$ *and* $H = (V_H, E_H)$ *be two graphs. The* ***Direct product*** *of* G *and* H *, denoted by* $G \times H$ *, has* $V(G \times H) = \{(g, h) / g \in G; h \in H\}$ *as the vertex set and* $E(G \times H) = \{(g_1, h_1)(g_2, h_2) / g_1g_2 \in E(G)$ *and* $h_1h_2 \in E(H)\}$. [7]

Definition 1.6. *Let* $G = (V_G, E_G)$ *and* $H = (V_H, E_H)$ *be two graphs. The* ***Strong product*** *of* G *and* H *, denoted by* $G \otimes H$ *, has* $V(G \otimes H) = \{(g, h) / g \in G; h \in H\}$ *as the vertex set and* $E(G \otimes H) = E(G \circ H) \cup E(G \times H)$. [7] .

Definition 1.7. *Let* $G = (V, E)$ *be a graph. A subset* S *of* V *is called an* ***independent set*** *if there is no edge between any two vertices of* S *.*

Definition 1.8. *The cardinality of a maximum independent set of a graph* G *is called* ***independence number*** *of* G *and is denoted by* $\beta_0(G)$ *or simply* β_0 *.*

Definition 1.9. *Let* $V = \{x_1, x_2, \ldots x_n\}$ *and* $E = \{(x_i, x_{i+1}) / 0 \leq i \leq n-1\}$ *. The graph* (V, E) *is called a* ***path*** *on n vertices and is denoted by* P_n *.*

Definition 1.10. *Let* $V = \{x_1, x_2, \dots x_n\}$ *and* $E = \{(x_i, x_{i+1}) / 0 \leq i \leq n-1\} \cup \{(x_1, x_n)\}$. *The graph* (V, E) *is called a* ***cycle*** *on n vertices and is denoted by* C_n .

The following results are from [4] and [5] respectively.

Lemma 1.11. *For a disconnected graph* G , $an(G) = max\{an(G[C]) : C \in Comp(G)\}$, *where Comp(*G *) denotes the set of all components of* G .

Lemma 1.12. *Let* $G = (V, E)$ *be a graph. If* $G - N[v]$ *is connected* $\forall\ v \in V$, *then* $an(G) = \beta_0$.

The following theorem is from [1].

Theorem 1.13. *Indepedence number for some product graphs are given below :*
1. $\beta_0 (P_m \otimes P_n) = \lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$
2. $\beta_0 (C_m \otimes C_n) = \beta_0 (C_m) \beta_0 (C_n)$ *where* $\beta_0 (C_n) = \lfloor \frac{n}{2} \rfloor$
3. $\beta_0 (P_m \otimes C_n) = \beta_0 (P_m) \beta_0 (C_n)$ *where* $\beta_0 (P_m) = \lceil \frac{m}{2} \rceil$

Definition 1.14. *Let* $G = (V_G, E_G)$ *and* $H = (V_H, E_H)$ *be two graphs. The* ***Strong product*** *of* G *and* H , *denoted by* $G \otimes H$, *has* $V(G \otimes H) = \{(g, h) / g \in G; h \in H\}$ *as the vertex set and* $E(G \otimes H) = E(G \circ H) \cup E(G \times H)$. [7] .

Definition 1.15. *Let* $G = (V, E)$ *be a graph. A subset* S *of* V *is called an* ***independent set*** *if there is no edge between any two vertices of* S .

Definition 1.16. *The cardinality of a maximum independent set of a graph* G *is called* ***independence number*** *of* G *and is denoted by* $\beta_0(G)$ *or simply* β_0 .

Definition 1.17. *Let* $V = \{x_1, x_2, \dots x_n\}$ *and* $E = \{(x_i, x_{i+1}) / 0 \leq i \leq n-1\}$. *The graph* (V, E) *is called a* ***path*** *on n vertices and is denoted by* P_n .

Definition 1.18. *Let* $V = \{x_1, x_2, \dots x_n\}$ *and* $E = \{(x_i, x_{i+1}) / 0 \leq i \leq n-1\} \cup \{(x_1, x_n)\}$. *The graph* (V, E) *is called a* ***cycle*** *on n vertices and is denoted by* C_n .

The following results are from [4] and [5] respectively.

Lemma 1.19. *For a disconnected graph G , $an(G) = max\{an(G[C]) : C \in Comp(G)\}$, where Comp(G) denotes the set of all components of G .*

Lemma 1.20. *Let $G = (V, E)$ be a graph. If $G - N[v]$ is connected $\forall\ v \in V$, then $an(G) = \beta_0$.*

The following theorem is from [1].

Theorem 1.21. *Indepedence number for some product graphs are given below :*

1. $\beta_0(P_m \otimes P_n) = \lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$

2. $\beta_0(C_m \otimes C_n) = \beta_0(C_m)\beta_0(C_n)$ *where* $\beta_0(C_n) = \lfloor \frac{n}{2} \rfloor$

3. $\beta_0(P_m \otimes C_n) = \beta_0(P_m)\beta_0(C_n)$ *where* $\beta_0(P_m) = \lceil \frac{m}{2} \rceil$

2 Main Results

Theorem 2.1. $an(P_m \times C_n) = \begin{cases} (k-1)\,l & if \quad m = 2k\ ,\ n = 2l \\ \lfloor \frac{m}{2} \rfloor \frac{n}{2} & if \quad m = 2k+1\ ,\ n = 2l \\ \lfloor \frac{m}{2} \rfloor n & if \quad n = 2l+1 \end{cases}$

Proof. We draw $P_m \times C_n$ in such a way that it has m columns and n rows.

Case (i): $m = 2k; n = 2l$

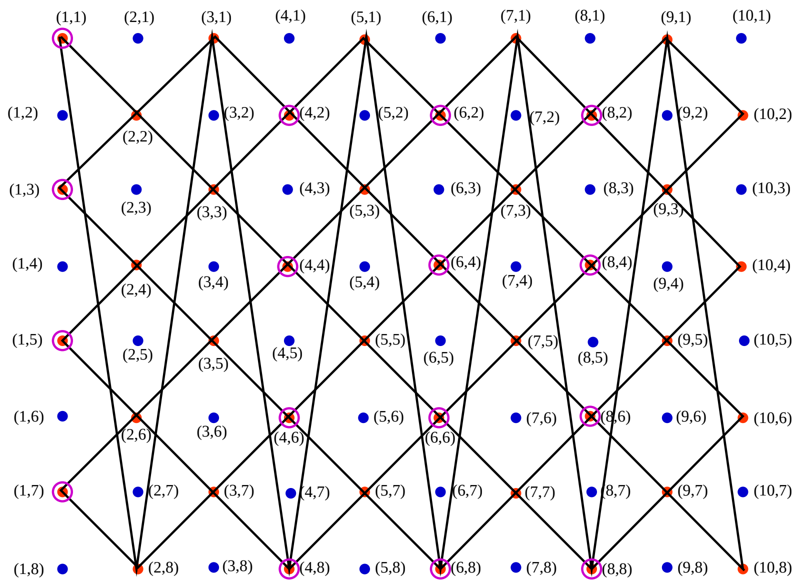


Fig. 1

For even m and n, $P_m \times C_n$ has two components - one being the upside down image of other. Therefore, it is enough if we find asteroidal number for one component. Let us consider the component containing (1,1), say G_1. Collect all the vertices in the first column of G_1 for Γ. None of the second column vertices can be chosen for Γ, since they are adjacent with those of first column vertices lying in Γ. Third column vertices are also to be avoided, since removal of any third column vertex and its neighbourhood will leave a first column vertex alone. Now, choosing all the vertices $(in\ G_1)$ of first/second, fourth, sixth, eighth, $\ldots (m-2)^{th}$ columns or first/third, fifth, seventh, ninth, $\ldots (m-1)^{th}$ columns, we get maximum asteroidal sets of cardinality $(k-1)\,l$.

Case(ii): $m = 2k+1$; $n = 2l$

For even m and odd n, $P_m \times C_n$ has two components - one being the upside down image of other. Therefore, it is enough if we find asteroidal number for one component. Let us consider the component containing (1,1), say G_1. The procedure to choose maximum asteroidal set is the same as that for previous case.

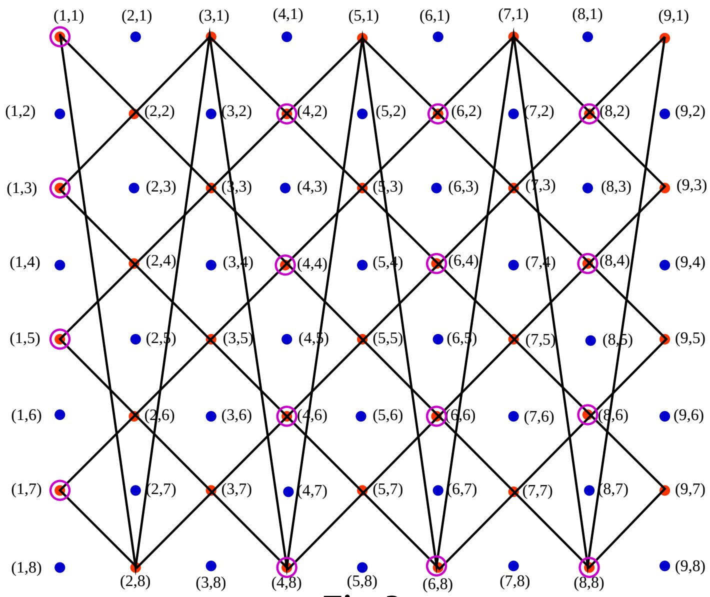


Fig. 2

Choosing all the vertices $(in\ G_1)$ of first/second, fourth, sixth, eighth, $\ldots (m-1)^{th}$ columns or first/third, fifth, seventh, ninth, $\ldots (m-2)^{th}$ columns, we get maximum asteroidal sets of cardinality $\left\lfloor \frac{m}{2} \right\rfloor \frac{n}{2}$.

Case (iii) : $n = 2l + 1$

For odd n, irrespective of m, $P_m \times C_n$ is connected. Choosing all the vertices of first/second, fourth, sixth, eighth, $\ldots$ columns or first/third, fifth, seventh, ninth, $\ldots$ columns $(\textit{the last column will be } m-1 \textit{ or } m-2 \textit{ depending on } m)$, we get maximum asteroidal sets of cardinality $\left\lfloor \frac{m}{2} \right\rfloor n$.

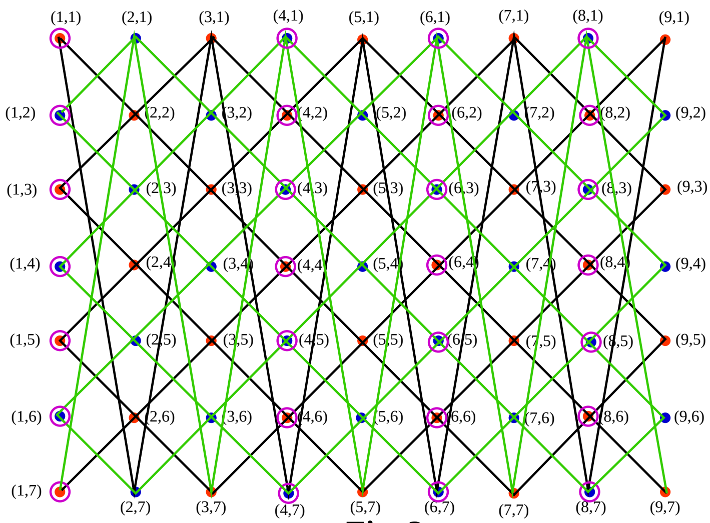


Fig. 3

□

Now we deal with direct product of cycles.

Theorem 2.2. $an(C_m \times C_n) = \begin{cases} \frac{mn}{4} & if \quad m,n \ are \ both \ even \\ \beta_0\,(C_m \times C_n) & otherwise \end{cases}$

Proof. $C_m \times C_n$ is drawn such that number of columns is greater than number of rows.

Case (i): For even m and n, the graph has two components - one being the mirror image of the other. Hence it is enough if we find asteroidal number for one component. Let G_1 be the component containing $(1,1)$. Choosing all the vertices $(of\ G_1)$ in the first, third,...last but one columns or second, fourth,....last columns, we get maximum asteroidal sets of cardinality $\frac{mn}{4}$.

Case (ii): If at least one of m, n is odd, the graph $C_m \times C_n$ is connected and satisfies the condition for Lemma 1.12. Hence asteroidal number in this case will be independence number.

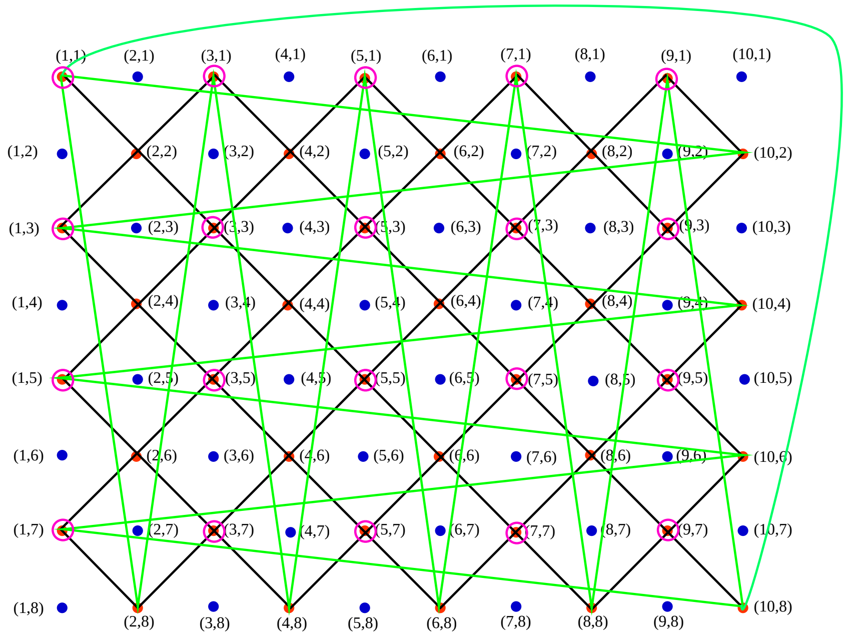


Fig. 4

$\square$

In the following we make a simple observation on Cartesian and Strong products of cycles and paths.

Theorem 2.3. *Asteroidal number is the same as that of independence number for the following graphs :*

$(i)\, C_m \circ C_n$

$(ii)\, P_m \otimes P_n$

$(iii)\, C_m \otimes C_n$

$(iv)\, P_m \otimes C_n$

Proof. Since $G - N[u]$ is connected $\forall u \in G$,where G may denote any graph from $(i)\, to\, (iv)$,by Lemma 1.11, the result holds. Also we see that $an\,(P_m \otimes P_n) = \left\lceil \frac{m}{2} \right\rceil \left\lceil \frac{n}{2} \right\rceil$, $an\,(C_m \otimes C_n) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor$, $an\,(P_m \otimes C_n) = \left\lceil \frac{m}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor$ (By theorem 1.13). $\square$

Now we collect some more standard graphs and investigate their asteroidal numbers.

Definition 2.4. *The complete bipartite graph $K_{1,n}$ is called a **star**.*

Definition 2.5. *The **bistar** $B_{m,n}$ is the graph obtained from K_2 by joining m pendent edges to one end of K_2 and n pendent edges to the other end of K_2.*

Definition 2.6. *The graph obtained by joining a single pendent edge to each vertex of a path is called a **comb**.*

Definition 2.7. *A **Triangular snake** denoted by T_n is obtained from a path $v_1v_2.....v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $1 \leq i \leq n-1$.*

Definition 2.8. *A **Quadrilateral snake** is obtained from a path $u_1u_2...u_n$ by joining u_i, u_{i+1} to new vertices v_i, w_i respectively and joining v_i and w_i.*

Definition 2.9. *A **Ladder** graph, denoted by L_n is a graph obtained from the cartesian product of P_n and P_1.*

Definition 2.10. *The **corona product** $G_1 \odot G_2$ of two graphs G_1 and G_2 is defined as the graph G obtained by taking one copy of G_1 (which has p_1 vertices) and p_1 copies of G_2 and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2.*

Definition 2.11. *An **asteroidal triple** of a graph $G = (V, E)$ is a set of three vertices, such that there exists a path between any two of them avoiding the neighbourhood of the third. Graphs without asteroidal triples are called asteroidal triple free graphs.* [2]

In the following we enumerate some asteroidal triple free graphs.

Theorem 2.12. *The following graphs are asteroidal triple free graphs and hence their asteroidal number is two:*

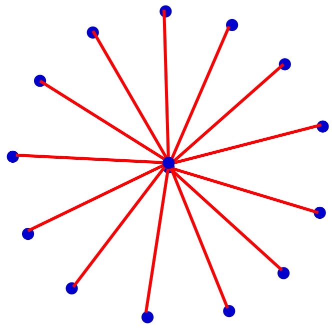
1. *Star graphs*
2. *Bistar graphs*
3. *Comb*
4. *Triangular snake*
5. *Quadrilateral snake*

6. Ladder graph

7. $P_n \odot mK_1$

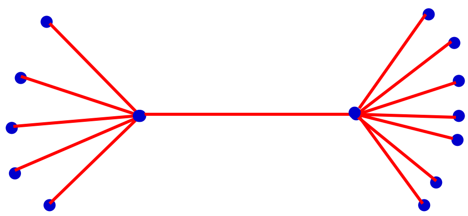
8. $P_m \odot K_m$

Proof. 1. Since removal of any pendent vertex and its neighbourhood $(see\ Fig\ 1)$ will result in a graph full of isolated vertices, asteroidal number of star graph is 2. Any two pendent vertices can be taken for a maximum asteroidal set.



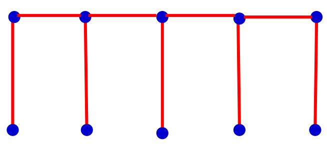
Star

2. Removal of a pendent vertex and its neighbourhood will result in a union of a star and isolated vertices. Hence it is asteroidal triple free.

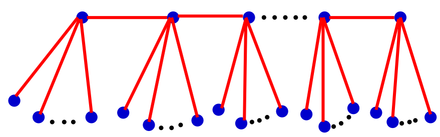


Bistar

For 3,7 and 8, consider any three vertices that are mutually non adjacent. For one vertex $(among\ the\ three)$, the removal of it's closed neighbourhood will disconnect the path thereby leaving the other two vertices in two different components. Hence combs have asteroidal number two.



Comb



$P_n \odot mK_1$

The argument for 4,5 and 6 is the same as that for 3.

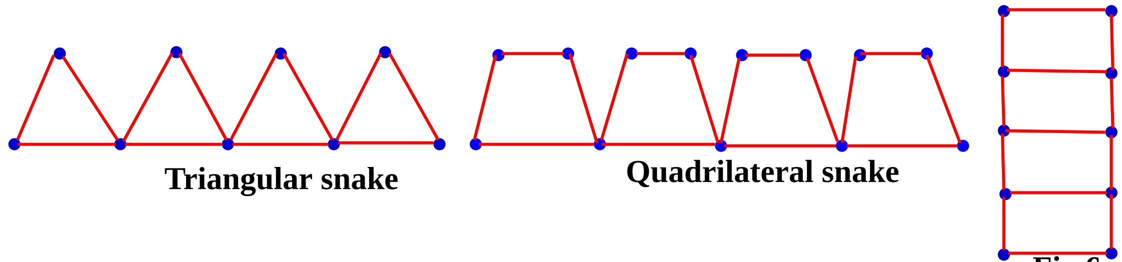


Fig 6

□

Definition 2.13. *The graph* $C_n + K_1$ *is called a* ***wheel*** *with n spokes and is denoted by* W_n .

Definition 2.14. ***Helm graph****, denoted by* H_n *is a graph obtained from a wheel by adjoining a pendent edge to each node of a cycle.*

Definition 2.15. *The* ***web graph*** $W_{n,r}$ *is a graph consisting of r concentric copies of the cycle graph* C_n *, with corresponding vertices connected by "spokes".*

Definition 2.16. *The* ***Mobius ladder*** M_n *is a cubic circulant graph with an even number n of vertices, formed from an n-cycle by adding edges (called "rungs") connecting opposite pairs of vertices in the cycle.*

Definition 2.17. *The graph* $C_n \odot K_1$ *is called a* ***crown****.*

Definition 2.18. *A* ***dragon*** *is a graph formed by joining an end vertex of a path* P_m *to a vertex of the cycle* C_n *. It is denoted as* $C_n@P_m$ *.*

Theorem 2.19. *Asteroidal number for some graphs is given below :*

1. $an\,(W_n) = \left\lfloor \frac{n}{2} \right\rfloor$

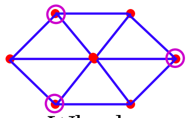
2. $an\,(W_{n,r}) = r \left\lfloor \frac{n}{2} \right\rfloor$

3. $an\,(H_n) = an\,(C_n \odot K_1) = n$

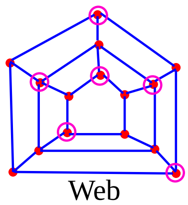
4. $an(M_n) = \begin{cases} \frac{n}{2} & if \quad n = 2k \ ; \ k \ contains \ an \ odd \ factor \\ \frac{n}{2} - 1 & otherwise \end{cases}$

5. $an\,(C_n@P_m) = \left\lfloor \frac{m}{2} \right\rfloor$

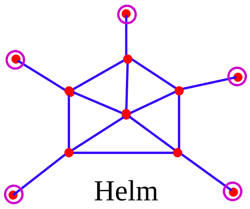
Proof. For a wheel W_n, alternate vertices of the cycle form a maximum asteroidal set and hence the cardinality is $\lfloor \frac{n}{2} \rfloor$. For a web graph $W_{n,r}$, alternate vertices in each cycle are chosen in such a way they are mutually non adjacent. For a helm graph H_n and crown graph, all the pendent vertices form a maximum asteroidal set and hence cardinality is n.



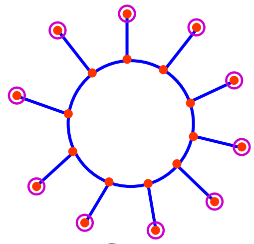
Wheel



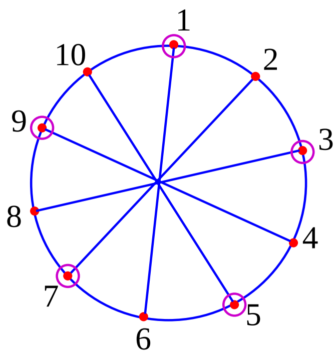
Web

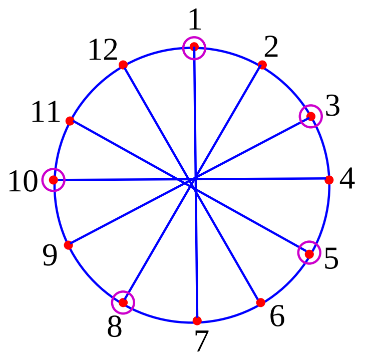


Helm

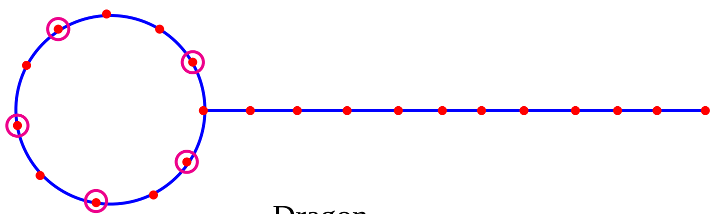


Crown





In Mobius Ladder graph M_n, when n has no odd factor, on choosing alternate vertices of the cycle, one vertex has to be left out since it is adjacent with an already chosen vertex. This problem doesn't arise if n has no odd factor and so alternate vertices form a maximum asteroidal set.



Dragon

Avoiding the vertex common to the path and the cycle, we choose alternate vertices on the cycle C_m for maximum asteroidal set. □

References

[1] Antoaneta Klobucar, Independent Sets and independent dominating sets in the strong product of paths and cycles *Mathematical Communications 10*, 2005,23-30.

[2] Corneil.D.G, Olariu.S and Stewart.L.K, Asteroidal triple free graphs *SIAM J. Discrete Math.*, 10(1997), pp.399-430.

[3] Ekkehard Kohler, Recognizing graphs without asteroidal triples, *J. Discrete Algorithms 2(2004)* pp.439-452.

[4] Haiko Muller, Asteroidal Sets in Graphs, Report submitted to Friedrich-Schiller University, Jena, on 3.2.1999.

[5] Hepzibai Jeyakumar, Studies in Graph theory-Asteroidal Sets in Graphs, Ph.D thesis, Manonmaniam Sundaranar University, Tirunelveli, India, February 2003.

[6] Lekkerkerker.C.G and Boland.J.C,Representation of a finite graph by a set of intervals on the real line, *Fund. Math.*, 51(1962), pp.45-64.

[7] Richard Hammack, Wilfried Imrich, Sandi Klavzar *Handbook of Product Graphs*, Second edition, CRC Press, 2011.

[8] Walter.J.R,Representations of chordal graphs as subtrees of a tree, *J. Graph Theory*, 2(1978), pp.265-267.

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