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RESEARCH PAPER

Vibration of inhomogeneous fibrous laminated plates using an efficient and simple polynomial refined theory

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ABSTRACT

In this article, a reliable model for the vibration of cross-ply and angle-ply laminated plates that own inhomogeneous elastic properties is considered. The methodology includes a theoretical study of free vibration behavior of composite plates with the inhomogeneous fibrous distribution of the volume fraction using a sinusoidal model by the use of the advanced refined theory of shear deformation of nth-higher-order. The micromechanical typical is proposed to represent the elastic and physical properties of the inhomogeneous laminated composite plate. The effects of inhomogeneity, lamination schemes, aspect ratio, and the number and order of layers on dimensionless vibration frequencies are investigated.

Keywords**:** Vibrations; advanced refined nth-order shear deformation theory; inhomogeneous fibrous; Hamilton's principle.

1. Introduction

The news advanced in the composite material used for the aerospace, motorized industry, marine, civil engineering applications, and other high-performance engineering applications to high performance motivated researchers in structure to develop a precise arithmetic model. Because of their mechanical advantages of specific resistance and specific module compared to traditional materials, these materials improved the resistance to shocks and fatigue, and the flexibility of design to assure the response realistic of the structure. However, the present development permits us to soften the hypothesis that fibers are right for every layer in a composite of fibers laminated. New industrial technology, as the direction of fibers, makes it possible to direct fibers along a wished path. The laminates with variable fiber paths produce unique boundary conditions that produce the transverse stresses and compression local that develop simultaneously. Many studies have been shown to predict the laminates with variable fiber spacing, see, for example, Martin and Leissa [1] who discussed the problem of plane stress of a composite plate with variable content of fibers. Leissa and Martin [2] initially a concept of rigidity variable by varying the spacing of fibers to make progress the presentation of the vibrations and the buckling of the plates anticipated by using the Ritz method. Pandey and Sherbourne [3, 4] studied the stability and pre-buckling stress-field analysis of inhomogeneous, fibrous composite plates. Shiau and Lee [5] presented the concentration of stress around the cavities in the laminated plates at a variable spacing of fibers. Benatta et al. [6] presented the volume fraction of fibers (FVF) across the direction of the thickness of an FG beam and established the influences of various graduations of FVF on the bending of the beam. Bedjilili et al. [7] investigated the vibration of composite beams with variable FVF through the thickness. The

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vibration of a fiber-reinforced composite cylinder with variable FVF has been discussed by Kargarnovin and Hashemi [8]. Kuo [9, 10] used the method of the finite element to study the aero-thermoelastic and stability of laminates to the variable spacing of fibers. Ganesan et al. [11] studied the hybrid fiber effect on the characteristics of cracking and stiffening in traction of the concrete geopolymer. However; few of the studies in the literature [12-15] have considered the composites with variable fiber spacing.

On the other hand, recently to surmount the limits of the traditional approach, these last years, of the new and modified approaches, were proposed to study the behavior in bending, buckling, and vibration of the laminated plate. Draiche et al. [16] proposed a trigonometric four-variable plate theory for analyzing the vibration of rectangular composite plates with patch mass. Thai [17] presented a nonlocal refined beam theory to treat the bending, buckling, and vibration of such beams. Zenkour [18, 19] proposed a theory of quasi-3D for functionally graded (FG) single-layered and sandwich plates with porosities. Fahsi et al. [20] studied the buckling, bending, and vibration behaviors of FG beams under elastic foundation by the new quasi-3D theory, including the effect of Porosity. Derbale et al. [21] developed a new refined theory of shear deformation nth higher-order to obtain the critical mechanical load and critical buckling temperature of simply supported laminated composite beams. Bouazza and Zenkour [22] presented the higher-order approach solution for vibration analysis of FG-CNT rectangular plates. The multi-quadric multiple radial functions (RBF) method was applied to analyze composite plates laminated by Ferreira et al. [23]. Inverse multi-quadric RBFs were used to analyze composite plates by Xiang and Wang [24]. Additional investigations are presented to be concerned with modeling for the thermoelastic buckling, bending, and vibration of different structures [25-37]. Some researchers used the refined four-variable theories for the analysis of the thick FG plate behavior [38-45]. A review of FG thick cylindrical and conical shells presented by Nejad et al. [46]. Mahboobeh et al. [47] studied the FG rotating thick cylindrical pressure vessels with exponentially-varying properties using the power series method of Frobenius.

The fundamental objective of this work is the analysis of free vibration of the laminated plates of inhomogeneous fibrous, using the new refined simplified theory of shear deformation of nth order to two variables. In this approach, we combine the idea of the theory of refined plates established by [16-20] that the author includes w_b and w_s to model the transverse displacement (transverse displacement of bending and shearing) instead of the hypothesis of constant displacement w_0 with the idea of the theory of the shear deformation of order n established by [48-50], the natural frequencies are determined by using the Navier method.

2. Theoretical formulation

2.1 Distribution of fiber

Strategies of Inhomogeneous composite plates are constructed via the function of fiber distribution sinusoidal. A similar function was employed by Pandey and Sherbourne [3] for describing various thickness distributions. Presently, the plates are in the composite of fibers with a variable fiber volume fraction in thickness

$$V_f(z) = \left(V_f\right)_{\mathrm{avg}} \left[\frac{1+(N_{\mathrm{v}}-1)\sin(\pi z)}{1+\frac{2}{\pi}(N_{\mathrm{v}}-1)}\right]. \tag{1}$$

The plate was analyzed as an inhomogeneous orthotropic material over the entire thickness. N_{v} is the ratio of the volume fraction of fiber in the center of the laminate $(z = 0)$ to the side $(z = \pm h/2)$ corresponds to the degree of non-uniformity and, also, the concave nature of convex of the variation of fiber according to $N_{\mathrm{v}} > 1$ or $N_V < 1$, respectively. The case of $N_{\mathrm{v}} < 1$ indicates a concentration of fibers more elevated on sides than on the center and vice versa when $N_{\mathrm{v}} > 1$. The case of $N_{\mathrm{v}} = 1$ which represents uniform variation with a volume fraction of fiber $\left(V_f\right)_{\mathrm{avg}}$.

Besides, to determine the characteristics of the properties of materials, using Eq. (2) which found in the literature (Jones [51]), it is possible to determine the values equivalent to the properties of the material based on the mixture laws, as follows:

$$
\begin{gathered}
E_1 = E_f\left[V_f + R_1\left(1 - V_f\right)\right], \quad E_2 = E_f\left(V_f + \frac{1-V_f}{R_1}\right)^{-1}, \\
\nu_{12} = \nu_f\left[V_f + R_2\left(1 - V_f\right)\right], \quad \nu_{21} = \frac{E_1}{E_2}\nu_{12}, \\
G_{12}(z) = G_f\left(V_f + \frac{1-V_f}{R_3}\right)^{-1},
\end{gathered} \tag{2}
$$

in which

$$
R_1 = \frac{E_m}{E_f}, \quad R_2 = \frac{\nu_m}{\nu_f}, \quad R_3 = \frac{G_m}{G_f}. \tag{3}
$$

2.2 Present new refined simple nth-higher-order shear deformation theory

In this survey, simplification hypotheses supplementary are brought to the theory of the shearing strain of the n order with a reduced number of unknowns. The field of displacement of the traditional theory of shear deformation of n order is given by [48-50]

$$
\begin{gathered}
u_1(x,y,z) = u_0(x,y) + z\phi_x(x,y) - f(z)\left(\phi_x(x,y) + \frac{\partial w_0(x,y)}{\partial x}\right), \\
u_2(x,y,z) = v_0(x,y) + z\phi_y(x,y) - f(z)\left(\phi_y(x,y) + \frac{\partial w_0(x,y)}{\partial y}\right), \\
u_3(x,y,z) = w_0(x,y), \quad f(z) = \frac{1}{n}\left(\frac{2}{h}\right)^{n-1} z^n, \quad n = 3,5,7,\dots,
\end{gathered} \tag{4}
$$

where u_0, v_0, w_0, ϕ_x and ϕ_y are five unknown functions of displacements and rotations of the mid-plane of the plate ($z = 0$) and h its thickness. By dividing the displacement w_0 into two parts; of bending w_b and shearing w_s and one put other hypotheses $\phi_x = -\frac{\partial w_b}{\partial x}$ and $\phi_y = -\frac{\partial w_b}{\partial y}$, the field of displacement of the current approach can be considered as follows:

$$
\begin{gathered}
u_1(x,y,z) = u_0(x,y) - z\frac{\partial w_\mathrm{b}}{\partial x} - f(z)\frac{\partial w_\mathrm{s}}{\partial x}, \\
u_2(x,y,z) = v_0(x,y) - z\frac{\partial w_\mathrm{b}}{\partial y} - f(z)\frac{\partial w_\mathrm{s}}{\partial y}, \\
u_3(x,y,z) = w_b(x,y) + w_s(x,y), \quad n = 3,5,7,\dots.
\end{gathered} \tag{5}
$$

There are thus more sophisticated analyzes while minimizing the number of unknown variables with only four unknown functions. However, the conventional n-order theory of shear deformation using five unknown functions. The displacement field of this new approach becomes as follows:

$$
\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z\begin{Bmatrix} \kappa_x^\mathrm{b} \\ \kappa_y^\mathrm{b} \\ \kappa_{xy}^\mathrm{b} \end{Bmatrix} + f(z)\begin{Bmatrix} \kappa_x^\mathrm{s} \\ \kappa_y^\mathrm{s} \\ \kappa_{xy}^\mathrm{s} \end{Bmatrix}, \quad \{\gamma_{xz}, \gamma_{yz}\} = g(z)\{\gamma_{xz}^0, \gamma_{yz}^0\}, \quad \varepsilon_z = 0, \tag{6}
$$

where

$$
\begin{gathered}
\varepsilon_x^0 = \frac{\partial u_0}{\partial x}, \quad \varepsilon_y^0 = \frac{\partial v_0}{\partial y}, \quad \gamma_{xy}^0 = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}, \quad \kappa_\eta^\zeta = -\frac{\partial^2 w_\zeta}{\partial \eta^2}, \quad \kappa_{xy}^\zeta = -2\frac{\partial^2 w_\zeta}{\partial x \partial y}, \\
\gamma_{\eta z}^0 = \frac{\partial w_s}{\partial \eta}, \quad g(z) = 1 - f'(z), \quad \eta = x, y, \quad \zeta = \mathrm{b}, \mathrm{s}.
\end{gathered} \tag{7}
$$

The behavior of laminated plates, therefore, obeys the constitutive law as follows:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & Q_{44} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}, \tag{8}$$

where

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}, \quad Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}, \quad Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}, \quad Q_{66} = G_{12}, \quad Q_{55} = G_{13}, \quad Q_{44} = G_{23}. \tag{9}$$

We deduce from it the constitutive relations of the global coordinate system, by transforming the constitutive relations of an arbitrary layer k in the base of orthotropic in the global coordinate system, the behavior equations of laminated is written as follows:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix}^{(k)} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} & 0 & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} & 0 & 0 \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} & 0 & 0 \\ 0 & 0 & 0 & \bar{Q}_{55} & \bar{Q}_{45} \\ 0 & 0 & 0 & \bar{Q}_{45} & \bar{Q}_{44} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}^{(k)}. \tag{10}$$

In the previous equation, we introduced the standard notations used for the components of shear and normal stress. $\bar{Q}_{ij}$ are the transformed material constants given as [51, 52]. The moments and shear forces are defined as

$$\begin{gathered} \{N_x, N_y, N_{xy}\} = \int_{-h/2}^{h/2} \{\sigma_x, \sigma_x, \tau_{xy}\} \mathrm{d}z = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \{\sigma_x, \sigma_x, \tau_{xy}\} \mathrm{d}z\,, \\ \{M_x^b, M_y^b, M_{xy}^b\} = \int_{-h/2}^{h/2} \{\sigma_x, \sigma_x, \tau_{xy}\} z\mathrm{d}z = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \{\sigma_x, \sigma_x, \tau_{xy}\} z\mathrm{d}z\,, \\ \{M_x^s, M_y^s, M_{xy}^s\} = \int_{-h/2}^{h/2} \{\sigma_x, \sigma_x, \tau_{xy}\} f(z) \mathrm{d}z = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \{\sigma_x, \sigma_x, \tau_{xy}\} f(z) \mathrm{d}z\,, \\ \{Q_x^s, Q_y^s\} = \int_{-h/2}^{h/2} \{\tau_{xz}, \tau_{yz}\} g(z) \mathrm{d}z = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \{\tau_{xz}, \tau_{yz}\} g(z) \mathrm{d}z\,. \end{gathered} \tag{11}$$

Using expressions (6)-(10) in Eq. (11), expressions for stress resultants (N_x, N_y, N_{xy}) moments $(M_x^\zeta, M_y^\zeta, M_{xy}^\zeta)$, $(\zeta = b, s)$ and shear forces (Q_x^s, Q_y^s) can be obtained. These expressions are:

$$\begin{Bmatrix} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} \\ \begin{Bmatrix} M_x^b \\ M_y^b \\ M_{xy}^b \end{Bmatrix} \\ \begin{Bmatrix} M_x^s \\ M_y^s \\ M_{xy}^s \end{Bmatrix} \end{Bmatrix} = \begin{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} & \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} & \begin{bmatrix} B_{11}^s & B_{12}^s & B_{16}^s \\ B_{12}^s & B_{22}^s & B_{26}^s \\ B_{16}^s & B_{26}^s & B_{66}^s \end{bmatrix} \\ \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} & \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} & \begin{bmatrix} D_{11}^s & D_{12}^s & D_{16}^s \\ D_{12}^s & D_{22}^s & D_{26}^s \\ D_{16}^s & D_{26}^s & D_{66}^s \end{bmatrix} \\ \begin{bmatrix} B_{11}^s & B_{12}^s & B_{16}^s \\ B_{12}^s & B_{22}^s & B_{26}^s \\ B_{16}^s & B_{26}^s & B_{66}^s \end{bmatrix} & \begin{bmatrix} D_{11}^s & D_{12}^s & D_{16}^s \\ D_{12}^s & D_{22}^s & D_{26}^s \\ D_{16}^s & D_{26}^s & D_{66}^s \end{bmatrix} & \begin{bmatrix} H_{11}^s & H_{12}^s & H_{16}^s \\ H_{12}^s & H_{22}^s & H_{26}^s \\ H_{16}^s & H_{26}^s & H_{66}^s \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} \\ \begin{Bmatrix} \kappa_x^b \\ \kappa_y^b \\ \kappa_{xy}^b \end{Bmatrix} \\ \begin{Bmatrix} \kappa_x^s \\ \kappa_y^s \\ \kappa_{xy}^s \end{Bmatrix} \end{Bmatrix}, \tag{12a}$$

$$\begin{Bmatrix} Q_y^s \\ Q_x^s \end{Bmatrix} = \begin{bmatrix} A_{44}^s & A_{45}^s \\ A_{45}^s & A_{55}^s \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^0 \\ \gamma_{xz}^0 \end{Bmatrix}, \tag{12b}$$

where A_{ij}, B_{ij}, … etc. are the plate stiffness, defined by

$$\begin{gathered} \{A_{ij}, B_{ij}, D_{ij}, B_{ij}^s, D_{ij}^s, H_{ij}^s\} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \bar{Q}_{ij}^{(k)} \{1, z, z^2, f(z), zf(z), [f(z)]^2\} \mathrm{d}z\,, \\ A_{pq}^s = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \bar{Q}_{pq}^{(k)} [g(z)]^2 \mathrm{d}z\,, \quad i,j = 1,2,6, \quad p,q = 4,5. \end{gathered} \tag{13}$$

We retain here Hamilton's principle. This principle considers that the sum of the variation of kinetic and potential energies and the work of non-conservative forces between two times t_1 and t_2 is zero, that is:

$$\delta^{(1)} \int_{t_1}^{t_2} (T-U)\mathrm{d}t = 0, \tag{14}$$

where T is the kinetic energy given by

$$T = \frac{1}{2}\iint_{\Omega} \int_{-h/2}^{h/2} \rho \dot{u}_i \dot{u}_i \mathrm{d}z\, \mathrm{d}\Omega, \tag{15}$$

and U is the total potential energy represented as

$$U = \frac{1}{2}\iint_{\Omega} \left[\int_{-h/2}^{h/2} \left(\sigma_x\varepsilon_x + \sigma_x\varepsilon_x + \tau_{yz}\gamma_{yz} + \tau_{xz}\gamma_{xz} + \tau_{xy}\gamma_{xy}\right)\mathrm{d}z - q(w_\mathrm{b} + w_\mathrm{s})\right]\mathrm{d}\Omega = 0. \tag{16}$$

Using Eqs. (5), (6), (12), (15) and (16), in Eq. (14), we then obtain the following equations of movement: The work of the external forces is given by the following formula ($q = 0$):

$$\delta u_0: \ \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_1\ddot{u}_0 - I_2\frac{\partial \ddot{w}_b}{\partial x} - I_4\frac{\partial \ddot{w}_s}{\partial x}, \tag{17a}$$

$$\delta v_0: \ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = I_1\ddot{v}_0 - I_2\frac{\partial \ddot{w}_b}{\partial y} - I_4\frac{\partial \ddot{w}_s}{\partial y}, \tag{17b}$$

$$\delta w_\mathrm{b}: \ \frac{\partial^2 M_x^b}{\partial x^2} + 2\frac{\partial^2 M_{xy}^b}{\partial x\partial y} + \frac{\partial^2 M_y^b}{\partial y^2} = I_1(\ddot{w}_\mathrm{b} + \ddot{w}_\mathrm{s}) + I_2\left(\frac{\partial \ddot{u}_0}{\partial x} + \frac{\partial \ddot{v}_0}{\partial y}\right) - I_3\nabla^2\ddot{w}_\mathrm{b} - I_5\nabla^2\ddot{w}_\mathrm{s}, \tag{17c}$$

$$\delta w_b: \ \frac{\partial^2 M_x^s}{\partial x^2} + 2\frac{\partial^2 M_{xy}^s}{\partial x\partial y} + \frac{\partial^2 M_y^s}{\partial y^2} + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = I_1(\ddot{w}_\mathrm{b} + \ddot{w}_\mathrm{s}) + I_4\left(\frac{\partial \ddot{u}_0}{\partial x} + \frac{\partial \ddot{v}_0}{\partial y}\right)$$
$$-I_5\nabla^2\ddot{w}_\mathrm{b} - I_6\nabla^2\ddot{w}_\mathrm{s}, \tag{17d}$$

where

$$\{I_1, I_2, I_3, I_4, I_5, I_6\} = \int_{-h/2}^{h/2} \rho\{1, z, z^2, f(z), zf(z), [f(z)]^2\}\mathrm{d}z. \tag{18}$$

The following approximate solution is seen to satisfy both the differential equation and the boundary conditions.

2.2.1 Cross-ply laminates

$$\begin{Bmatrix} (w_\mathrm{b}, w_\mathrm{s}) \\ u_0 \\ v_0 \end{Bmatrix} = \sum_{m=1}^{\infty}\sum_{l=1}^{\infty} \begin{Bmatrix} (W_{\mathrm{b}ml}, W_{sml})\sin(\alpha x)\sin(\beta y) \\ U_{ml}\cos(\alpha x)\sin(\beta y) \\ V_{ml}\sin(\alpha x)\cos(\beta y) \end{Bmatrix} \mathrm{e}^{-i\omega t}. \tag{19}$$

2.2.2 Antisymmetric angle-ply laminates

$$\begin{Bmatrix} (w_\mathrm{b}, w_\mathrm{s}) \\ u_0 \\ v_0 \end{Bmatrix} = \sum_{m=1}^{\infty}\sum_{l=1}^{\infty} \begin{Bmatrix} (W_{\mathrm{b}ml}, W_{sml})\sin(\alpha x)\sin(\beta y) \\ U_{ml}\sin(\alpha x)\sin(\beta y) \\ V_{ml}\cos(\alpha x)\sin(\beta y) \end{Bmatrix} \mathrm{e}^{-\mathrm{i}\omega t}. \tag{20}$$

3. Results and discussion

In this part, diverse examples are illustrated and analyzed to check the precision and the efficiency of this approach in the computation of the behavior of isotropic, orthotropic, and laminates in free vibration. For purposes of checking, the numerical outcomes obtained by the current model are compared with available results in the published references.

3.1 Comparison studies

We will first study the case of an isotropic square plate, a comparative study is carried out between the present theory and the various theories of plates, the exact three-dimensional elasticity solutions of Srinivas and Rao [25], higher-order shear deformation theory by Reddy and Phan [26] and analytical approach by Hosseini Hashemi et al. [27] in Table 1 where. The following plate parameters are adapted to the comparison: $E_1/E_2 = 1$ and $\nu = 0.3$. The natural frequencies factors $\bar{\omega} = \omega(b^2/\pi^2)\sqrt{\rho h/D}$ are presented for isotropic plates for various modes. Shapes of vibration of modes are defined per m and n, where these whole numbers respectively indicate the number of half-waves the x- and y-directions.

Table1. Comparison of non-dimensional out-of-plane natural frequencies $\bar{\omega}$ for simply-supported isotropic square plate ($\nu = 0.3$, $h/a = 0.1$)

a/b	Mode	3D [25]	TSDT [26]	TSDT [27]	Present			
					$n=3$	$n=5$	$n=7$	$n=9$
1	(1,1)	0.0932	0.0931	0.0930	0.0917	0.0926	0.0929	0.0930
	(1,2)	0.226	0.2222	0.2220	0.2151	0.2198	0.2213	0.2220
	(2,2)	0.3421	0.3411	0.3406	0.3256	0.3358	0.3256	0.3358
	(1,3)	0.4171	0.4158	0.4151	0.3937	0.4082	0.4129	0.4152
	(2,3)	0.5239	0.5221	0.5208	0.4890	0.5105	0.4890	0.5105
	(1,4)	---	0.6545	0.6525	0.6059	0.6372	0.6476	0.6528
	(3,3)	0.6889	0.6862	0.6840	0.6335	0.6674	0.6787	0.6843
	(2,4)	0.7511	0.7481	0.7454	0.6872	0.7262	0.6872	0.7262
	(3,4)	---	0.8949	0.8908	0.8132	0.8649	0.8826	0.8914
	(1,5)	0.9268	0.9230	0.9187	0.8371	0.8914	0.9100	0.9193
$1/\sqrt{2}$	(1,1)	0.0704	0.0703	0.0704	0.0696	0.0701	0.0703	0.0704
	(1,2)	0.1376	0.1373	0.1373	0.1345	0.1364	0.1370	0.1373
	(2,1)	0.2018	0.2014	0.2012	0.1955	0.1994	0.2007	0.2013
	(1,3)	0.2431	0.2426	0.2424	0.2343	0.2398	0.2416	0.2424
	(2,2)	0.2634	0.2628	0.2625	0.2531	0.2596	0.2616	0.2626
	(2,3)	0.3612	0.3601	0.3600	0.3430	0.3543	0.3579	0.3597
	(1,4)	0.3800	0.3789	0.3783	0.3601	0.3725	0.3764	0.3784
	(3,1)	0.3987	0.3974	0.3968	0.3770	0.3904	0.3948	0.3969
	(3,2)	0.4535	0.4519	0.4511	0.4263	0.4431	0.4485	0.4512
	(2,4)	0.4890	0.4873	0.4863	0.4581	0.4771	0.4834	0.4865

Table 2. Comparison non-dimensional natural frequencies $\hat{\omega}$ of a simply-supported square orthotropic plate with different thickness-to-side ratios.

h/a	CPT [28]	FSDT [28]	SHT [28]	FEM [29]	Present			
					$n=3$	$n=5$	$n=7$	$n=9$
0.1	0.03800	0.03615	0.03617	0.3719	0.0355	0.0361	0.0363	0.0364
0.2	0.14844	0.12597	0.12628	0.13045	0.1199	0.1256	0.1274	0.1283
0.3	0.32181	0.24098	0.24226	0.24277	0.2233	0.2397	0.2454	0.2483
0.4	0.54541	0.36561	0.36885	0.36712	0.3324	0.3631	0.3742	0.3799
0.5	0.80626	0.49366	0.50021	0.48719	0.4434	0.4897	0.5070	0.5160

In the second part, we are interested in verifying the results of the present theory of an orthotropic plate with simple boundary conditions with results available in the references published for different ratios thickness/side, the numerical values are shown in Table 2. In this case, the natural frequency parameter is determined as $\hat{\omega} = \omega h\sqrt{\rho/E_1}$. The frequencies of Reddy [28] and Liu [29] are reported in Table 2. The plate parameters are:

$$E_1 = 4.0714E_2, \quad E_3 = E_2, \quad G_{12} = G_{13} = 0.4071\, E_2, \quad G_{23} = 0.3571\, E_2,$$
$$\nu_{12} = 0.277, \quad \nu_{13} = 0.068, \quad \nu_{23} = 0.4. \tag{21}$$

Based on the comparison of the dimensionless fundamental natural frequency for a square orthotropic composite plate with simple supports as a function of h/a. From the results indicated in Table 2, we can observe the non-dimensional natural frequencies moreover increase regularly when ratios thickness/side (h/a) increased from 0.1 to 0.5. As the increase of the thickness-to-side ratio, the difference between the values of the present theory and the classical theory of the plates increases. It is also noted that the transverse shear strain has a certain effect on the natural frequencies.

Thirdly, in Table 3 the results of non-dimensionalized fundamental frequencies $\tilde{\omega} = \omega(a^2/h)\sqrt{\rho/E_2}$, of the composite plate with simply-supported the stacking sequence (0°/90°/90°/0°) are compared with the 3D elasticity solutions by Noor [30], the theory of shear deformation the higher-order (HSDT) by Phan and Reddy [31], and the finite element of three-dimensional (3D-FEM) by Rao and Sinha [32]. The material parameters are assumed to be: $E_1/E_2 =$ open, $G_{12} = G_{13} = 0.6\, E_2$, $G_{23} = 0.5\, E_2$, $\nu_{12} = 0.25$. The comparisons are well justified.

Table 3. Comparison of non-dimensional natural frequencies $\tilde{\omega}$ of simply-supported four-layered square cross-ply (0°/90°/90°/0°) laminates plate, $(a/h = 5)$, Material I.

E_1/E_2	3D elasticity [30]	HSDT [31]	3D-FEM [32]	Present
3	6.6815	6.5597	6.5778	6.6003
10	8.2103	8.2718	8.2791	8.5731
20	9.5603	9.5263	9.5033	10.1516
30	10.272	10.272	10.2132	11.1132
40	10.752	10.787	10.6916	11.7710

Table 4. Comparison of results with the non-dimensional fundamental frequency $\breve{\omega}$ for a simply-supported square laminated plate.

Stacking sequence	Mode	DSC [33]	CLPT [34]	EFG [35]	Present			
					$n = 3$	$n = 5$	$n = 7$	$n = 9$
(0°/0°/0)	1	15.171	15.17	15.18	15.1684	15.1685	15.1685	15.1685
	2	33.248	33.32	33.34	33.2390	33.2392	33.2393	33.2395
(15°/-15°/15°)	1	15.469	15.40	15.41	15.5282	15.5282	15.5282	15.5283
	2	34.153	34.12	34.15	34.3804	34.3806	34.3808	34.3809
(30°/-30°/30°)	1	16.058	15.87	15.88	16.2236	16.2237	16.2237	16.2238
	2	36.060	35.92	35.95	37.1311	37.1313	37.1316	37.1318
(45°/-45°/45°)	1	16.348	16.10	16.11	16.5604	16.5605	16.5605	16.5606
	2	37.146	37.00	37.04	40.1739	40.1741	40.1744	40.1747

Finally, The non-dimensionalized frequencies, for simply supported square laminated composite plates of three-layer with various orientations (0°/0°/0°), (15°/-15°/15°), (30°/-30°/30°), and (45°/-45°/45°) are presented in Table4 compared with the ones obtained from the discrete singular convolution (DSC) by Secgin and Sarigul [33], classical laminated plate theory (CLPT) by Dai et al. [34] and element free Galerkin method (EFG) by Chen et al. [35]. In this example, the non-dimensional frequency is given as $\breve{\omega} = \omega a^2\sqrt{\rho h/D_{01}}$ employing another arbitrary rigidity expression (i.e., $D_{01} = E_1 h^3/\big(12(1-\nu_{12}\nu_{21})\big)$. In the reason of comparison of the results, the properties of the materials and the geometrical parameters are identical as in Secginand Sarigul [33]: thickness $h = 0.06$ m, length $a = b = 10$ m, elastic constants ratio $E_1/E_2 = 2.45$; $G_{12}/E_2 = 0.48$; Poisson's ratio $\nu_{12} = 0.23$ and mass density of $\rho = 8000$ kg/m^3.

3.2 Parametric studies

In this part, all calculations are made for graphite-epoxy composite plates (Leissa and Martin [2], Pandey and Sherbourne [4]) with the constants of the materials following: $E_f = 275.8$ GPa, $E_m = 3.44$ GPa, $\nu_f = 0.20$ and $\nu_m = 0.35$. Average fiber volume fraction, $(V_f)_{\text{avg}}$, is taken as 50%.

The sinusoidal fiber distribution function according to the value of N_V simulates a variety of distributions as shown in Figs. 1-3. It is right to underline that the total quantity of fibers is constant and equal to the one of a plate to uniform distribution and fraction by volume of fibers, $(V_f)_{\text{avg}}$. This function is very practical to analyze the advantages of a distribution non-uniform on a uniform for a given quantity of fibers.

Studies of parameters are led to analyze non-homogeneous effect, side-to-thickness ratios a/h, the effect of fiber material type, and lamination angle on the nondimensional natural frequencies of cross-ply plates.

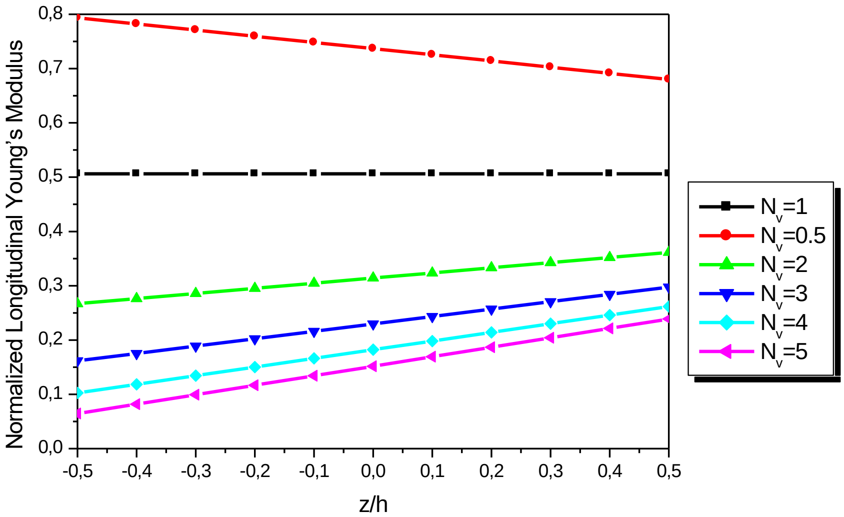

Fig. 1. Normalized longitudinal Young's modulus for various values of N_{v}.

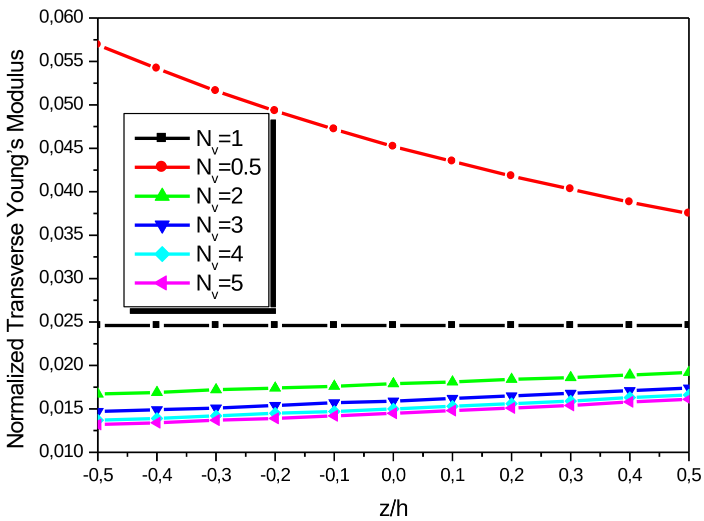

Fig. 2. Normalized transverse Young's modulus for various values of N_{v}.

The obtained results in Figure 4 show that the non-dimensional natural frequencies $\varpi = \omega(a^2/h)\sqrt{\rho/D_{11}}$ because the inclusion of the non-homogeneous effect is meaningful for all ratios of thickness considered. The uniform distribution with a fiber corresponds to ($N_{\text{v}} = 1$), the higher fiber concentration at the edges than the center with ($N_{\text{v}} = 0.5$) and ($N_{\text{v}} = 2,3,4,5$) implies higher fiber concentration at the center than the edges.

The variation of the non-dimensional natural frequencies ϖ for simply supported four-layered square $(\theta/-\theta/-\theta/\theta)_s$ laminates plate having sinusoidal fiber distributions and homogeneous plate, for the value of thickness ratio ($a/h = 10$), presented in Figure 5. Here, the non-dimensional natural frequencies decrease as θ rises from 0 to about 20 degrees and after rises as θ rises until $\theta = 70°$ besides, thereafter, decreases as θ increases.

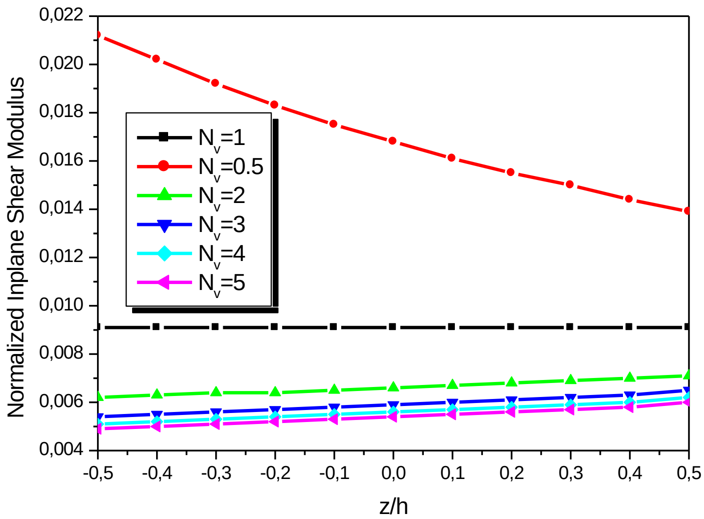

Fig. 3. Normalized in-plane shear modulus for various values of N_{v}.

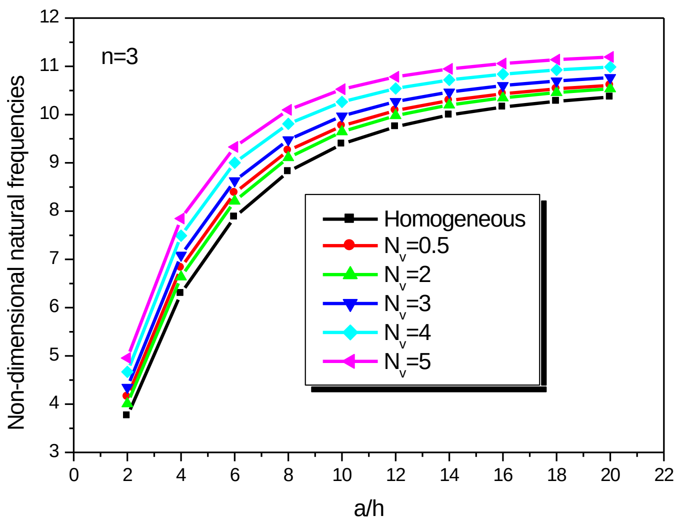

Fig. 4. Effect of the aspect ratio on the non-dimensional natural frequencies ϖ of simply-supported three-layered square cross-ply (0°/90°/0°) laminates plate for various values of N_{v}.

The non-dimensional natural frequencies fibrous composite plate ϖ for simply-supported with different ply orientation an angle is shown in Figure 6. Two different lamination schemes $(\theta/-\theta)_s$ and $(\theta/-\theta)_2$, being the ply orientation angle, are considered. For the sinusoidal fiber distributions corresponding to $N_v = 0.5$, the non-dimensional natural frequencies are asymmetric for both the lamination schemes, and the non-dimensional natural frequencies are seen when the ply orientation angle is $70°$.

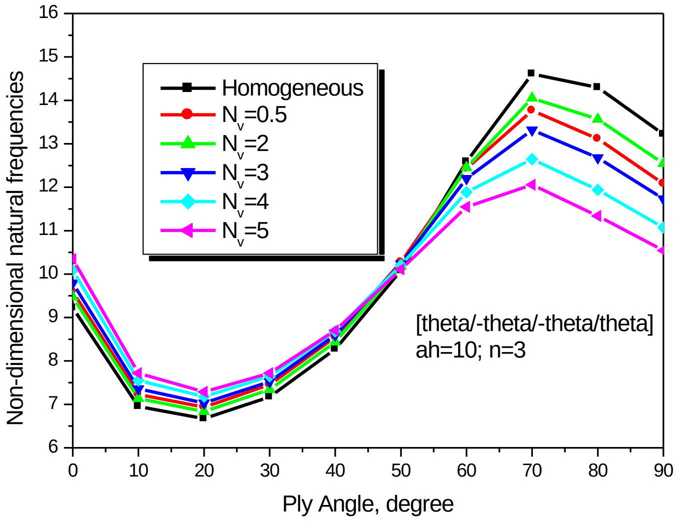

Fig. 5. Effect of ply angle on non-dimensional natural frequencies ϖ for of simply-supported four-layered square cross-ply $(\theta/-\theta/-\theta/\theta)$ having sinusoidal fiber distributions.

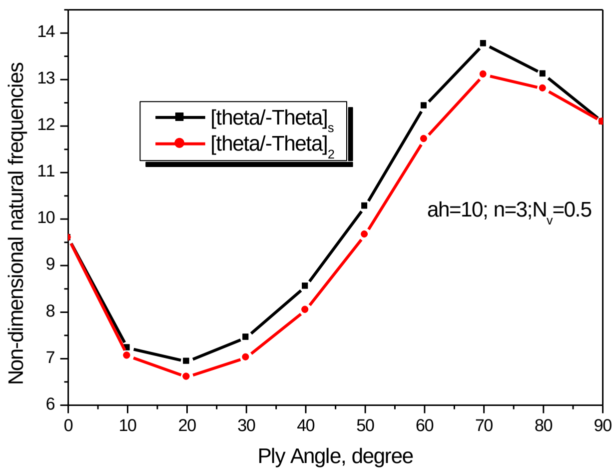

Fig. 6. Effect of lamination angle on non-dimensional natural frequencies ϖ of the laminated square plate having sinusoidal fiber distributions ($N_v = 0.5$).

4. Conclusions

To analyze the problems of non-homogeneous laminated plates in free vibrations. In this study examples of composite plates whose Young's moduli vary continuously piecewise in the direction of the thickness. We will determine numerically, for the different examples, non-dimensionalized frequencies, from analytical expressions by Navier's method. Finally, the effects of various parameters are presented. The numerical results support the following conclusions:

- A good agreement between the results of this theory and the values of the literature, as shown in the section on comparative studies.
- Unlike other theories, transverse displacement is considered to be the combined effect of the bending and shear component, therefore, the effects of transverse shear deformation and/or normal transverse deformation are taken into account. This approach does not require correction factors.
- The classical theory of the plates seems a particular case of the current theory.
- It is found that decreasing the value of the side-to-thickness ratio leads to an increase in the non-dimensional frequencies.
- It is noted that the reduction in the value of the ratio side/thickness involves a rise in the non-dimensional frequencies.
- The values of the non-dimensional parameter of frequency in fibrous composite plates in the homogeneous and inhomogeneous cases are affected by the order and the number of the plies appreciably.
- The ordering and the sequence of the layers influence the non-dimensional frequency parameter.

Conflict of interest statement

The authors declare that they have no conflict of interest.

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