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Linear optimization on the intersection of two fuzzy relational inequalities defined with Yager family of t-norms

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ABSTRACT

In this paper, optimization of a linear objective function with fuzzy relational inequality constraints is investigated where the feasible region is formed as the intersection of two inequality fuzzy systems and Yager family of t-norms is considered as fuzzy composition. Yager family of t-norms is a parametric family of continuous nilpotent t-norms which is also one of the most frequently applied one. This family of t-norms is strictly increasing in its parameter and covers the whole spectrum of t-norms when the parameter is changed from zero to infinity. The resolution of the feasible region of the problem is firstly investigated

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when it is defined with max-Yager composition. Based on some theoretical results, conditions are derived for determining the feasibility. Moreover, in order to simplify the problem, some procedures are presented. It is shown that a lower bound is always attainable for the optimal objective value. Also, it is proved that the optimal solution of the problem is always resulted from the unique maximum solution and a minimal solution of the feasible region. A method is proposed to generate random feasible max-Yager fuzzy relational inequalities and an algorithm is presented to solve the problem. Finally, an example is described to illustrate these algorithms.

1 Introduction

In this paper, we study the following linear problem in which the constraints are formed as the intersection of two fuzzy systems of relational inequalities defined by Yager family of t-norms:

$$\begin{aligned} &\min\ Z = c^T x \\ &A\varphi x \leq b^1 \\ &D\varphi x \geq b^2 \\ &x \in [0,1]^n \end{aligned} \tag{1}$$

where $I_1 = \{1,2,..,m_1\}$, $I_2 = \{m_1+1, m_1+2,.., m_1+m_2\}$ and $J = \{1,2,..,n\}$. $A = (a_{ij})_{m_1 \times n}$ and $D = (d_{ij})_{m_2 \times n}$ are fuzzy matrices such that $0 \leq a_{ij} \leq 1$ ($\forall i \in I_1$ and $\forall j \in J$) and $0 \leq d_{ij} \leq 1$ ($\forall i \in I_2$ and $\forall j \in J$). $b^1 = (b_i^1)_{m_1 \times 1}$ is an m_1–dimensional fuzzy vector in $[0,1]^{m_1}$ (i.e., $_{0 \leq b_i^1 \leq 1}, \forall i \in I_1$) , $b^2 = (b_i^2)_{m_2 \times 1}$ is an m_2–dimensional fuzzy vector in $[0,1]^{m_2}$ (i.e., $_{0 \leq b_i^2 \leq 1}, \forall i \in I_2$), and c is a vector in R^n. Moreover, "φ " is the max-Yager composition, that is, $\varphi(x,y) = T_Y^p(x,y) = \max\left\{1 - ((1-x)^p + (1-y)^p)^{1/p}, 0\right\}$ in which $p > 0$.
By these notations, problem (1) can be also expressed as follows:

$$\begin{aligned} &\min\ Z = c^T x \\ &\max_{j \in J}\{T_Y^p(a_{ij}, x_j)\} \leq b_i^1\ ,\ i \in I_1 \\ &\max_{j \in J}\{T_Y^p(d_{ij}, x_j)\} \geq b_i^2\ ,\ i \in I_2 \\ &x \in [0,1]^n \end{aligned} \tag{2}$$

Especially, by setting $A = D$ and $b^1 = b^2$, the above problem is converted to max- Yager fuzzy relational equations. As mentioned, the family $\left\{T_Y^p\right\}$ is strictly increasing in p. It can be easily shown that Yager t-norm $T_Y^p(x,y)$ converges to the basic fuzzy intersection $\min\{x,y\}$ as p goes to infinity and converges to Drastic product t-norm [8] as p approaches zero. Also, it is interesting to note that $T_Y^1(x,y) = \max\{x+y-1, 0\}$, that is, the Yager t-norm is converted to Lukasiewicz t-norm if $p = 1$.
The theory of fuzzy relational equations (FRE) as a generalized version of Boolean relation equations was firstly proposed by Sanchez and applied in problems of the medical diagnosis [40]. Nowadays, it is well known that many issues associated with a body

knowledge can be treated as FRE problems [36]. In addition to the preceding applications, FRE theory has been applied in many fields, including fuzzy control, discrete dynamic systems, prediction of fuzzy systems, fuzzy decision making, fuzzy pattern recognition, fuzzy clustering, image compression and reconstruction, fuzzy information retrieval, and so on. Generally, when inference rules and their consequences are known, the problem of determining antecedents is reduced to solving an FRE [34]. We refer the reader to [26] in which the authors provided a good overview of FRE and classified basic FREs by investigating the relationship among operators used in the definition of fuzzy relational equations.

The solvability determination and the finding of solutions set are the primary (and the most fundamental) subject concerning with FRE problems. Di Nola et al. proved that the solution set of FRE (if it is nonempty) defined by continuous max-t-norm composition is often a non-convex set that is completely determined by one maximum solution and a finite number of minimal solutions [5]. This non-convexity property is one of two bottlenecks making major contribution to the increase of complexity in problems that are related to FRE, especially in the optimization problems subjected to a system of fuzzy relations. The other bottleneck is concerned with detecting the minimal solutions for FREs. Chen and Wang [2] presented an algorithm for obtaining the logical representation of all minimal solutions and deduced that a polynomial-time algorithm to find all minimal solutions of FRE (with max-min composition) may not exist. Also, Markovskii showed that solving max-product FRE is closely related to the covering problem which is an NP-hard problem [33]. In fact, the same result holds true for a more general t-norms instead of the minimum and product operators [2,3,29,30]. Over the last decades, the solvability of FRE defined with different max-t compositions have been investigated by many researchers [35,37,38,41,43,44,46,49,52]. Moreover, some researchers introduced and improved theoretical aspects and applications of fuzzy relational inequalities (FRI)[13,15,16,22,27,51]. Li and Yang [27] studied a FRI with addition-min composition and presented an algorithm to search for minimal solutions. They applied FRI to meet a data transmission mechanism in a BitTorrent-like Peer-to-Peer file sharing systems. Ghodousian and Khorram [13] focused on the algebraic structure of two fuzzy relational inequalities $A\varphi x \le b^1$and $D\varphi x \ge b^2$, and studied a mixed fuzzy system formed by the two preceding FRIs, where $\varphi$is an operator with (closed) convex solutions. Generally, if $\varphi$is an operator with closed convex solutions, the solutions set of $D\varphi x \ge b^2$ is determined by a finite number of maximal solutions as well as the same number of minimal ones. In particular, if $\varphi$is a continuous non-decreasing function (specially, a continuous t-norm), all maximal solutions overlap each other [13]. Guo et al. [15] investigated a kind of FRI problems and the relationship between minimal solutions and FRI paths. They also introduced some rules for reducing the problems and presented an algorithm for solving optimization problems with FRI constraints.

The problem of optimization subject to FRE and FRI is one of the most interesting and on-going research topic among the problems related to FRE and FRI theory [1,8,11-23,28,31,39,42,47,51]. Fang and Li [9] converted a linear optimization problem subjected to FRE constraints with max-min operation into an integer programming problem and solved it by branch and bound method using jump-tracking technique. In [24] an application of optimizing the linear objective with max-min composition was employed for the streaming media provider seeking a minimum cost while fulfilling the requirements assumed by a three-tier framework. Wu et al. [45] improved the method used by Fang and Li, by decreasing the search domain and presented a simplification process by three rules resulted from a necessary condition. Chang and Shieh [1] presented new theoretical results concerning the linear optimization problem constrained by fuzzy max–min relation equations. They improved an upper bound on the optimal objective value, some rules for simplifying the problem and proposed a rule for reducing the solution tree. The topic of the linear optimization problem was also investigated with max-product operation [11,18,32]. Loetamonphong and Fang defined two sub-problems by separating negative and non-negative coefficients in the objective function and then obtained the optimal solution by combining those of the two sub-problems [32]. The maximum solution of FRE is the optimum of the sub-problem having negative coefficients. Another sub-problem was converted into a binary programming problem and solved by branch and bound method. Also, in [18] and [11] some necessary conditions of the feasibility and simplification techniques were presented for solving FRE with max-product composition. Moreover, some generalizations of the linear optimization with respect to FRE have been studied with the replacement of max-min and max-product compositions with different fuzzy compositions such as max-average composition [21,47], max-star composition [14,23] and max-t-norm composition [19,28,42]. For example, Li and Fang [28] solved the linear optimization problem subjected to a system of sup-t equations by reducing it to a 0-1 integer optimization problem. In [19] a method was presented for solving linear optimization problems with the max-Archimedean t-norm fuzzy relation equation constraint. In [42], the authors solved the same problem whit continuous Archimedean t-norm and used the covering problem rather than the branch-and-bound methods for obtaining some optimal variables.

Recently, many interesting generalizations of the linear programming subject to a system of fuzzy relations have been introduced and developed based on composite operations used in FRE, fuzzy relations used in the definition of the constraints, some developments on the objective function of the problems and other ideas [6,10,16, 25,31,48]. For example, Wu et al. [48] represented an efficient method to optimize a linear fractional programming problem under FRE with max-Archimedean t-norm composition. Dempe and Ruziyeva [4] generalized the fuzzy linear optimization problem by considering fuzzy coefficients. Dubey et al. studied linear programming problems involving interval uncertainty modeled using intuitionistic fuzzy set [6]. The linear optimization of bipolar FRE was studied by some researchers where FRE defined with max-min composition [10] and max-Lukasiewicz composition [25,31]. In [31], the authors presented an algorithm without translating the original problem into a 0-1 integer linear problem.

The optimization problem subjected to various versions of FRI could be found in the

literature as well [12,13,15,16,22,50,51]. Yang [50] applied the pseudo-minimal index algorithm for solving the minimization of linear objective function subject to FRI with addition-min composition. Xiao et al. [51] introduced the latticized linear programming problem subject to max-product fuzzy relation inequalities with application in the optimization management model of wireless communication emission base stations. Ghodousian and Khorram [12] introduced a system of fuzzy relational inequalities with fuzzy constraints (FRI-FC) in which the constraints were defined with max-min composition. They used this fuzzy system to convincingly optimize the educational quality of a school (with minimum cost) to be selected by parents.
The remainder of the paper is organized as follows. In section 2, some preliminary notions and definitions and three necessary conditions for the feasibility of problem (1) are presented. In section 3, the feasible region of problem (1) is determined as a union of the finite number of closed convex intervals. Two simplification operations are introduced to accelerate the resolution of the problem. Moreover, a necessary and sufficient condition based on the simplification operations is presented to realize the feasibility of the problem. Problem (1) is resolved by optimization of the linear objective function considered in section 4. In addition, the existence of an optimal solution is proved if problem (1) is not empty. The preceding results are summarized as an algorithm and, finally in section 5 an example is described to illustrate. Additionally, in section 5, a method is proposed to generate feasible test problems for problem (1).

2. Basic properties of max-Yager FRI

This section describes the basic definitions and structural properties concerning problem (1) that are used throughout the paper. For the sake of simplicity, let $S_{T_Y^p}(A,b^1)$ and $S_{T_Y^p}(D,b^2)$ denote the feasible solutions sets of inequalities $A\varphi x \le b^1$ and $D\varphi x \ge b^2$, respectively, that is, $S_{T_Y^p}(A,b^1) = \left\{x \in [0,1]^n \ : \ A\varphi x \le b^1\right\}$ and $S_{T_Y^p}(D,b^2) = \left\{x \in [0,1]^n \ : \ D\varphi x \ge b^2\right\}$. Also, let $S_{T_Y^p}(A,D,b^1,b^2)$ denote the feasible solutions set of problem (1). Based on the foregoing notations, it is clear that $S_{T_Y^p}(A,D,b^1,b^2) = S_{T_Y^p}(A,b^1) \bigcap S_{T_Y^p}(D,b^2)$.

Definition 1. For each $i \in I_1$ and each $j \in J$, we define $S_{T_Y^p}(a_{ij},b_i^1) = \left\{x \in [0,1] \ : \ T_Y^p(a_{ij},x) \le b_i^1\right\}$. Similarly, for each $i \in I_2$ and each $j \in J$, $S_{T_Y^p}(d_{ij},b_i^2) = \left\{x \in [0,1] \ : \ T_Y^p(d_{ij},x) \ge b_i^2\right\}$. Furthermore, the notations
$J_i^1 = \left\{j \in J \ : \ S_{T_Y^p}(a_{ij},b_i^1) \ne \emptyset\right\}$, $\forall i \in I_1$, and $J_i^2 = \left\{j \in J \ : \ S_{T_Y^p}(d_{ij},b_i^2) \ne \emptyset\right\}$, $\forall i \in I_2$, are used in the text.

Remark 1. From the least-upper-bound property of R, it is clear that $\inf_{x\in[0,1]} \left\{S_{T_Y^p}(a_{ij},b_i^1)\right\}$ and $\sup_{x\in[0,1]} \left\{S_{T_Y^p}(a_{ij},b_i^1)\right\}$ exist, if $S_{T_Y^p}(a_{ij},b_i^1) \ne \emptyset$. Moreover, since T_Y^p is a t-norm, its monotonicity property implies that $S_{T_Y^p}(a_{ij},b_i^1)$ is actually a connected subset of $[0,1]$. Additionally, due to the continuity of T_Y^p, we must have

$\inf_{x\in[0,1]}\left\{S_{T_Y^p}(a_{ij},b_i^1)\right\} = \min_{x\in[0,1]}\left\{S_{T_Y^p}(a_{ij},b_i^1)\right\}$ and $\sup_{x\in[0,1]}\left\{S_{T_Y^p}(a_{ij},b_i^1)\right\} = \max_{x\in[0,1]}\left\{S_{T_Y^p}(a_{ij},b_i^1)\right\}$. Therefore,
$S_{T_Y^p}(a_{ij},b_i^1) = \left[\min_{x\in[0,1]}\left\{S_{T_Y^p}(a_{ij},b_i^1)\right\}, \max_{x\in[0,1]}\left\{S_{T_Y^p}(a_{ij},b_i^1)\right\}\right]$, i.e., $S_{T_Y^p}(a_{ij},b_i^1)$ is a closed subinterval of $[0,1]$. By the similar argument, if $S_{T_Y^p}(d_{ij},b_i^2) \neq \emptyset$, then we have $S_{T_Y^p}(d_{ij},b_i^2) = \left[\min_{x\in[0,1]}\left\{S_{T_Y^p}(d_{ij},b_i^2)\right\}, \max_{x\in[0,1]}\left\{S_{T_Y^p}(d_{ij},b_i^2)\right\}\right] \subseteq [0,1]$.

From Definition 1 and Remark 1, the following two corollaries are resulted.

Corollary 1. For each $i \in I_1$ and each $j \in J$, $S_{T_Y^p(a_{ij},b_i^1)\neq\emptyset}$. Also, $S_{T_Y^p}(a_{ij},b_i^1) = \left[0, \max_{x\in[0,1]}\left\{S_{T_Y^p}(a_{ij},b_i^1)\right\}\right]$.

Proof. Since $T_Y^p(a_{ij},0) = 0$, we have $T_Y^p(a_{ij},0) \leq b_i^1$, $\forall i \in I_1$ and $\forall j \in J$. Therefore, $0 \in S_{T_Y^p}(a_{ij},b_i^1)$ and then $\min_{x\in[0,1]}\left\{S_{T_Y^p}(a_{ij},b_i^1)\right\} = 0$, $\forall i \in I_1$ and $\forall j \in J$. Now, by noting Remark 1 we also have, $S_{T_Y^p}(a_{ij},b_i^1) = \left[0, \max_{x\in[0,1]}\left\{S_{T_Y^p}(a_{ij},b_i^1)\right\}\right]$, $\forall i \in I_1$ and $\forall j \in J$. This completes the proof.

Corollary 2. If $S_{T_Y^p(d_{ij},b_i^2)\neq\emptyset}$ for some $i \in I_2$ and $j \in J$, then $S_{T_Y^p}(d_{ij},b_i^2) = \left[\min_{x\in[0,1]}\left\{S_{T_Y^p}(d_{ij},b_i^2)\right\}, 1\right]$.

Proof. Noting Remark 1, it is sufficient to show that $1 \in S_{T_Y^p}(d_{ij},b_i^2)$. Suppose that $S_{T_Y^p}(d_{ij},b_i^2) \neq \emptyset$. Therefore, there exists some $x \in [0,1]$ such that $T_Y^p(d_{ij},x) \geq b_i^2$. Now, the monotonicity property of T_Y^p implies $T_Y^p(d_{ij},1) \geq T_Y^p(d_{ij},x) \geq b_i^2$ that means $1 \in S_{T_Y^p}(d_{ij},b_i^2)$.

Remark 2. Corollary 1 together with Definition 1 implies $J_i^1 = J$, $\forall i \in I_1$.

Definition 2. For each $i \in I_1$ and each $j \in J$, we define

$$U_{ij} = \begin{cases} 1 & a_{ij} < b_i^1 \\ 1 - \sqrt[p]{(1-b_i^1)^p - (1-a_{ij})^p} & a_{ij} \geq b_i^1 \end{cases}$$

Also, for each $i \in I_2$ and each $j \in J$, we set

$$L_{ij} = \begin{cases} +\infty & d_{ij} < b_i^2 \\ 0 & b_i^2 = 0\ ,\ d_{ij} \geq b_i^2 \\ 1 - \sqrt[p]{(1-b_i^2)^p - (1-d_{ij})^p} & b_i^2 \neq 0\ ,\ d_{ij} \geq b_i^2 \end{cases}$$

Remark 3. From Definition 2, if $a_{ij} = b_i^1$, then $U_{ij} = 1$. Also, we have $L_{ij} = 1$, if $d_{ij} = b_i^2 \neq 0$.

Lemma 1 below shows that U_{ij} and L_{ij} stated in Definition 2, determine the maximum and minimum solutions of sets $S_{T_Y^p}(a_{ij},b_i^1)$ $(i \in I_1)$ and $S_{T_Y^p}(d_{ij},b_i^2)(i \in I_2)$, respectively.

Lemma 1. (a) $U_{ij} = \max_{x\in[0,1]} \left\{S_{T_Y^p}(a_{ij},b_i^1)\right\}$, $\forall i \in I_1$ and $\forall j \in J$. (b) If $S_{T_Y^p}(d_{ij},b_i^2)\neq\emptyset$ for some $i \in I_2$ and $j \in J$, then $L_{ij} = \min_{x\in[0,1]} \left\{S_{T_Y^p}(d_{ij},b_i^2)\right\}$.

Proof. (a) Let $i \in I_1$,$j \in J$ and $x \in S_{T_Y^p}(a_{ij},b_i^1)$. Firstly, suppose that $a_{ij} \le b_i^1$. In this case, $U_{ij}=1$ from Definition 2 and Remark 3. Since $x\in S_{T_Y^p}(a_{ij},b_i^1)$, then $x\in[0,1]$and therefore $x \le U_{ij}$. Hence, it is sufficient to show that $U_{ij} \in S_{T_Y^p}(a_{ij},b_i^1)$. But, the identity law of T_Y^p implies $T_Y^p(a_{ij},U_{ij}) = T_Y^p(a_{ij},1) = a_{ij} \le b_i^1$. Therefore, $U_{ij} \in S_{T_Y^p}(a_{ij},b_i^1)$ and $x \le U_{ij}$ ($\forall x\in S_{T_Y^p}(a_{ij},b_i^1)$) that mean $U_{ij}=\max_{x\in[0,1]}\left\{S_{T_Y^p}(a_{ij},b_i^1)\right\}$. Otherwise, suppose that $a_{ij} > b_i^1$. In this case, $U_{ij} = 1-\sqrt[p]{(1-b_i^1)^p-(1-a_{ij})^p}$. Since $T_Y^p(a_{ij},U_{ij}) = b_i^1$ and T_Y^p has the monotonicity property, we have $U_{ij}\in S_{T_Y^p}(a_{ij},b_i^1)$ and $T_Y^p(a_{ij},x) \ge b_i^1$ for each $x > U_{ij}$. Therefore, U_{ij} must be the maximum of the set $S_{T_Y^p}(a_{ij},b_i^1)$.

(b) Let $i \in I_2$,$j \in J$ and $x \in S_{T_Y^p}(d_{ij},b_i^2)$. Since $S_{T_Y^p}(d_{ij},b_i^2)\neq\emptyset$, then we must have $d_{ij} \ge b_i^2$ (because, if $d_{ij} < b_i^2$, then $T_Y^p(d_{ij},x) \le T_Y^p(d_{ij},1) = d_{ij} < b_i^2$, $\forall x \in [0,1]$). If $b_i^2 = 0$,then $L_{ij} = 0$ from Definition 2 and Remark 3. Therefore, $T_Y^p(d_{ij},L_{ij}) = T_Y^p(d_{ij},0) = 0 = b_i^2$and obviously $L_{ij} = 0 \le x$, $\forall x\in S_{T_Y^p}(d_{ij},b_i^2)$. Consequently, $L_{ij} = \min_{x\in[0,1]}\left\{S_{T_Y^p}(d_{ij},b_i^2)\right\}$. Otherwise, suppose that $b_i^2 \neq 0$. In this case, we have $L_{ij} = 1-\sqrt[p]{(1-b_i^2)^p-(1-d_{ij})^p}$. Again, since $T_Y^p(d_{ij},L_{ij}) = b_i^2$ and T_Y^p has the monotonicity property, we have $L_{ij}\in S_{T_Y^p}(d_{ij},b_i^2)$ and $T_Y^p(d_{ij},x) \le b_i^2$ for each $x < L_{ij}$. Therefore, L_{ij} must be the minimum of the set $S_{T_Y^p}(d_{ij},b_i^2)$. This completes the proof.

Lemma 1 together with the corollaries 1 and 2 results in the following consequence.

Corollary 3. (a) For each $i \in I_1$ and $j \in J$, $S_{T_Y^p}(a_{ij},b_i^1) = [0,U_{ij}]$. (b) If $S_{T_Y^p}(d_{ij},b_i^2)\neq\emptyset$ for some $i \in I_2$ and $j \in J$, then $S_{T_Y^p}(d_{ij},b_i^2) = [L_{ij},1]$.

Definition 3. For each $i \in I_1$, let $S_{T_Y^p}(a_i,b_i^1) = \left\{x \in [0,1]^n \ : \ \max_{j=1}^n\left\{T_Y^p(a_{ij},x_j)\right\} \le b_i^1\right\}$. Similarly, for each $i \in I_2$, we define $S_{T_Y^p}(d_i,b_i^2) = \left\{x \in [0,1]^n \ : \ \max_{j=1}^n\left\{T_Y^p(d_{ij},x_j)\right\} \ge b_i^2\right\}$.

According to Definition 3 and the constraints stated in (2), sets $S_{T_Y^p}(a_i,b_i^1)$ and $S_{T_Y^p}(d_i,b_i^2)$ actually denote the feasible solutions sets of the i'th inequality $\max_{j\in J}\{T_Y^p(a_{ij},x_j)\} \le b_i^1$ $(i \in I_1)$ and $\max_{j\in J}\{T_Y^p(d_{ij},x_j)\} \ge b_i^2$ $(i \in I_2)$ of problem (1), respectively. Based on (2) and

Definitions 1 and 3, it can be easily concluded that for a fixed $i \in I_1$, $S_{T_Y^p}(a_i, b_i^1) \neq \emptyset$ iff $S_{T_Y^p}(a_{ij}, b_i^1) \neq \emptyset$, $\forall j \in J$. On the other hand, by Corollary 1 we know that $S_{T_Y^p}(a_{ij}, b_i^1) \neq \emptyset$, $\forall i \in I_1$ and $\forall j \in J$. As a result, $S_{T_Y^p}(a_i, b_i^1) \neq \emptyset$ for each $i \in I_1$. However, in contrast to $S_{T_Y^p}(a_i, b_i^1)$, set $S_{T_Y^p}(d_i, b_i^2)$ may be empty. Actually, for a fixed $i \in I_2$, $S_{T_Y^p}(d_i, b_i^2)$ is nonempty if and only if $S_{T_Y^p}(d_{ij}, b_i^2)$ is nonempty for at least some $j \in J$. Additionally, for each $i \in I_2$ and $j \in J$ we have $S_{T_Y^p}(d_{ij}, b_i^2) \neq \emptyset$ if and only if $d_{ij} \geq b_i^2$. These results have been summarized in the following lemma. Part (b) of the lemma gives a necessary and sufficient condition for the feasibility of set $S_{T_Y^p}(d_i, b_i^2)$ ($\forall i \in I_2$). It is to be noted that the lemma 2 (part (b)) also provides a necessary condition for problem (1).

Lemma 2. (a) $S_{T_Y^p}(a_i, b_i^1) \neq \emptyset$, $\forall i \in I_1$. (b) For a fixed $i \in I_2$, $S_{T_Y^p}(d_i, b_i^2) \neq \emptyset$ iff $\bigcup_{j=1}^{n} S_{T_Y^p}(d_{ij}, b_i^2) \neq \emptyset$. Additionally, for each $i \in I_2$ and $j \in J$, $S_{T_Y^p}(d_{ij}, b_i^2) \neq \emptyset$ iff $d_{ij} \geq b_i^2$.

Definition 4. For each $i \in I_2$ and $j \in J_i^2$, we define $S_{T_Y^p}(d_i, b_i^2, j) = [0,1] \times ... \times [0,1] \times [L_{ij}, 1] \times [0,1] \times ... \times [0,1]$, where $[L_{ij}, 1]$ is in the j'th position.

In the following lemma, the feasible solutions set of the i'th fuzzy relational inequality is characterized.

Lemma 3. (a) $S_{T_Y^p}(a_i, b_i^1) = [0, U_{i1}] \times [0, U_{i2}] \times ... \times [0, U_{in}]$, $\forall i \in I_1$. (b) $S_{T_Y^p}(d_i, b_i^2) = \bigcup_{j \in J_i^2} S_{T_Y^p}(d_i, b_i^2, j)$, $\forall i \in I_2$.

Proof. (a) Fix $i \in I_1$ and let $x \in S_{T_Y^p}(a_i, b_i^1)$. By Definition 3, $x_j \in [0,1]$ for each $j \in J$, and $\max_{j=1}^{n}\left\{T_Y^p(a_{ij}, x_j)\right\} \leq b_i^1$. The latter inequality implies $T_Y^p(a_{ij}, x_j) \leq b_i^1$, $\forall j \in J$. Thus, by Definition 1 and Corollary 3 we have $x_j \in S_{T_Y^p}(a_{ij}, b_i^1) = [0, U_{ij}]$, $\forall j \in J$, which necessitates $x \in [0, U_{i1}] \times [0, U_{i2}] \times ... \times [0, U_{in}]$. Conversely, suppose that $x \in [0, U_{i1}] \times [0, U_{i2}] \times ... \times [0, U_{in}]$. Then, by Corollary 3, $x_j \in [0, U_{ij}] = S_{T_Y^p}(a_{ij}, b_i^1)$, $\forall j \in J$, which implies $x_j \in [0,1]$ and $T_Y^p(a_{ij}, x_j) \leq b_i^1$, $\forall j \in J$. Thus, $x \in [0,1]^n$ and $\max_{j=1}^{n}\left\{T_Y^p(a_{ij}, x_j)\right\} \leq b_i^1$. Therefore, by Definition 3, $x \in S_{T_Y^p}(a_i, b_i^1)$.

(b) Fix $i \in I_2$ and let $x \in S_{T_Y^p}(d_i, b_i^2)$. By Definition 3, $x \in [0,1]^n$ and $\max_{j=1}^{n}\left\{T_Y^p(d_{ij}, x_j)\right\} \geq b_i^2$. Then there exists some $j_0 \in J_i^2$ such that $T_Y^p(d_{ij_0}, x_{j_0}) \geq b_i^2$. Therefore, from Definition 1 and Corollary 3, it is concluded that $x_{j_0} \in S_{T_Y^p}(d_{ij_0}, b_i^2) = [L_{ij_0}, 1]$. Now, from Definition 4 we have $x \in S_{T_Y^p}(d_i, b_i^2, j_0)$. Thus, $x \in \bigcup_{j \in J_i^2} S_{T_Y^p}(d_i, b_i^2, j)$. Conversely, suppose that $x \in \bigcup_{j \in J_i^2} S_{T_Y^p}(d_i, b_i^2, j)$. Then there exists some $j_0 \in J_i^2$ such that $x \in S_{T_Y^p}(d_i, b_i^2, j_0)$. Therefore, by Definition 4, $x \in [0,1]^n$ and $x_{j_0} \in S_{T_Y^p}(d_{ij_0}, b_i^2) = [L_{ij_0}, 1]$, which implies $T_Y^p(d_{ij_0}, x_{j_0}) \geq b_i^2$. Thus, $x \in [0,1]^n$ and $\max_{j=1}^{n}\left\{T_Y^p(d_{ij}, x_j)\right\} \geq b_i^2$, which requires $x \in S_{T_Y^p}(d_i, b_i^2)$.

Definition 5. Let $\overline{X}(i) = [U_{i1}, U_{i2}, ..., U_{in}]$, $\forall i \in I_1$. Also, let $\underline{X}(i,j) = [\underline{X}(i,j)_1, \underline{X}(i,j)_2, ..., \underline{X}(i,j)_n]$,

$\forall i \in I_2$ and $\forall j \in J_i^2$, where

$$\underline{X}(i,j)_k = \begin{cases} L_{ij} & k = j \\ 0 & k \neq j \end{cases}$$

Lemma 3 together with Definitions 4 and 5, results in Theorem 1, which completely determines the feasible region for the i'th relational inequality.

Theorem 1. (a) $S_{T_Y^p}(a_i, b_i^1) = [\mathbf{0}, \overline{X}(i)]$, $\forall i \in I_1$. (b) $S_{T_Y^p}(d_i, b_i^2) = \bigcup_{j \in J_i^2} [\underline{X}(i,j), \mathbf{1}]$, $\forall i \in I_2$, where $\mathbf{0}$ and $\mathbf{1}$ are n–dimensional vectors with each component equal to zero and one, respectively.

Theorem 1 gives the upper and lower bounds for the feasible solutions set of the i'th relational inequality. Actually, for each $i \in I_1$, vectors $\mathbf{0}$ and $\overline{X}(i)$ are the unique minimum and the unique maximum of set $S_{T_Y^p}(a_i, b_i^1)$. In addition, for each $i \in I_2$, set $S_{T_Y^p}(d_i, b_i^2)$ has the unique maximum (i.e., vector $\mathbf{1}$), but the finite number of minimal solutions $\underline{X}(i,j)(\forall j \in J_i^2)$. Furthermore, part (b) of Theorem 1 presents another feasible necessary condition for problem (1) as stated in the following corollary.

Corollary 4. If $S_{T_Y^p}(A, D, b^1, b^2) \neq \emptyset$, then $\mathbf{1} \in S_{T_Y^p}(d_i, b_i^2)$, $\forall i \in I_2$ (i.e., $\mathbf{1} \in \bigcap_{i \in I_2} S_{T_Y^p}(d_i, b_i^2) = S_{T_Y^p}(D, b^2)$).

Proof. Let $S_{T_Y^p}(A, D, b^1, b^2) \neq \emptyset$. Then, $S_{T_Y^p}(D, b^2) \neq \emptyset$, and therefore, $S_{T_Y^p}(d_i, b_i^2) \neq \emptyset$, $\forall i \in I_2$. Now, Theorem 1 (part (b)) implies $\mathbf{1} \in S_{T_Y^p}(d_i, b_i^2)$, $\forall i \in I_2$.

Lemma 4 describes the shape of the feasible solutions set for the fuzzy relational inequalities $A\varphi x \leq b^1$ and $D\varphi x \geq b^2$, separately.

Lemma 4. (a) $S_{T_Y^p}(A, b^1) = \bigcap_{i \in I_1} [0, U_{i1}] \times \bigcap_{i \in I_1} [0, U_{i2}] \times \ldots \times \bigcap_{i \in I_1} [0, U_{in}]$. (b) $S_{T_Y^p}(D, b^2) = \bigcap_{i \in I_2} \bigcup_{j \in J_i^2} S_{T_Y^p}(d_i, b_i^2, j)$.

Proof. The proof is obtained from Lemma 3 and equations $S_{T_Y^p}(A, b^1) = \bigcap_{i \in I_1} S_{T_Y^p}(a_i, b_i^1)$ and $S_{T_Y^p}(D, b^2) = \bigcap_{i \in I_2} S_{T_Y^p}(d_i, b_i^2)$.

Definition 6. Let $e : I_2 \to J_i^2$ so that $e(i) = j \in J_i^2$, $\forall i \in I_2$, and let E_D be the set of all vectors e. For the sake of convenience, we represent each $e \in E_D$ as an m_2–dimensional vector $e = [j_1, j_2, ..., j_{m_2}]$ in which $j_k = e(k)$, $k = 1, 2, ..., m_2$.

Definition 7. Let $e = [j_1, j_2, ..., j_{m_2}] \in E_D$. We define $\overline{X} = \min_{i \in I_1} \left\{ \overline{X}(i) \right\}$, that is, $\overline{X}_j = \min_{i \in I_1} \left\{ \overline{X}(i)_j \right\}$, $\forall j \in J$. Moreover, let $\underline{X}(e) = [\underline{X}(e)_1, \underline{X}(e)_2, ..., \underline{X}(e)_n]$, where $\underline{X}(e)_j = \max_{i \in I_2} \left\{ \underline{X}(i, e(i))_j \right\} = \max_{i \in I_2} \left\{ \underline{X}(i, j_i)_j \right\}$, $\forall j \in J$.

Based on Theorem 1 and the above definition, we have the following theorem characterizing the feasible regions of the general inequalities $A\varphi x \leq b^1$ and $D\varphi x \geq b^2$ in the most familiar way.

Theorem 2. (a) $S_{T_Y^p}(A, b^1) = [\mathbf{0}, \overline{X}]$, $\forall i \in I_1$. (b) $S_{T_Y^p}(D, b^2) = \bigcup_{e \in E_D} [\underline{X}(e), \mathbf{1}]$.

Proof. (a) By considering Definitions 5 and 7, for each $j \in J$ we have $\bigcap_{i \in I_1} [0, U_{ij}] =$

$\left[0, \min_{i\in I_1}\left\{U_{ij}\right\}\right] = \left[0, \min_{i\in I_1}\left\{\overline{X}(i)_j\right\}\right] = [0, \overline{X}_j]$. Therefore, part (a) of lemma 4 can be rewritten as $S_{T_Y^p}(A,b^1) = \left[0,\overline{X}_1\right]\times\left[0,\overline{X}_2\right]\times ... \times\left[0,\overline{X}_n\right] = [\mathbf{0},\overline{X}]$, where $\mathbf{0}$ is the zero vector. This proves part (a).
(b) From part (b) of lemma 4, $S_{T_Y^p}(D,b^2) = \bigcap_{i\in I_2}\bigcup_{j\in J_i^2} S_{T_Y^p}(d_i, b_i^2, j)$. Now, by Definitions 4 and 5, we have

$$S_{T_Y^p}(D,b^2) = \bigcap_{i\in I_2}\bigcup_{j\in J_i^2}[0,1]\times ... \times[0,1]\times[L_{ij},1]\times[0,1]\times ... \times[0,1] = \bigcap_{i\in I_2}\bigcup_{j\in J_i^2}[\underline{X}(i,j),\mathbf{1}].$$

Therefore, from Definitions 6 and 7 we have
$S_{T_Y^p}(D,b^2) = \bigcap_{i\in I_2}\bigcup_{e\in E_D}[\underline{X}(i,e(i)),\mathbf{1}] = \bigcup_{e\in E_D}\bigcap_{i\in I_2}[\underline{X}(i,e(i)),\mathbf{1}] = \bigcup_{e\in E_D}\left[\max_{i\in I_2}\{\underline{X}(i,e(i))\},\mathbf{1}\right] = \bigcup_{e\in E_D}[\underline{X}(e),\mathbf{1}]$where, the last equality is resulted from Definition 7. This completes the proof.
Corollary 5. Assume that $S_{T_Y^p}(A,D,b^1,b^2) \neq \emptyset$. Then, there exists some $e \in E_D$ such that $[\mathbf{0},\overline{X}]\bigcap[\underline{X}(e),\mathbf{1}] \neq \emptyset$.

Corollary 6. Assume that $S_{T_Y^p}(A,D,b^1,b^2) \neq \emptyset$. Then, $\overline{X} \in S_{T_Y^p}(D,b^2)$.
Proof. Let $S_{T_Y^p}(A,D,b^1,b^2) \neq \emptyset$. By Corollary 5, $[\mathbf{0},\overline{X}]\bigcap[\underline{X}(e'),\mathbf{1}] \neq \emptyset$ for some $e' \in E_D$. Thus, $\overline{X} \in [\underline{X}(e'),\mathbf{1}]$ that means $\overline{X} \in \bigcup_{e\in E_D}[\underline{X}(e),\mathbf{1}]$. Therefore, from Theorem 2 (part (b)), $\overline{X} \in S_{T_Y^p}(D,b^2)$.

3. Feasible solutions set and simplification operations

In this section, two operations are presented to simplify the matrices A and D, and a necessary and sufficient condition is derived to determine the feasibility of the main problem. At first, we give a theorem in which the bounds of the feasible solutions set of problem (1) are attained. As is shown in the following theorem, by using these bounds, the feasible region is completely found.
Theorem 3. Suppose that $S_{T_Y^p}(A,D,b^1,b^2) \neq \emptyset$. Then $S_{T_Y^p}(A,D,b^1,b^2) = \bigcup_{e\in E_D}[\underline{X}(e),\overline{X}]$.
Proof. Since $S_{T_Y^p}(A,D,b^1,b^2) = S_{T_Y^p}(A,b^1)\bigcap S_{T_Y^p}(D,b^2)$, then by Theorem 2, $S_{T_Y^p}(A,D,b^1,b^2) = [\mathbf{0},\overline{X}]\bigcap(\bigcup_{e\in E_D}[\underline{X}(e),\mathbf{1}])$ and the statement is established.
In practice, there are often some components of matrices A and D, which have no effect on the solutions to problem (1). Therefore, we can simplify the problem by changing the values of these components to zeros. We refer the interesting reader to [13] where a brief review of such these processes is given. Here, we present two simplification techniques based on the Yager family of t-norms.
Definition 8. If a value changing in an element, say a_{ij}, of a given fuzzy relation matrix A has no effect on the solutions of problem (1), this value changing is said to be an equivalence operation.
Corollary 7. Suppose that $i \in I_1$ and $T_Y^p(a_{ij_0}, x_{j_0}) < b_i, \forall x \in S_{T_Y^p}(A,b^1)$. In this case, it

is obvious that $\max_{j=1}^{n}\left\{T_Y^p(a_{ij},x_j)\right\} \le b_i^1$ is equivalent to $\max_{\substack{j=1 \\ j\ne j_0}}^{n}\left\{T_Y^p(a_{ij},x_j)\right\} \le b_i^1$, that is, "resetting a_{ij_0} to zero" has no effect on the solutions of problem (1) (since component a_{ij_0} only appears in the i'th constraint of problem (1)). Therefore, if $T_Y^p(a_{ij_0},x_{j_0}) < b_i^1, \forall x \in S_{T_Y^p}(A,b^1)$, then "resetting a_{ij_0} to zero" is an equivalence operation.

Lemma 5 (simplification of matrix A). Suppose that matrix $\tilde{A} = (\tilde{a}_{ij})_{m_1\times n}$ is resulted from matrix A as follows:

$$\tilde{a}_{ij} = \begin{cases} 0 & a_{ij} < b_i^1 \\ a_{ij} & a_{ij} \ge b_i^1 \end{cases}$$

for each $i \in I_1$ and $j \in J$. Then, $S_{T_Y^p}(A,b^1) = S_{T_Y^p}(\tilde{A},b^1)$.

Proof. From corollary 7, it is sufficient to show that $T_Y^p(a_{ij_0},x_{j_0}) < b_i^1, \forall x \in S_{T_Y^p}(A,b^1)$. But, from the monotonicity and identity laws of T_Y^p, we have $T_Y^p(a_{ij_0},x_{j_0}) \le T_Y^p(a_{ij_0},1) = a_{ij_0} < b_i^1$, $\forall x_{j_0} \in [0,1]$. Thus, $T_Y^p(a_{ij_0},x_{j_0}) < b_i^1, \forall x \in S_{T_Y^p}(A,b^1)$.

Lemma 5 gives a condition to reduce the matrix A. In this lemma, $\tilde{A}$ denote the simplified matrix resulted from A after applying the simplification process. Based on this notation, we define $\tilde{J}_i^1 = \left\{j \in J \ : \ S_{T_Y^p}(\tilde{a}_{ij},b_i^1) \ne \emptyset\right\}(\forall i \in I_1)$ where $\tilde{a}_{ij}$ denotes (i,j)'th component of matrix $\tilde{A}$. So, from Corollary 1 and Remark 2, it is clear that $\tilde{J}_i^1 = J_i^1 = J$. Moreover, since $S_{T_Y^p}(A,D,b^1,b^2) = S_{T_Y^p}(A,b^1) \bigcap S_{T_Y^p}(D,b^2)$, from Lemma 5 we can also conclude that $S_{T_Y^p}(A,D,b^1,b^2) = S_{T_Y^p}(\tilde{A},D,b^1,b^2)$.

By considering a fixed vector $e \in E_D$ in Theorem 3, interval $[\underline{X}(e),\overline{X}]$ is meaningful iff $\underline{X}(e) \le \overline{X}$. Therefore, by deleting infeasible intervals $[\underline{X}(e),\overline{X}]$ in which $\underline{X}(e) \nleq \overline{X}$, the feasible solutions set of problem (1) stays unchanged. In order to remove such infeasible intervals from the feasible region, it is sufficient to neglect vectors e generating infeasible solutions $\underline{X}(e)$ (i.e., solutions $\underline{X}(e)$ such that $\underline{X}(e) \nleq \overline{X}$). These considerations lead us to introduce a new set $E_D' = \left\{e \in E_D \ : \ \underline{X}(e) \le \overline{X}\right\}$ to strengthen Theorem 3. By this new set, Theorem 3 can be written as $S_{T_Y^p}(A,D,b^1,b^2) = \bigcup_{e\in E_D'}[\underline{X}(e),\overline{X}]$, if $S_{T_Y^p}(A,D,b^1,b^2) \ne \emptyset$.

Lemma 6. Let $I_j(e) = \{i \in I_2 \ : \ e(i) = j\}$ and $J(e) = \left\{j \in J \ : \ I_j(e) \ne \emptyset\right\}$, $\forall e \in E_D$. Then,

$$\underline{X}(e)_j = \begin{cases} \max\limits_{i\in I_j(e)}\left\{L_{i\,e(i)}\right\} & j \in J(e) \\ 0 & j \notin J(e) \end{cases}$$

Proof. From Definition 7, $\underline{X}(e)_j = \max\limits_{i\in I_2}\left\{\underline{X}(i,e(i))_j\right\}$, $\forall j \in J$. On the other hand, by Definition 5, we have

$$\underline{X}(i,e(i))_j = \begin{cases} L_{i\,e(i)} & j = e(i) \\ 0 & j \ne e(i) \end{cases}$$

Now, the result follows by combining these two equations.
Corollary 8. $e \in E'_D$ if and only if $L_{i\,e(i)} \le \overline{X}_{e(i)}$, $\forall i \in I_2$.
Proof. Firstly, from the definition of set E'_D, we note that $e \in E'_D$ if and only if $\underline{X}(e)_j \le \overline{X}_j$, $\forall j \in J$. Now, let $e \in E'_D$ and by contradiction, suppose that $L_{i_0\,e(i_0)} > \overline{X}_{e(i_0)}$ for some $i_0 \in I_2$. So, by setting $e(i_0) = j_0$, we have $j_0 \in J(e)$, and therefore lemma 6 implies $\underline{X}(e)_{j_0} = \max_{i \in I_{j_0}(e)} \left\{ L_{i\,e(i)} \right\} \ge L_{i_0\,e(i_0)} > \overline{X}_{e(i_0)}$. Thus, $\underline{X}(e)_{j_0} > \overline{X}_{e(i_0)}$ that contradicts $e \in E'_D$. The converse statement is easily proved by Lemma 6.
As mentioned before, to accelerate identification of the meaningful solutions $\underline{X}(e)$, we reduce our search to set E'_D instead of set E_D. As a result from Corollary 8, we can confine set J_i^2 by removing each $j \in J_i^2$ such that $L_{ij} > \overline{X}_j$ before selecting the vectors e to construct solutions $\underline{X}(e)$. However, lemma 7 below shows that this purpose can be accomplished by resetting some components of matrix D to zeros. Before formally presenting the lemma, some useful notations are introduced.
Definition 9 (simplification of matrix D). Let $\tilde{D} = (\tilde{d}_{ij})_{m_2 \times n}$ denote a matrix resulted from D as follows:

$$\tilde{d}_{ij} = \begin{cases} 0 & j \in J_i^2 \text{ and } L_{ij} > \overline{X}_j \\ d_{ij} & \text{otherwise} \end{cases}$$

Also, similar to Definition 1, assume that $\tilde{J}_i^2 = \left\{ j \in J \;:\; S_{T_Y^p}(\tilde{d}_{ij}, b_i^2) \ne \emptyset \right\}$ $(\forall i \in I_2)$ where $\tilde{d}_{ij}$ denotes (i,j)'th components of matrix $\tilde{D}$.
According to the above definition, it is easy to verify that $\tilde{J}_i^2 \subseteq J_i^2$, $\forall i \in I_2$. Furthermore, the following lemma demonstrates that the infeasible solutions $\underline{X}(e)$ are not generated, if we only consider those vectors e generated by the components of the matrix $\tilde{D}$, or equivalently vectors e generated based on the set $\tilde{J}_i^2$ instead of J_i^2.

Lemma 7. $E_{\tilde{D}} = E'_D$, where $E_{\tilde{D}}$ is the set of all functions $e : I_2 \to \tilde{J}_i^2$ so that $e(i) = j \in \tilde{J}_i^2$, $\forall i \in I_2$.
Proof. Let $e \in E'_D$. Then, by Corollary 8, $L_{i\,e(i)} \le \overline{X}_{e(i)}$, $\forall i \in I_2$. Therefore, we have $\tilde{d}_{i\,e(i)} = d_{i\,e(i)}$, $\forall i \in I_2$, that necessitates $\tilde{J}_i^2 = J_i^2$, $\forall i \in I_2$. Hence, $e(i) \in \tilde{J}_i^2$, $\forall i \in I_2$, and then $e \in E_{\tilde{D}}$. Conversely, let $e \in E_{\tilde{D}}$. Therefore, $e(i) \in \tilde{J}_i^2$, $\forall i \in I_2$. Since $\tilde{J}_i^2 \subseteq J_i^2$, $\forall i \in I_2$, then $e(i) \in J_i^2$, $\forall i \in I_2$, and therefore $e \in E_D$. By contradiction, suppose that $e \notin E'_D$. So, by Corollary 8, there is some $i_0 \in I_2$ such that $L_{i_0\,e(i_0)} > \overline{X}_{e(i_0)}$. Hence, $\tilde{d}_{i_0\,e(i_0)} = 0$ (since $e(i_0) \in J_{i_0}^2$ and $L_{i_0\,e(i_0)} > \overline{X}_{e(i_0)}$) and $L_{i_0\,e(i_0)} > 0$. The latter inequality together with Definition 2 and Remark 3 implies $b_{i_0}^2 > 0$. But in this case, $T_Y^p(\tilde{d}_{i_0\,e(i_0)}, x) = T_Y^p(0, x) = 0 < b_{i_0}^2$, $\forall x \in [0,1]$, that contradicts $e(i_0) \in J_{i_0}^2$.
By Lemma 7, we always have $\underline{X}(e) \le \overline{X}$ for each vector e, which is selected based on the components of matrix $\tilde{D}$. Actually, matrix $\tilde{D}$ as a reduced version of matrix D, removes all the infeasible intervals from the feasible region by neglecting those vectors e generating the infeasible solutions $\underline{X}(e)$. Also, similar to Lemma 5 we have $S_{T_Y^p}(A, D, b^1, b^2) = S_{T_Y^p}(A, \tilde{D}, b^1, b^2)$. This result and Lemma 5 can be summarized by

$S_{T_Y^p}(A,D,b^1,b^2) = S_{T_Y^p}(\tilde{A},\tilde{D},b^1,b^2)$.

Definition 10. Let $L = (L_{ij})_{m_2\times n}$ be a matrix whose (i,j)'th component is equal to L_{ij}. We define the modified matrix $L^* = (L^*_{ij})_{m_2\times n}$ from the matrix L as follows:

$$L^*_{ij} = \begin{cases} +\infty & L_{ij} > \overline{X}_j \\ L_{ij} & otherwise \end{cases}$$

As will be shown in the following theorem, matrix L^* is useful for deriving a necessary and sufficient condition for the feasibility of problem (1) and accelerating identification of the set $S_{T_Y^p}(A,D,b^1,b^2)$.

Theorem 4. $S_{T_Y^p}(A,D,b^1,b^2) \neq \emptyset$ iff there exists at least some $j \in J_i^2$ such that $L^*_{ij} \neq +\infty$, $\forall i \in I_2$.

Proof. Let $x \in S_{T_Y^p}(A,D,b^1,b^2)$. Then, from Corollary 5, there exists some $e' \in E_D$ such that $[\underline{X}(e'),\overline{X}] \neq \emptyset$. Therefore, $\underline{X}(e') \leq \overline{X}$ that implies $e' \in E'_D$. Now, by Corollary 8, we have$L_{i\,e'(i)} \leq \overline{X}_{e'(i)}$, $\forall i \in I_2$. Hence, by considering Definition 10, $L^*_{i\,e'(i)} \neq +\infty$,$\forall i \in I_2$. Conversely, suppose that $L^*_{ij_i} \neq +\infty$ for some $j_i \in J_i^2$, $\forall i \in I_2$. Then, from Definition 10 we have

$$L_{ij_i} \leq \overline{X}_{j_i}, \forall i \in I_2 \tag{3}$$

Consider vector $e' = [j_1, j_2, ..., j_m] \in E_D$. So, by noting Lemma 6, $\underline{X}(e')_{j_i} = \max\limits_{i\in I_j(e')} \left\{L_{i\,e'(i)}\right\} = \max\limits_{i\in I_j(e')} \left\{L_{ij_i}\right\}$, $\forall i \in I_2$, and $\underline{X}(e')_j = 0$ for each $j \in J - \{j_1, j_2, ..., j_m\}$. These equations together with (3) imply $\underline{X}(e') \leq \overline{X}$ that means $[\underline{X}(e'),\overline{X}] \neq \emptyset$. Now, the result follows from Corollary 5.

4. Optimization of the problem

According to the well-known schemes used for optimization of linear problems such as (1) [9,13,16,28], problem (1) is converted to the following two sub-problems:

$$\begin{aligned} \min\ & Z_1 = \textstyle\sum_{j=1}^n c_j^+ x_j \\ & A\varphi x \leq b^1 \\ & D\varphi x \geq b^2 \\ & x \in [0,1]^n \end{aligned} \tag{4}$$

and

$$\begin{aligned} \min\ & Z_2 = \textstyle\sum_{j=1}^n c_j^- x_j \\ & A\varphi x \leq b^1 \\ & D\varphi x \geq b^2 \\ & x \in [0,1]^n \end{aligned} \tag{5}$$

Where $c_j^+ = \max\{c_j, 0\}$ and $c_j^- = \min\{c_j, 0\}$ for $j = 1, 2, ..., n$. It is easy to prove that $\overline{X}$ is the optimal solution of (5), and the optimal solution of (4) is $\underline{X}(e')$ for some $e' \in E'_D$.

Theorem 5. Suppose that $S_{T_Y^p}(A,D,b^1,b^2) \neq \emptyset$, and $\overline{X}$ and $\underline{X}(e^*)$ are the optimal solutions of sub-problems (5) and (4), respectively. Then $c^T x^*$ is the lower bound of the optimal objective function in (1), where $x^* = [x_1^*, x_2^*, ..., x_n^*]$ is defined as follows:

$$x_j^* = \begin{cases} \overline{X}_j & c_j < 0 \\ \underline{X}(e^*)_j & c_j \geq 0 \end{cases} \tag{6}$$

for $j = 1, 2, ..., n$.
Proof. Let $x \in S_{T_Y^p}(A,D,b^1,b^2)$. Then, from Theorem 3 we have $x \in \bigcup_{e \in E_D} [\underline{X}(e), \overline{X}]$. Therefore, for each $j \in J$ such that $c_j \geq 0$, inequality $x_j^* \leq x_j$ implies $c_j^+ x_j^* \leq c_j^+ x_j$. In addition, for each $j \in J$ such that $c_j < 0$, inequality $x_j^* \geq x_j$ implies $c_j^- x_j^* \leq c_j^- x_j$. Hence, $\sum_{j=1}^n c_j x_j^* \leq \sum_{j=1}^n c_j x_j$.

Corollary 9. Suppose that $S_{T_Y^p}(A,D,b^1,b^2) \neq \emptyset$. Then, $x^* = [x_1^*, x_2^*, ..., x_n^*]$ as defined in (6), is the optimal solution of problem (1).
Proof. As in the poof of Theorem 5, $c^T x^*$ is the lower bound of the optimal objective function. According to the definition of vector x^*, we have $\underline{X}(e^*)_j \leq x_j^* \leq \overline{X}_j$, $\forall j \in J$, which implies $x^* \in \bigcup_{e \in E_D} [\underline{X}(e), \overline{X}] = S_{T_Y^p}(A,D,b^1,b^2)$.
We now summarize the preceding discussion as an algorithm.

Algorithm 1 (solution of problem (1))
Given problem (1):
1. Compute U_{ij} $(\forall i \in I_1$ and $\forall j \in J)$ and L_{ij} $(\forall i \in I_2$ and $\forall j \in J)$ by Definition 2.
2. If $\mathbf{1} \in S_{T_Y^p}(D,b^2)$, then continue; otherwise, stop, the problem is infeasible (Corollary 4).
3. Compute vectors $\overline{X}(i)$ $(\forall i \in I_1)$ from Definition 5, and then vector $\overline{X}$ from Definition 7.
4. If $\overline{X} \in S_{T_Y^p}(D,b^2)$, then continue; otherwise, stop, the problem is infeasible (Corollary 6).
5. Compute simplified matrices $\tilde{A}$ and $\tilde{D}$ from Lemma 5 and Definition 9, respectively.
6. Compute modified matrix L^* from Definition 10.
7. For each $i \in I_2$, if there exists at least some $j \in J_i^2$ such that $L_{ij}^* \neq +\infty$, then continue; otherwise, stop, the problem is infeasible (Theorem 4).
8. Find the optimal solution $\underline{X}(e^*)$ for the sub-problem (4) by considering vectors $e \in E_{\tilde{D}}$ and set $\tilde{J}_i^2$, $\forall i \in I_2$(Lemma 7).
9. Find the optimal solution $x^* = [x_1^*, x_2^*, ..., x_n^*]$ for the problem (1) by (6) (Corollary 9).

It should be noted that there is no polynomial time algorithm for complete solution of FRIs with the expectation $N \neq NP$. Hence, the problem of solving FRIs is an NP-hard problem in terms of computational complexity [2].

5. Construction of test problems and numerical example

In this section, we present a method to generate random feasible regions formed as the intersection of two fuzzy inequalities with Yager family of t-norms. In section 5.1, we prove that the max-Yager fuzzy relational inequalities constructed by the introduced method are actually feasible. In section 5.2, the method is used to generate a random test problem for problem (1), and then the test problem is solved by Algorithm 1 presented in section 4.
5.1. Construction of test problems
There are several ways to generate a feasible FRI defined with max-Yager composition. In what follows, we present a procedure to generate random feasible max-Yager fuzzy relational inequalities:
Algorithm 2 (construction of feasible Max-Yager FRI)

1. Generate randon scalars $a_{ij} \in [0,1]$, $i = 1,2,...,m_1$ and $j = 1,2,...,$n, and $b_i^1 \in [0,1]$, $i = 1,2,...,m_1$.
2. Compute $\overline{X}$ by Definition 7.
2. Randomly select m_2 columns $\{j_1, j_2,..., j_{m_2}\}$ from $J = \{1,2,...,n\}$.
2. For $i \in \{1,2,...,m_2\}$, assign a random number from $[0, \overline{X}_{j_i}]$ to b_i^2.
3. For $i \in \{1,2,...,m_2\}$, if $b_i^2 \neq 0$, then
   Assign a random number from interval $\left[\max\left\{b_i^2, 1 - \sqrt[p]{(1-b_i^2)^p - (1-\overline{X}_{j_i})^p}\right\}, 1\right]$ to d_{ij_i}.
   End
4. For $i \in \{1,2,...,m_2\}$
      For each $k \in \{1,2,...,m_2\} - \{i\}$
         Assign a random number from $[0\ ,\ 1]$ to d_{kj_i}.
      End
   End
5. For each $i \in \{1,2,...,m_2\}$ and each $j \notin \{j_1, j_2,..., j_{m_2}\}$
      Assign a random number from $[0,1]$ to d_{ij}.
   End

By the following theorem, it is proved that Algorithm 2 always generates random feasible max-Yager fuzzy relational inequalities.

Theorem 6. Problem (1) with feasible region constructed by Algorithm (2) has the nonempty feasible solutions set (i.e., $S_{T_Y^p}(A, D, b^1, b^2) \neq \emptyset$).
Proof. By considering the columns $\{j_1, j_2,..., j_{m_2}\}$ selected by Algorithm 2, let $e' = [j_1, j_2,..., j_{m_2}]$. We show that $e' \in E_D$ and $\underline{X}(e') \leq \overline{X}$. Then, the result follows from Corollary 5. From Algorithm 2, the following inequalities are resulted for each $i \in I_2$:

1. $b_i^2 \leq \overline{X}_{j_i}$.
2. $b_i^2 \leq d_{ij_i}$.

3. $1-\sqrt[p]{(1-b_i^2)^p-(1-\overline{X}_{j_i})^p}\le d_{ij_i}$.

By (I), we have $1-\sqrt[p]{(1-b_i^2)^p-(1-\overline{X}_{j_i})^p}\le 1$. This inequality together with $b_i^2\in[0,1]$, $\forall i\in I_2$, implies that the interval $\left[\max\left\{b_i^2, 1-\sqrt[p]{(1-b_i^2)^p-(1-\overline{X}_{j_i})^p}\right\},1\right]$ is meaningful. Also, by (II), $e'(i)=j_i\in J_i^2$, $\forall i\in I_2$. Therefore, $e'\in E_D$. Moreover, since the columns $\{j_1,j_2,...,j_{m_2}\}$ are distinct, sets $I_{j_i}(e')$ $(i\in I_2)$ are all singleton, i.e.,

$$I_{j_i}(e')=\{i\},\forall i\in I_2 \tag{7}$$

As a result, we also have $J(e')=\{j_1,j_2,...,j_{m_2}\}$ and $I_j(e')=\emptyset$ for each $j\notin\{j_1,j_2,...,j_{m_2}\}$. On the other hand, from Definition 5, we have $\underline{X}(i,e'(i))_{e'(i)}=\underline{X}(i,j_i)_{j_i}=L_{ij_i}$ and $\underline{X}(i,e'(i))_j=0$ for each $j\notin J-\{j_i\}$. This fact together with (7) and Lemma 6 implies $\underline{X}(e')_{j_i}=L_{ij_i}$, $\forall i\in I_2$, and $\underline{X}(e')_j=0$ for $j\notin\{j_1,j_2,...,j_{m_2}\}$. So, in order to prove $\underline{X}(e')\le\overline{X}$, it is sufficient to show that $\underline{X}(e')_{j_i}\le\overline{X}_{j_i}$, $\forall i\in I_2$. But, from Definition 2 and Remark 3,

$$\underline{X}(e')_{j_i}=L_{ij_i}=\begin{cases}0 & b_i^2=0\\ 1-\sqrt[p]{(1-b_i^2)^p-(1-d_{ij_i})^p} & b_i^2\neq 0\end{cases} \tag{8}$$

Now, inequality (III) implies

$$1-\sqrt[p]{(1-b_i^2)^p-(1-d_{ij_i})^p}\le\overline{X}_{j_i} \tag{9}$$

Therefore, by relations (8) and (9), we have $\underline{X}(e')_{j_i}\le\overline{X}_{j_i}$, $\forall i\in I_2$. This completes the proof.

5.2. Numerical example

Consider the following linear optimization problem (1) in which the feasible region has been randomly generated by Algorithm 2 presented in section 5.1.

$$\min\ Z = -4.6323\,x_1 + 2.4489\,x_2 + 6.0913\,x_3 - 7.9206\,x_4 + 4.5848\,x_5 + 2.9718\,x_6 - 0.5069\,x_7 + 8.6582\,x_8$$

$$\begin{bmatrix} 0.0964 & 0.2050 & 0.7567 & 0.2050 & 0.8179 & 0.2755 & 0.1249 & 0.9049 \\ 0.5991 & 0.6213 & 0.5421 & 0.4340 & 0.7084 & 0.9516 & 0.6172 & 0.2817 \\ 0.2336 & 0.1740 & 0.2821 & 0.1422 & 0.0432 & 0.3467 & 0.3555 & 0.6139 \\ 0.0323 & 0.2895 & 0.2449 & 0.3756 & 0.1459 & 0.2973 & 0.3629 & 0.6619 \\ 0.5799 & 0.0185 & 0.2863 & 0.7936 & 0.2333 & 0.4044 & 0.0685 & 0.2000 \\ 0.8422 & 0.7015 & 0.9631 & 0.8128 & 0.2467 & 0.3022 & 0.8672 & 0.9600 \\ 0.5569 & 0.9521 & 0.2307 & 0.9038 & 0.1703 & 0.7573 & 0.4579 & 0.6651 \\ 0.8399 & 0.7490 & 0.5373 & 0.5404 & 0.2351 & 0.3597 & 0.0776 & 0.5413 \end{bmatrix} \varphi x \le \begin{bmatrix} 0.8690 \\ 0.5570 \\ 0.0214 \\ 0.4827 \\ 0.8080 \\ 0.7360 \\ 0.5723 \\ 0.0090 \end{bmatrix}$$

$$\begin{bmatrix} 0.4067 & 0.1700 & 0.3225 & 0.4529 & 0.4168 & 0.5262 & 0.8422 & 0.3476 \\ 0.4631 & 0.3712 & 0.4638 & 0.0580 & 0.2803 & 0.2466 & 0.6319 & 0.7873 \\ 0.2027 & 0.0398 & 0.0990 & 0.1063 & 0.5981 & 0.9493 & 0.2954 & 0.7177 \\ 0.8695 & 0.7092 & 0.5710 & 0.9984 & 0.5084 & 0.5429 & 0.6220 & 0.0280 \\ 0.5979 & 0.9655 & 0.3259 & 0.8663 & 0.0365 & 0.7809 & 0.0475 & 0.0668 \\ 0.9166 & 0.6413 & 0.4505 & 0.6152 & 0.0637 & 0.5219 & 0.9946 & 0.9271 \\ 0.0230 & 0.1741 & 0.5778 & 0.9603 & 0.3229 & 0.9319 & 0.2068 & 0.0878 \\ 0.8994 & 0.0622 & 0.6030 & 0.0269 & 0.0984 & 0.1471 & 0.6074 & 0.3324 \end{bmatrix} \varphi x \ge \begin{bmatrix} 0.0850 \\ 0.0338 \\ 0.2418 \\ 0.1997 \\ 0.0306 \\ 0.0189 \\ 0.0799 \\ 0.0536 \end{bmatrix}$$

$$x \in [0,1]^n$$

where $|I_1| = |I_2| = |J| = 8$ and $\varphi(x,y) = T_Y^p(x,y) = \max\left\{1 - ((1-x)^p + (1-y)^p)^{1/p}, 0\right\}$ in which $p = 3$. Moreover, $Z_1 = 2.4489\,x_2 + 6.0913\,x_3 + 4.5848\,x_5 + 2.9718\,x_6 + 8.6582\,x_8$ is the objective function of sub-problem (4) and $Z_2 = -4.6323\,x_1 - 7.9206\,x_4 - 0.5069\,x_7$ is that of sub-problem (5). By Definition 2, matrices $U = (U_{ij})_{8\times 8}$ and $L = (L_{ij})_{8\times 8}$ are as follows:

$$U = \begin{bmatrix} 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 0.9098 \\ 0.8116 & 0.7703 & 1.0000 & 1.0000 & 0.6666 & 0.5597 & 0.7772 & 1.0000 \\ 0.3915 & 0.4752 & 0.3350 & 0.5290 & 0.7945 & 0.2714 & 0.2636 & 0.1008 \\ 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 0.6085 \\ 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\ 0.7884 & 1.0000 & 0.7386 & 0.8138 & 1.0000 & 1.0000 & 0.7719 & 0.7391 \\ 1.0000 & 0.5750 & 1.0000 & 0.5833 & 1.0000 & 0.6478 & 1.0000 & 0.7340 \\ 0.0220 & 0.0413 & 0.1236 & 0.1220 & 0.3699 & 0.2436 & 0.6376 & 0.1215 \end{bmatrix}$$

$$L = \begin{bmatrix} 0.3035 & 0.6149 & 0.3850 & 0.2666 & 0.2950 & 0.2173 & 0.0988 & 0.3585 \\ 0.1967 & 0.2665 & 0.1963 & 0.7851 & 0.3554 & 0.3950 & 0.1067 & 0.0575 \\ \infty & \infty & \infty & \infty & 0.3571 & 0.2435 & 0.7200 & 0.2963 \\ 0.2104 & 0.2544 & 0.3244 & 0.1997 & 0.3684 & 0.3431 & 0.2946 & \infty \\ 0.1179 & 0.0312 & 0.3033 & 0.0398 & 0.8930 & 0.0556 & 0.8193 & 0.7372 \\ 0.0225 & 0.0868 & 0.1873 & 0.0975 & 0.7070 & 0.1433 & 0.0189 & 0.0216 \\ \infty & 0.5945 & 0.1825 & 0.0807 & 0.3770 & 0.0824 & 0.5338 & 0.8798 \\ 0.0589 & 0.8727 & 0.1409 & \infty & 0.7122 & 0.5897 & 0.1388 & 0.3291 \end{bmatrix}$$

Therefore, by Corollary 3 we have, for example:

$S_{T_Y^p}(a_{25}, b_2^1) = [0, U_{25}] = [0, 0.6666]$ and $S_{T_Y^p}(a_{74}, b_7^1) = [0, U_{74}] = [0, 0.5833]$. $S_{T_Y^p}(d_{44}, b_4^2) = [L_{44}, 1] = [0.1997, 1]$ and $S_{T_Y^p}(d_{85}, b_8^2) = [L_{85}, 1] = [0.7122, 1]$. Also, from Definition 1, $J_1^2 = J_2^2 = J_5^2 = J_6^2 = \{1, 2, ..., 8\}$, $J_3^2 = \{5, 6, 7, 8\}$, $J_4^2 = \{1, 2, ..., 7\}$, $J_7^2 = \{2, 3, ..., 8\}$and $J_8^2 = \{1, 2, 3, 5, 6, 7, 8\}$. Moreover, the only components of matrix Dsuch that $d_{ij} < b_i^2$ are as follows: d_{31}, d_{32}, d_{33} and d_{34} (in the third row), d_{48} (in the fourth row), d_{71} (in the seventh row) and d_{84} (in the eighth row). Therefore, by Lemma 2 (part (b)), $S_{T_Y^p}(d_i, b_i^2) = \bigcup_{j=1}^{n} S_{T_Y^p}(d_{ij}, b_i^2) \neq \emptyset$, $\forall i \in I_2$.
By Definition 5, we have

$$\overline{X}(1) = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0.9098]$$

$$\overline{X}(2) = [0.8116 \quad 0.7703 \quad 1 \quad 1 \quad 0.6666 \quad 0.5597 \quad 0.7772 \quad 1]$$

$$\overline{X}(3) = [0.3915 \quad 0.4752 \quad 0.3350 \quad 0.5290 \quad 0.7945 \quad 0.2714 \quad 0.2636 \quad 0.1008]$$

$$\overline{X}(4) = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0.6085]$$

$$\overline{X}(5) = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]$$

$$\overline{X}(6) = [0.7884 \quad 1 \quad 0.7386 \quad 0.8138 \quad 1 \quad 1 \quad 0.7719 \quad 0.7391]$$

$$\overline{X}(7) = [1 \quad 0.5750 \quad 1 \quad 0.5833 \quad 1 \quad 0.6478 \quad 1 \quad 0.7340]$$

$$\overline{X}(8) = [0.0220 \quad 0.0413 \quad 0.1236 \quad 0.1220 \quad 0.3699 \quad 0.2436 \quad 0.6376 \quad 0.1215]$$

Also, for example

$$\underline{X}(3,5) = [0 \quad 0 \quad 0 \quad 0 \quad 0.3571 \quad 0 \quad 0 \quad 0], \underline{X}(3,6) = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.2435 \quad 0 \quad 0],$$

$$\underline{X}(3,7) = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.7200 \quad 0], \underline{X}(3,8) = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.2963].$$

Therefore, by Theorem 1, $S_{T_Y^p}(a_i, b_i^1) = [\mathbf{0}, \overline{X}(i)]$, $\forall i \in I_1$, and for example $S_{T_Y^p}(d_3, b_3^2) = \bigcup_{j=5}^{8} [\underline{X}(3, j), \mathbf{1}]$, for the third row of matrix D(i.e., $i = 3 \in I_2$). From Corollary 4, the necessary condition holds for the feasibility of the problem. More precisely, we have

$$D\varphi\mathbf{1} = \begin{bmatrix} 0.8422 \\ 0.7873 \\ 0.9493 \\ 0.9984 \\ 0.9655 \\ 0.9946 \\ 0.9603 \\ 0.8994 \end{bmatrix} \geq \begin{bmatrix} 0.0850 \\ 0.0338 \\ 0.2418 \\ 0.1997 \\ 0.0306 \\ 0.0189 \\ 0.0799 \\ 0.0536 \end{bmatrix} = b^2$$

that means $\mathbf{1} \in S_{T_Y^p}(D, b^2)$.
From Definition 7,

$$\overline{X} = [0.0220 \quad 0.0413 \quad 0.1236 \quad 0.1220 \quad 0.3699 \quad 0.2436 \quad 0.2636 \quad 0.1008]$$

which determines the feasible region of the first inequalities, i.e., $S_{T_Y^p}(A, b^1) = [\mathbf{0}, \overline{X}]$ (Theorem 2, part (a)). Also,

$$D\varphi\overline{X} = \begin{bmatrix} 0.2469 \\ 0.1767 \\ 0.2527 \\ 0.2009 \\ 0.2125 \\ 0.2636 \\ 0.2405 \\ 0.1655 \end{bmatrix} \geq \begin{bmatrix} 0.0850 \\ 0.0338 \\ 0.2418 \\ 0.1997 \\ 0.0306 \\ 0.0189 \\ 0.0799 \\ 0.0536 \end{bmatrix} = b^2$$

Therefore, we have $\overline{X} \in S_{T_Y^p}(D, b^2)$, which satisfies the necessary feasibility condition stated in Corollary 6. On the other hand, from Definition 6, we have $|E_D| = 5619712$. Therefore, the number of all vectors $e \in E_D$ is equal to 5619712. However, each solution $\underline{X}(e)$ generated by vectors $e \in E_D$ is not necessary a feasible solution. For example, for $e' = [1, 5, 6, 7, 2, 6, 6, 7]$, we attain from Definition 7

$$\begin{aligned} \underline{X}(e') &= \max_{i \in I_2} \{\underline{X}(i, e'(i))\} \\ &= \max \{\underline{X}(1,1), \underline{X}(2,5), \underline{X}(3,6), \underline{X}(4,7), \underline{X}(5,2), \underline{X}(6,6), \underline{X}(7,6), \underline{X}(8,7)\} \end{aligned}$$

where

$$\underline{X}(1,1) = [0.3035 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$$
$$\underline{X}(2,5) = [0 \quad 0 \quad 0 \quad 0 \quad 0.3554 \quad 0 \quad 0 \quad 0]$$
$$\underline{X}(3,6) = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.2435 \quad 0 \quad 0]$$
$$\underline{X}(4,7) = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.2946 \quad 0]$$
$$\underline{X}(5,2) = [0 \quad 0.0312 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$$
$$\underline{X}(6,6) = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.1433 \quad 0 \quad 0]$$
$$\underline{X}(7,6) = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.0824 \quad 0 \quad 0]$$
$$\underline{X}(8,7) = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.1388 \quad 0]$$

Therefore, $\underline{X}(e') = [0.3035 \quad 0.0312 \quad 0 \quad 0.7851 \quad 0.3554 \quad 0.2435 \quad 0.1388 \quad 0]$. It is obvious that $\underline{X}(e') \not\leq \overline{X}$ (actually, $\underline{X}(e')_1 > \overline{X}_1$ and $\underline{X}(e')_4 > \overline{X}_4$) which means $\underline{X}(e') \notin S_{T_Y^p}(A, D, b^1, b^2)$ from Theorem 3.
From the first simplification (Lemma 5), "resetting the following components a_{ij} to zeros" are equivalence operations: a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16} , a_{17}; a_{23}, a_{24}, a_{28}; a_{71}, a_{72}, a_{73}, a_{74}, a_{75}, a_{76} , a_{77};$a_{5j} (j = 1, 2, ..., 8)$; a_{62}, a_{65}, a_{66}; a_{71}, a_{73}, a_{75} , a_{77}. So, matrix $\tilde{A}$ is resulted as follows:

$$\tilde{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.9049 \\ 0.5991 & 0.6213 & 0 & 0 & 0.7084 & 0.9516 & 0.6172 & 0 \\ 0.2336 & 0.1740 & 0.2821 & 0.1422 & 0.0432 & 0.3467 & 0.3555 & 0.6139 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.6619 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.8422 & 0 & 0.9631 & 0.8128 & 0 & 0 & 0.8672 & 0.9600 \\ 0 & 0.9521 & 0 & 0.9038 & 0 & 0.7573 & 0 & 0.6651 \\ 0.8399 & 0.7490 & 0.5373 & 0.5404 & 0.2351 & 0.3597 & 0.0776 & 0.5413 \end{bmatrix}$$

Also, by Definition 9, we can change the value of components d_{11}, d_{12}, d_{13}, d_{14}, d_{18}; d_{21}, d_{22}, d_{23}, d_{24}, d_{26}; d_{37}, d_{38}; d_{41}, d_{42}, d_{43}, d_{44}, d_{46}, d_{47}; d_{51}, d_{53}, d_{55}, d_{57}, d_{58}; d_{61}, d_{62}, d_{63}, d_{65}; d_{72}, d_{73}, d_{75}, d_{77}, d_{78}; d_{81}, d_{82}, d_{83}, d_{85}, d_{86}, d_{88} to zeros. For example, since $8 \in J_3^2$ and $L_{38} = 0.2963 > 0.1008 = \overline{X}_8$, then $\tilde{d}_{38} = 0$. Simplified matrix $\tilde{D}$ is obtained as follows:

$$\tilde{D} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.4168 & 0.5262 & 0.8422 & 0 \\ 0 & 0 & 0 & 0 & 0.2803 & 0 & 0.6319 & 0.7873 \\ 0.2027 & 0.0398 & 0.0990 & 0.1063 & 0.5981 & 0.9493 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5084 & 0 & 0 & 0.0280 \\ 0 & 0.9655 & 0 & 0.8663 & 0 & 0.7809 & 0 & 0 \\ 0 & 0 & 0 & 0.6152 & 0 & 0.5219 & 0.9946 & 0.9271 \\ 0.0230 & 0 & 0 & 0.9603 & 0 & 0.9319 & 0 & 0 \\ 0 & 0 & 0 & 0.0269 & 0 & 0 & 0.6074 & 0 \end{bmatrix}$$

Additionally, $\tilde{J}_1^2 = \{5,6,7\}$, $\tilde{J}_2^2 = \{5,7,8\}$, $\tilde{J}_3^2 = \{5,6\}$, $\tilde{J}_4^2 = \{5\}$, $\tilde{J}_5^2 = \{2,4,6\}$, $\tilde{J}_6^2 = \{4,6,7,8\}$, $\tilde{J}_7^2 = \{4,6\}$ and $\tilde{J}_8^2 = \{7\}$. Based on these results and Lemma 7, we have $|E_{\tilde{D}}| = \left|E'_D\right| = 432$. Therefore, the simplification processes reduced the number of the minimal candidate solutions from 5619712 to 432, by removing 5619280 infeasible points $\underline{X}(e)$. Consequently, the feasible region has 432 minimal candidate solutions, which are feasible. In other words, for each $e \in E_{\tilde{D}}$, we have $\underline{X}(e) \in S_{T_Y^p}(A, D, b^1, b^2)$. However, each feasible solution $\underline{X}(e)$ $(e \in E_{\tilde{D}})$ may not be a minimal solution for the problem. For example, by selecting $e' = [6, 8, 5, 5, 2, 4, 6, 7]$, the corresponding solution is obtained as

$\underline{X}(e') = [0 \quad 0.0312 \quad 0 \quad 0.0975 \quad 0.3684 \quad 0.2173 \quad 0.1388 \quad 0.0575]$. Although $\underline{X}(e')$ is feasible (because of the inequality $\underline{X}(e') \le \overline{X}$) but it is not actually a minimal solution. To see this, let $e'' = [5, 5, 5, 5, 4, 7, 4, 7]$. Then, $\underline{X}(e'') = [0 \quad 0 \quad 0 \quad 0.0807 \quad 0.3684 \quad 0 \quad 0.1388 \quad 0]$. Obviously, $\underline{X}(e'') \le \underline{X}(e')$ which shows that $\underline{X}(e')$ is not a minimal solution.

Now, we obtain the modified matrix L^* according to Definition 10:

$$L^* = \begin{bmatrix} \infty & \infty & \infty & \infty & 0.2950 & 0.2173 & 0.0988 & \infty \\ \infty & \infty & \infty & \infty & 0.3554 & \infty & 0.1067 & 0.0575 \\ \infty & \infty & \infty & \infty & 0.3571 & 0.2435 & \infty & \infty \\ \infty & \infty & \infty & \infty & 0.3684 & \infty & \infty & \infty \\ \infty & 0.0312 & \infty & 0.0398 & \infty & 0.0556 & \infty & \infty \\ \infty & \infty & \infty & 0.0975 & \infty & 0.1433 & 0.0189 & 0.0216 \\ \infty & \infty & \infty & 0.0807 & \infty & 0.0824 & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & 0.1388 & \infty \end{bmatrix}$$

As is shown in matrix L^*, for each $i \in I_2$ there exists at least some $j \in J_i^2$

such that $L^*_{ij} \neq +\infty$. Thus, by Theorem 4 we have $S_{T_Y^p}(A, D, b^1, b^2) \neq \emptyset$. Finally, vector $\overline{X}$ is optimal solution of sub-problem (5). For this solution, $Z_2 = -4.6323\,\overline{X}_1 - 7.9206\,\overline{X}_4 - 0.5069\,\overline{X}_7 = -1.2018$. Also, $Z = c^T\overline{X} = 2.9448$. In order to find the optimal solution $\underline{X}(e^*)$ of sub-problems (4), we firstly compute all minimal solutions by making pairwise comparisons between all solutions $\underline{X}(e)$ $(\forall e \in E_{\tilde{D}})$, and then we find $\underline{X}(e^*)$ among the resulted minimal solutions. Actually, the feasible region has two minimal solutions as follows:

$$e_1 = [5\ ,\ 5\ ,\ 5\ ,\ 5\ ,\ 4\ ,\ 7\ ,\ 4\ ,\ 7]$$
$$\underline{X}(e_1) = [0 \quad 0 \quad 0 \quad 0.0807 \quad 0.3684 \quad 0 \quad 0.1388 \quad 0]$$

$$e_2 = [5\ ,\ 5\ ,\ 5\ ,\ 5\ ,\ 6\ ,\ 7\ ,\ 6\ ,\ 7]$$
$$\underline{X}(e_2) = [0 \quad 0 \quad 0 \quad 0 \quad 0.3684 \quad 0.0824 \quad 0.1388 \quad 0]$$

By comparison of the values of the objective function for the minimal solutions, $\underline{X}(e_1)$ is optimal in (4) (i.e., $e^* = e_1$). For this solution,

$$\begin{aligned} Z_1 &= \textstyle\sum_{j=1}^{n} c_j^+ \underline{X}(e_1)_j \\ &= 2.4489\,\underline{X}(e_1)_2 + 6.0913\,\underline{X}(e_1)_3 + 4.5848\,\underline{X}(e_1)_5 + 2.9718\,\underline{X}(e_1)_6 + 8.6582\,\underline{X}(e_1)_8 \\ &= 1.6893 \end{aligned}$$

Also, $Z = c^T\underline{X}(e_1) = 0.97931$. Thus, from Corollary 9, $x^* = [0.022 \quad 0 \quad 0 \quad 0.122 \quad 0.3684 \quad 0 \quad 0.2636 \quad 0]$ and then $Z^* = c^T x^* = 0.48727$.

Conclusion

In this paper, we proposed an algorithm to find the optimal solution of linear problems subjected to two fuzzy relational inequalities with Yager family of t-norms. The feasible solutions set of the problem is completely resolved and a necessary and sufficient condition and three necessary conditions were presented to determine the feasibility of the problem. Moreover, depending on the max-Yager composition, two simplification operations were proposed to accelerate the solution of the problem. Finally, a method was introduced for generating feasible random max-Yager inequalities. This method was used to generate a test problem for our algorithm. The resulted test problem was then solved by the proposed algorithm. As future works, we aim at testing our algorithm in other type of linear optimization problems whose constraints are defined as FRI with other well-known t-norms.

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