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# Tenacity and some related results

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**ABSTRACT**

Conceptually graph vulnerability relates to the study of graph intactness when some of its elements are removed. The motivation for studying vulnerability measures is derived from design and analysis of networks under hostile environment. Graph tenacity has been an active area of research since the the concept was introduced in 1992.

The tenacity T(G) of a graph G is defined as

$$T(G) = \min_{A \subset V(G)} \{ \frac{| A | + \tau(G-A)}{\omega(G-A)} \}$$

where $\tau(G-A)$ denotes the order (the number of vertices) of a largest component of G-A and $\omega(G-A)$ is the number of components of G-A.

In this paper we discuss tenacity and its properties in vulnerability calculation.

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## 1 Introduction

We consider only finite undirected graphs without loops and multiple edges. Let G be a graph. We denote by V(G), E(G) and $| V(G) |$ the set of vertices, the set of edges and the

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order of a graph G, respectively. A set of vertices in G is independent if no two of them are adjacent. The largest number of vertices in any such set is called the vertex independence number of G and is denoted by $\alpha(G)$ or $\alpha$. The vertex connectivity, $\kappa = \kappa(G)$, of a finite, undirected, simple graph G (without loops or multiple edges) is the minimum number of vertices whose removal results in a disconnected graph or results in the trivial graph $K_1$. Graph G is called n-connected if $\kappa \geq n$. Analogously, the edge-connectivity, $\lambda = \lambda(G)$, of a finite, undirected, connected simple graph G is the minimum number of edges whose removal results in a disconnected or trivial graph $K_1$. A graph G is called n-edge- connected if $\lambda(G) \geq n$. Given a graph G, the graph $G^2$ has $V(G^2) = V(G)$ and $uv \in E(G^2)$ if and only if $uv \in E(G)$ or the distance from u to v is 2. The complement $\overline{G}$ of a graph G also has $V(G)$ as its vertex set, but two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. The line graph of a graph G, denoted $L(G)$, is the graph whose vertices are the edges of G, and two vertices are adjacent whenever the corresponding edges of G are adjacent.

The concept of tenacity of a graph G was introduced as a useful measure of the "vulnerability" of G. we compared integrity, connectivity, binding number, toughness, and tenacity for several classes of graphs. The results suggest that tenacity is a most suitable measure of stability or vulnerability in that for many graphs it is best able to distinguish between graphs that intuitively should have different levels of vulnerability. In [3-25], they studied more about this new invariant. The tenacity of a graph G, T(G), is defined by $T(G) = min\{\frac{|S|+\tau(G-S)}{\omega(G-S)}\}$, where the minimum is taken over all vertex cutsets S of G. We define $\tau(G-S)$ to be the number of the vertices in the largest component of the graph G - S, and $\omega(G-S)$ be the number of components of G - S. A connected graph G is called T-tenacious if $\mid S \mid + \tau(G-S) \geq T\omega(G-S)$ holds for any subset S of vertices of G with $\omega(G-S) > 1$. If G is not complete, then there is a largest T such that G is T-tenacious ; this T is the tenacity of G. On the other hand, a complete graph contains no vertex cutset and so it is T-tenacious for every T. Accordingly, we define $T(K_p) = \infty$ for every p $(p \geq 1)$. A set $S \subseteq V(G)$ is said to be a T-set of G if $T(G) = \frac{|S|+\tau(G-S)}{\omega(G-S)}$.

Any undefined terms can be found in the standard references on graph theory, including Bondy and Murty [1].

**Vulnerability Calculation**

Let $C_n = (v_1 v_2 \cdots v_n)$ be the n-cycle and define the k-th power of the n-cycle, $C_n^k$ by

$$C_n^k = C_n + \{v_i v_j \mid\mid i - j \mid \leq k\}.$$

We have the following four propositions.

**Proposition 1** : If G is a spanning subgraph of H, then $T(G) \leq T(H)$.

**Proposition 2** : For any graph G, $T(G) \geq \frac{\kappa(G)+1}{\alpha(G)}$.

**Proposition 3** : If G is not complete, then $T(G) \leq \frac{n-\alpha(G)+1}{\alpha(G)}$.

**Proposition 4** : If $k \leq n-k$, then $T(K_{k,n-k}) = \frac{k+1}{n-k}$.

We can prove the following two lemmas:

**Lemma 1** : If A is a minimal T-set for $C_n^k$, then A consists of the union of sets of k consecutive vertices such that there exists at least one vertex not in A between any two sets of consecutive vertices in A.

Lemma 1 gives us an indication of the size of the cut-set for the tenacity of $C_n^k$ ; the next lemma gives us the size of the largest component.
**Lemma 2** : There is a T-set, A, for $C_n^k$ such that all components of $C_n^k$ have order $\tau(C_n^k - A)$ or $\tau(C_n^k - A) - 1$.

These two lemmas allow us to determine precisely the tenacity of the power of cycles.
**Theorem 1** : Let $C_n^k$ be a power of cycles and $n = r(k+1)+s$, for $0 \leq s < k+1$. Then $T(C_n^k) = k + \frac{1+\lceil \frac{s}{r} \rceil}{r}$.
**Proof** : Let A be a minimal T-set of $C_n^k$. By lemma 1 and lemma 2, $\mid A \mid = k\omega$, and $\tau(C_n^k - A) = \lceil \frac{n-k\omega}{\omega} \rceil$. Thus, from the definition of tenacity we have

$$T = min\{\frac{k\omega + \lceil \frac{n-k\omega}{\omega} \rceil}{\omega} \mid 2 \leq \omega \leq r\}.$$

Now consider the function $f(\omega) = \frac{k\omega + \lceil \frac{n-k\omega}{\omega} \rceil}{\omega} = k + \frac{\lceil \frac{n}{\omega} - k \rceil}{\omega}$. Let $\omega_1$ and $\omega_2$ be any two integers in [2 , r] with $\omega_1 \leq \omega_2$, then $\lceil \frac{n}{\omega_2} \rceil \leq \lceil \frac{n}{\omega_1} \rceil$. Thus $f(\omega_2) = k + \frac{\lceil \frac{n}{\omega_2} - k \rceil}{\omega_2} \leq k + \frac{\lceil \frac{n}{\omega_1} - k \rceil}{\omega_1} = f(\omega_1)$. Hence the function $f(\omega)$ is a non increasing function and the minimum value occurs at the boundary. Thus $\omega = r$ and $\lceil \frac{n-k\omega}{\omega} \rceil = \lceil \frac{r(k+1)+s-kr}{r} \rceil = 1 + \lceil \frac{s}{r} \rceil$. Therefore, $T(C_n^k) = k + \frac{1+\lceil \frac{s}{r} \rceil}{r}$.$\square$

Now we can discuss about tenacity and its operation on graphs. If the removal of a vertex from a graph results in a complete graph, the tenacity becomes infinite. On the other hand, the removal of a vertex cannot lower by too much. We can easily prove the following two theorems and corollaries:
**Theorem 2**: For any nontrivial, incomplete graph G with n vertices and any vertex v, $T(G - v) \geq T(G) - \frac{1}{2}$.

The following theorem allow us to find the tenacity of several important classes of graphs.
**Theorem 3**: If G is a bipartite, r-regular, r-connected graph on n vertices, then $T(G) = \frac{n+2}{n}$.

This result gives several interesting corollaries.

**Corollary 1**: If $G_1$ is a bipartite, d-regular, d-connected graph with $n_1$ vertices and $G_2$ is a bipartite, q-regular, q-connected graph with $n_2$ vertices, then $T(G_1 \times G_2) = \frac{n_1 n_2 + 2}{n_1 n_2}$.
**Corollary 2**: For any integer n, $T(Q_n) = \frac{2^n+2}{2^n}$.
**Corollary 3**: For any integers n and m, $T(C_n \times C_m) = \frac{nm+2}{nm}$.
**Corollary 4**: For any even integer n, $T(C_n \times K_2) = \frac{n+1}{n}$.

In 1972, Chvátal [2] introduced the concept of the toughness of a graph. It measures in a simple way how tightly various pieces of a graph hold together; therefore he called it toughness. Let G be a graph and t a real number such that the implication $\omega(G-A) > 1 \Rightarrow \mid A \mid \geq t \cdot \omega(G-A)$ holds for each set A of vertices of G. Then G will be said to be t-tough.
**Proposition 5**: $G \subset H \Rightarrow t(G) \leq t(H)$.

Thus toughness is a nondecreasing invariant whose values range from zero to infinity. A graph G is disconnected if and only if $t(G) = 0$; G is complete if and only if $t(G) = +\infty$.
**Theorem 4**: $t(K_m \times K_n) = \frac{1}{2}(m+n) - 1, (m, n \geq 2)$.

Without attempting to obtain the best possible result, we can prove quite easily the following relation between T(G) and t(G). This result gives us a number of corollaries.
**Theorem 5**: For any graph G, $T(G) \geq t(G) + \frac{1}{\alpha(G)}$.
**Proof**: Let $A \subseteq V(G)$ be a t-set and $B \subseteq G$ be a T-set. Then $\frac{|B|+\tau(G-B)}{\omega(G-B)} \geq \frac{|B|}{\omega(G-B)} + \frac{1}{\omega(G-B)} + \geq \frac{|A|}{\omega(G-A)} + \frac{1}{\alpha(G)}$.□

We next obtain some bounds on the tenacity of products of graphs.
**Theorem 6**: If $n \geq m$, then $\frac{m^2+mn-2m+2}{2m} \leq T(K_m \times K_n) \leq \frac{mn-n+\lceil \frac{n}{m} \rceil}{m}$.
**Proof**: By Theorem 4, $t(K_m \times K_n) = \frac{m+n-2}{2}$. It is easy to see that $\alpha(K_m \times K_n) = m$. Let $V(K_n) = \{1, 2, 3, \cdots, n\}$ and $V(K_m) = \{1, 2, 3, \cdots, m\}$. Then $V(K_m \times K_n) = \{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$. Also let $n = am + b$, for $0 \leq b < m$, so if $b = 0$ then $a = \lceil \frac{n}{m} \rceil = \frac{n}{m}$ and otherwise $a + 1 = \lceil \frac{n}{m} \rceil$. Now, if $b = 0$, then define the sets $W_i$ as $W_i = \{(i, ia - a + 1), \cdots, (i, ia)\}$ for $1 \leq i \leq m$ , otherwise define the sets $W_i$ as follows:

$$W_i = \begin{cases} \{(i, ia + i - a), \cdots, (i, ia + i)\} & 1 \leq i \leq b \\ \{(i, ia + b - a + 1), \cdots, (i, ia + b)\}, & b + 1 \leq i \leq m, \end{cases}$$

and let $W = \bigcup_{i=1}^{m} W_i$. Define $A = V(G) - W$ and so $\mid A \mid = mn - n$. It is easy to see that the $W_i$, $1 \leq i \leq m$, are the components of G-A and so $\tau(G-A) = \lceil \frac{n}{m} \rceil$ and $\omega(G-A) = m$. The result follows.□
The following our conjecture proved recently:
**Conjecture** : If $n \geq m \geq 2$ then $T(K_m \times K_n) = \frac{mn-n+\lceil \frac{n}{m} \rceil}{m}$.
**Corollary 5**: For any integer n, $T(K_n \times K_n) = n - 1 + \frac{1}{n}$.
**Corollary 6:** For any graph G, $T(G^2) > \kappa(G)$.
**Corollary 7:** Let G be a non-empty graph and let m be the largest integer such that $K_{1,m}$ is an induced subgraph of G. Then $T(G) \geq \frac{\kappa(G)}{m} + \frac{1}{\alpha(G)}$.
**Theorem 7:** If G is connected and a non-complete $K_{1,3}$-free graph then $T(G) > \frac{\kappa(G)}{2}$.
**Proof:** Suppose G is a non-complete, $K_{1,3}$-free graph with connectivity $\kappa(G)$. Let A be a T-set, and let $W_1, W_2, \cdots, W_m$ be the components of G-A.
Since G has a connectivity $\kappa(G)$, it is $\kappa(G)$-connected and so there exist $\kappa(G)$ internally disjoint paths from $u_i \in W_i$ to $u_j \in W_j$ for all $1 \leq i,\ j \leq m$ with $i \neq j$. Each of these

paths must contain a vertex of A. Then for each i there are at least $\kappa(G)$ edges coming from $W_i$ to distinct vertices of A. Thus in all there are at least m$\kappa(G)$ edges from G-A to A counting at most one from any component $W_i$ to a particular vertex of A.
Suppose $v \in A$ is adjacent to vertices $u_1, u_2$, and $u_3$ in distinct components of G-A. Then, since $\{u_1, u_2, u_3\}$ is an independent set the graph induced by $\{v, u_1, u_2, u_3\}$ is a $K_{1,3}$, a contradiction. Hence we can conclude any vertex of A is adjacent to at most two components of G-A. Thus, there are at most 2 $\mid A \mid$ edges coming from G-A to vertices of A, counting at most one edge from any component to a particular vertex of A. Hence $m\kappa(G) \leq 2 \mid A \mid$, or $\frac{m\kappa(G)}{2} \leq \mid A \mid$. Therefore $\frac{m\kappa(G)}{2} < \mid A \mid +1$, or $\frac{|A|+1}{m} > \frac{\kappa(G)}{2}$. Thus $T(G) = \frac{|A|+\tau(G-A)}{\omega(G-A)} \geq \frac{|A|+1}{m} > \frac{\kappa(G)}{2}$.□

The effect of removing a vertex is considered first. If the removal of a vertex from a graph results in a complete graph, the tenacity becomes infinite. On the other hand, the removal of a vertex cannot lower T by too much.
**Theorem 8:** For any nontrivial non-complete graph G on n vertices and any vertex v, $T(G-v) \geq T(G) - \frac{1}{2}$.
**Proof:** Let $G' = G - v$. If $G' = K_{n-1}$, then $T(G') = \infty$, and the theorem holds. Hence, assume $G' \neq K_{n-1}$. Let A' be a T-set for G', and let $\mid A' \mid = m$, then $T(G') = \frac{m+\tau(G'-A')}{\omega(G'-A')}$. Now define $A = A' \cup \{v\}$. Clearly A is a disconnecting set for G and so $T(G) \leq \frac{|A|+\tau(G-A)}{\omega(G-A)}$. But $\mid A \mid = m+1$ and $G - A = G' - A'$, so $T(G) \leq \frac{m+1+\tau(G'-A')}{\omega(G'-A')} = \frac{m+\tau(G'-A')}{\omega(G'-A')} + \frac{1}{\omega(G'-A')} = T(G') + \frac{1}{\omega(G'-A')} \leq T(G') + \frac{1}{2}$, since $\omega(G'-A') \geq 2$. Hence $T(G) \leq T(G') + \frac{1}{2}$.□

We next obtain some bounds on the tenacity of a graph.
**Proposition 6:** If G is connected, then $T(G) \geq \frac{1}{\Delta(G)}$.
**Proof:** $K_n$ is a special case, otherwise the removal of any vertex of a connected graph G yields at most $\Delta(G)$ components. Similarly, the removal of any n vertices yields at most $n\Delta(G)$ components. Then, from the definition we have $T(G) \geq \frac{n+1}{n\Delta(G)} \geq \frac{1}{\Delta(G)}$.□
**Lemma 3:** If A is a minimal T-set for the graph G then, for each vertex v of A, the induced subgraph $< V(G) - A + v >$ has fewer components than does G-A.
**Proof:** Let $A' = A - v$. If G-A' has at least as many components as G-A, then $\mid A' \mid = \mid A \mid -1$ and $\tau(G - A') \leq \tau(G - A) + 1$. Therefore $\frac{|A'|+\tau(G-A')}{\omega(G-A)} = \frac{|A|-1+\tau(G-A')}{\omega(G-A)} \leq \frac{|A|-1+\tau(G-A)+1}{\omega(G-A)} = T(G)$, contrary to our choice of A.□
**Theorem 9:** Let $G = G_1 + G_2$, where $\mid V(G) \mid = n$, $\mid V(G_i) \mid = p_i$, $T(G) = T$ and $T(G_i) = T_i$ for $i = 1, 2$. Then if $G \neq K_n$ we have

$$min\{\frac{[n+\tau(G_1-A_1)]T_1}{p_1+\tau(G_1-A_1}, \frac{[n+\tau(G_2-A_2)]T_2}{p_2+\tau(G_2-A_2)}\} < T \leq min\{\frac{n-\alpha_1+1}{\alpha_1}, \frac{n-\alpha_2+1}{\alpha 2}\},$$

where $\alpha_i$ is the independence number of $G_i$, and $A_i$ is a disconnecting set of $G_i$ for $i = 1, 2$.
**Proof:** Because of the structure of G, the graph cannot be disconnected without removing one of $V(G_1)$ or $V(G_2)$. Having removed the appropriate set, we can then disconnect the graph by disconnecting the remaining graph, either $G_1$ or $G_2$. Candidates for T

are of the form $\frac{n_1+p_2+\tau(G_1-A_1)}{\omega(G_1-A_1)}$ or $\frac{n_2+p_1+\tau(G_2-A_2)}{\omega(G_2-A_2)}$ where $n_i =\mid A_i \mid$ for $i = 1, 2$. Then $T = min\{\frac{n_1+p_2+\tau(G_1-A_1)}{\omega(G_1-A_1)}, \frac{n_2+p_1+\tau(G_2-A_2)}{\omega(G_2-A_2)}\}$, where the minimum is taken over all $A_1$ and $A_2$ as defined. Also $T_1 \leq \frac{n_1+\tau(G_1-A_1)}{\omega(G_1-A_1)}$ which implies $\omega(G_1 - A_1) \leq \frac{n_1+\tau(G_1-A_1)}{T_1}$. Thus $\frac{n_1+p_2+\tau(G_1-A_1)}{\omega(G_1-A_1)} \geq \frac{[n_1+p_2+\tau(G_1-A_1)]T_1}{n_1+\tau(G_1-A_1)}$. Similarly, $\frac{n_2+p_1+\tau(G_2-A_2)}{\omega(G_2-A_2)} \geq \frac{[n_2+p_1+\tau(G_2-A_2)]T_2}{n_2+\tau(G_2-A_2)}$. Thus $T \geq min\{[1+\frac{p_2}{n_1+\tau(G_1-A_1)}]T_1, [1+\frac{p_1}{n_2+\tau(G_2-A_2)}]T_2\}$. Also we know that $n_1 < p_1$ and $n_2 < p_2$, therefore $T > min\{\frac{[n+\tau(G_1-A_1)]T_1}{p_1+\tau(G_1-A_1)}, \frac{[n+\tau(G_2-A_2)]T_2}{p_2+\tau(G_2-A_2)}\}$. From Proposition 3, we observe that two candidates for T are $\frac{(p_1-\alpha_1)+1+p_2}{\alpha_1}$ and $\frac{(p_2-\alpha_2)+p_1}{\alpha_2}$, which yield $T \leq min\{\frac{n-\alpha_1+1}{\alpha_1}, \frac{n-\alpha_2+1}{\alpha_2}\}$.□

**Theorem 10:** Let G be a graph with n vertices and $G \neq K_n$, then $T(G) + T(\overline{G}) \geq \frac{1}{n-1}$.

**Proof:** We observe that at least one of G or $\overline{G}$ is connected. Suppose $\overline{G}$ is not connected. We proved (Proposition 6) that $T(G) \geq \frac{1}{\Delta(G)} \geq \frac{1}{n-1}$ for any graph G. Thus, $T(G)+T(\overline{G}) \geq \frac{1}{n-1}$. Now suppose G is not connected but $\overline{G}$ is connected. Again by Proposition 6, we have $T(\overline{G}) \geq \frac{1}{n-1}$. Therefore $T(G) + T(\overline{G}) \geq \frac{1}{n-1}$.□

**Theorem 11:** Let G be a graph with $0 < T(G) < \infty$, and let $\lambda(G) = \lambda$, then $T(L(G)) > \frac{\lambda}{2}$.

**Proof:** Assume there exist vertex cutsets A for L(G) such that A is a t-set. By Theorem 5, $T(L(G)) > t(L(G))$. Let E be those edges of G which are incident to vertices of A. Then E is an edge-cutset of G. Thus we have $t(L(G)) = min\{\frac{|A|}{\omega(L(G)-A)}\} \geq min\{\frac{|E|}{\omega(G-E)}\} = t'(G)$, where A is a cutset and E is an edge cutset of G.
Using the result of Chvátal [2] we have $t'(G) = min\{\frac{|E|}{\omega(G-E)}\} = \frac{\lambda}{2}$. Therefore $T(L(G)) > \frac{\lambda}{2}$.□

The binding number of a graph G was defined by Woodall in [26] as

$$bind(G) = \min_{A}\{\frac{\mid N(A) \mid}{\mid A \mid}\}$$

where $\phi \neq A \subseteq V(G)$ and $N(A) \neq V(G)$. The binding number was called the melting-point of the graph. the reason for the name "binding number" is that, roughly speaking, if bind(G) is large, then the vertices of G are bound tightly together, in the sense that G has many edges fairly well distributed.

**Theorem 12:** For any graph G, $T(G) \geq bind(G) - 1$.

**Proof:** Let bind(G) = c. If $c < 1$, then $c - 1 < 0$ and the result follows since T(G) is nonnegative. Consider $c \geq 1$. Suppose that A is a subset of V(G) such that $\omega = \omega(G - A) \geq 2$. We want to prove that $\frac{|A|+1}{\omega} > (c - 1)$. If each of the $\omega$ components of G-A has at least two vertices, let S consist of the vertices in all the components except the smallest, so that

$$\mid S \mid \geq \frac{\mid V(G) - A \mid (\omega - 1)}{\omega} \geq \frac{2\omega(\omega - 1)}{\omega} = 2(\omega - 1) \geq \omega.$$

If, on the other hand, V(G)-A contains an isolated vertex, let $S = V(G) - A$. So that

$| S |=| V(G) - A |\geq \omega$. In either case $N(S) \neq V(G)$, and since $bind(G) = c \geq 1$,

$$| S | + | A | +1 >| S | + | A |\geq| N(S) |\geq c | S | .$$

It follows that $| A | +1 > (c-1) | S |\geq (c-1)\omega$. Therefore $\frac{|A|+1}{\omega} > c-1$, so $T > c-1$.□

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