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A Survey on Tenacity Parameter Part I

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ABSTRACT

If we think of the graph as modeling a network, the vulnerability measure the resistance of the network to disruption of operation after the failure of certain stations or communication links. In assessing the "vulnerability" of a graph one determines the extent to which the graph retains certain properties after the removal of vertices and / or edges. Many graph theoretical parameters have been used to describe the vulnerability of communication networks, including connectivity, integrity, toughness, binding number, tenacity and... .
In this paper we survey and discuss tenacity and its properties in vulnerability calculation and we will compare different measures of vulnerability with tenacity for several classes of graphs.

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1 PRELIMINARIES

Throughout this paper we will let n be the number of vertices of G, and we use $\alpha(G)$ to denote the independence number of G. Let A be a subset of V(G). The neighborhood

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of A, N(A), consists of all vertices of G adjacent to at least one vertex of A. We define G-A to be the graph induced by the vertices of V-A. Also, for any graph G, τ(G) is the number of vertices in a largest component of G and ω(G) is the number of components of G. A cutset of a connected graph G is a collection of vertices whose removal results in a disconnected graph.
Since we are primarily interested in the case where disruption of the graph is caused by the removal of a vertex or vertices (and the resulting loss of all edges incident with the removed vertices), we shall restrict our discussion to vertex stability measures. In the interest of completeness, however, we have included several related measures of edge stability.
The first two measures provide information about how easily the graph can be broken-up by the removal of specific sets of vertices.
The vertex connectivity [15, 16, 17], $\kappa = \kappa(G)$, of a finite, undirected, connected, simple graph G (without loops or multiple edges) is the minimum number of vertices whose removal results in a disconnected graph or results in the trivial graph K_1. Graph G is called n-connected if $\kappa \geq n$. Analogously, the edge-connectivity [15, 16, 17], $\lambda = \lambda(G)$, of a finite, undirected, connected simple graph G is the minimum number of edges whose removal results in a disconnected or trivial graph K_1. A graph G is called n-edge- connected if $\lambda(G) \geq n$.
A collection of vertices in V(G) is called a cutset if their removal disconnects G, and a collection of edges in V(G) is called an edge-cutset if their removal disconnects G.
The binding number of a graph G was defined by Woodall in [50], as

$$bind(G) = \min_{A}\{\frac{\mid N(A) \mid}{\mid A \mid}\}$$

where $\phi \neq A \subseteq V(G)$ and $N(A) \neq V(G)$. In [51, 52], the binding number was called the melting-point of the graph. the reason for the name "binding number" is that, roughly speaking, if bind(G) is large, then the vertices of G are bound tightly together, in the sense that G has many edges fairly well distributed.
We stat some of the results in [50].

(1) $bind(K_n) = n - 1$ for $n \geq 1$.

(2) $bind(K_{a,b}) = min(\frac{a}{b}, \frac{b}{a})$ for $(a \geq 1, b \geq 1)$.

(3) If $G = C_n$, with $n \geq 3$, then $bind(G) = \begin{cases} 1, \text{for } n \text{ } even, \\ \frac{n-1}{n-2}, \text{for } n \text{ } odd. \end{cases}$

(4) If $G = P_n$, with $n \geq 1$, then $bind(G) = \begin{cases} 1, \text{for } n \text{ } even \\ \frac{n-1}{n+1}, \text{for } n \text{ } odd. \end{cases}$

Kane, Mohanty and Hales [25], studied the binding numbers of four types of product graphs : cartesian product, tensor product, strong cartesian product and lexicographic product. Since it is difficult to determine the binding numbers of products of arbitrary

graphs, they restricted themselves to products of two graphs which could be any one of the following types of graphs : complete graph (K_n), complete bipartite graph ($K_{m,n}$), cycle (C_n) and path (P_n).
In [50] Woodall proved that, if $bind(G) \geq c$, then G contains at least $\frac{|V(G)|c}{c+1}$ disjoint edges if $0 \leq c \leq \frac{1}{2}$, at least $\frac{|V(G)|(3c-2)}{3c} - \frac{2(c-1)}{c}$ disjoint edges if $1 \leq c \leq \frac{4}{3}$, a Hamiltonian circuit if $c \geq \frac{3}{2}$, and a circuit of length at least $\frac{3(|V(G)|-1)(c-1)}{c}$ if $1 < c \leq \frac{3}{2}$.

The next set of measures also take into consideration the structure of the graph G-A. In particular, they reflect how badly the graph G-A has been disconnected. Since we must ultimately face the reconnection problem - repairing a broken network - these measures could prove to be very useful.

The concept of integrity of a graph G was introduced in [3, 4], as a useful measure of the vulnerability of a graph G. If we think of the graph as modeling a network, vulnerability parameters measure the resistance of the network to disruption of operation after the failure of certain stations. The vertex integrity of a graph G, is defined as $I(G) = min\{| A | + \tau(G - A)\}$, where the minimum is taken over all $A \subseteq V(G)$ and $\tau(G - A)$ is the maximum order of a component of G-A.
The integrity is a measure which deals with the first fundamental question. How many vertices can still communicates? Integrity has been studied in numerous papers, including [3, 12].
In [4], Barefoot, Entringer and Swart compared integrity, connectivity, binding number and toughness for several classes of graphs. The integrities of the several classes of graphs calculated in [4], were determined using ad hoc methods. Any set A with the property that $| A | + \tau(G-A) = I(G)$ is called an I-set of G. The corresponding edge version called the edge-integrity I'(G) is defined as $I'(G) = min\{| E' | + \tau(G-E')\}$, where the minimum is taken over all $E' \subseteq E(G)$. Thus, for instance, a small edge-integrity is in some sence a measure of how a graph can be split into "small pieces" by the removal of a "few" edges. Bagga, Beineke, Lipman and Pippert in [2], first listed some basic facts about the edge integrity : In [20] Fellows and Stuekle studied the computational complexity of edge - integrity. In [2], a new lower bound on the edge integrity of graphs in general is given, but most of the results concern trees.

The toughness of a graph G was introduced by Chvátal in [10], who observed the relationship between this parameter and the existence of Hamilton cycles in the given graph, and several results regarding this invariant were obtained. The original approach to toughness is as follows. A connected graph G is called t-tough if $t\omega(G - A) \leq | A |$ for any subset A of V(G) with $\omega(G - A) > 1$, [13,23,37,38]. If G is not complete, then there is a largest t such that G is t-tough; this number is the toughness of G and denoted by t(G). Thus $t(G) = min\{\frac{|A|}{\omega(G-A)}\}$, where A is a cutset of G. Since a complete graph has no cutset A, we set $t(K_n) = \infty$ for all $n \geq 1$.

An alternate definition is easier to apply in some cases. Let G be an (n,e) gragh of connectivity κ, $G \neq K_n$, $\omega_p = max\{\omega(G-A)\}$,where $\mid A \mid = p$, and $t_p = \frac{p}{\omega_p}$. Then G is t-tough for $0 \leq t \leq \min(t_p)$, where $\kappa \leq p$. The toughness deals with the second fundemental question, namely, how difficult is it to reconnect the graph ? The toughness has been studied extensively; see for example [19, 38].

The tenacity is another vulnerability measure, incorporating ideas of both toughness and integrity and dealing with both of the above questions, [13, 14]. The tenacity of a graph G, T(G), is defined by $\mathrm{T(G)} = \min\{\frac{|\mathrm{A}|+\tau(\mathrm{G-A})}{\omega(\mathrm{G-A})}\}$, where the minimum is taken over all vertex cutset A of G, G-A is the graph induced by the vertices of V-A, $\tau(\mathrm{G}-\mathrm{A})$ is the number of vertices in the largest component of the graph induced by G-A and $\omega(\mathrm{G}-\mathrm{A})$ is the number of compnents of G-A. A connected graph G is called T-tenacious if $\mid \mathrm{A} \mid + \tau(\mathrm{G}-\mathrm{A}) \geq \mathrm{T}\omega(\mathrm{G}-\mathrm{A})$ holds for any subset A of vertices of G with $\omega(\mathrm{G}-\mathrm{A}) > 1$. If G is not complete, then there is a largest T such that G is T-tenacious; this T is the tenacity of G. On the other hand, a complete graph contains no vertex cutset and so it is T-tenacious for every T. Accordingly, we define $\mathrm{K_p} = \infty$ for every p ($\mathrm{p} \geq 1$). A set $\mathrm{A} \subseteq \mathrm{V(G)}$ is said to be a T-set of G if $\mathrm{T(G)} = \frac{|\mathrm{A}|+\tau(\mathrm{G-A})}{\omega(\mathrm{G-A})}$.

We also consider the edge-tenacity, T'(G), defined by $T'(G) = min\{\frac{|F|+\tau(G-F)}{\omega(G-F)}\}$, where the minimum is taken over all edge cutset F of G. A set $F \subseteq E(G)$ is said to be a T'-set of G if $T'(G) = \frac{|F|+\tau(G-F)}{\omega(G-F)}$.

We will compare integrity, connectivity, binding number, toughness and tenacity for several classes of graphs. The results suggest that tenacity is a most suitable measure of vulnerability in that it is best able to distinguish between graphs that intuitively should have different levels of vulnerability.

There exist other stability measures such as the edge-connectivity vector [29], the ratio of disruption [28], the complement of disruption, the cut frequency vector, cohesion [41, 43], and neighbor-connectivity [17].

2 Calculation of Vulnerability Measure

Let $C_n = (v_1 v_2 \cdots v_n)$ be the n-cycle and define the k-th power of the n-cycle, C_n^k by $C_n^k = C_n + \{v_i v_j \mid\mid i-j \mid \leq k\}$.
We wil calculate the five measures of vulnerability for the complete bipartite graph $K_{k,n-k}$, $k \leq n-k$, powers C_n^k of the n-cycle, and the graph $G_(n,k)$, $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$, has n vertices and vertex v which is adjacent to all vertices of the two complete subgraphs, copies of K_k and K_{n-k-1}, i.e $G_{n,k} \equiv K_1 + (K_k \cup K_{n-k-1})$.
These graphs were purposefully chosen, because they exhibit the widest possible rang of edge density and because they illustrate where the different measures of vulnerability differ in their effectiveness in measurring important structural characteristics of graphs.

Lemma 2.1 : If A is a minimal T-set for C_n^k, then A consists of the union of sets of k consecutive vertices such that there exists at least one vertex not in A between any two sets of consecutive vertices in A.

Proof : We assume C_n^k is labeled by $0, 1, 2, \cdots, n-1$. Let A be a minimal T-set for C_n^k and j be the least integer such that $S = \{j, j+1, \cdots, j+t-1\}$ is a maximal set of consecutive vertices such that $S \subseteq A$. Re-label the vertices of C_n^k as $v_1 = j, v_2 = j+1, \cdots, v_t = j+t-1, \cdots, v_n$. Since $A \neq V(C_n^k)$, $S \neq V(C_n^k)$ so v_n does not belong to A. Since A must leave at least two components, $t \neq n-1$, so $v_{t+1} \neq v_n$. Therefore $\{v_{t+1}, v_n\} \cap A = \phi$. Now suppose $t < k$. Choose v_i such that $1 \leq i \leq t$, and delete v_i from A yielding a new set $A' = A - \{v_i\}$ with $\mid A' \mid = \mid A \mid -1$. The edges $v_i v_n$ and $v_i v_t + 1$ are in $C_n^k - A'$. Consider a vertex v_p adjacent to v_i in $C_n^k - A'$, if $p \geq t+1$, then $p < t+k$, so v_p is also adjacent to v_{t+1} in $C_n^k - A'$ and if $p < n$ then $p \geq n-k+1$ and v_p is also adjacent to v_n in $C_n^k - A'$. Since $t < k$, then v_n and v_{t+1} are adjacent in $C_n^k - A$. Therefore we can conclude that deleting vertex v_i from A does not change the number of components, and so $\omega(C_n^k - A) = \omega(C_n^k - A')$ and the maximum order of a component of $C_n^k - A$ is $\tau(C_n^k - A') \leq \tau(C_n^k - A) + 1$.
Therefore $\frac{|A'| + \tau(C_n^k - A')}{\omega(C_n^k - A')} \leq \frac{|A| - 1 + \tau(C_n^k - A) + 1}{\omega(C_n^k - A)} = T(C_n^k)$, contrary to our choice of A. Thus we must have $t \geq k$.
Now suppose $t > k$. Delete v_t from the set A yielding a new set $A_1 = A - \{v_t\}$. Since $t > k$, the edge $v_t v_n$ is not in $C_n^k - A_1$. Consider a vertex v_p adjacent to v_t in $C_n^k - A_1$. Then $p \geq t+1$ and $p \leq t+k$. So v_p is also adjacent to $v_t + 1$ in $C_n^k - A_1$. Therefore deleting v_t from A yields $\omega(C_n^k - A) = \omega(C_n^k - A_1)$, $\tau(C_n^k - A_1) \leq \tau(C_n^k - A) + 1$. Therefore$\frac{|A_1| + \tau(C_n^k - A_1)}{\omega(C_n^k - A_1)} \leq \frac{|A| - 1 + \tau(C_n^k - A) + 1}{\omega(C_n^k - A)}$, again contrary to our choice of A. Thus $t = k$ and so A consists of the union of sets of exactly k consecutive vertices.

Lemma 2.1 gives us an indication of the size of the cut-set for the tenacity of C_n^k ; the next lemma gives us the size of the largest component.

Lemma 2.2 : There is a T-set, A, for C_n^k such that all components of C_n^k have order $\tau(C_n^k - A)$ or $\tau(C_n^k - A) - 1$.

Proof : Among all minimum order T-sets, consider those sets with maximum order, s, of the minimum order component. Among these sets let A be one with the fewest components of order s. Suppose $s \leq \tau(C_n^k - A) - 2$. Note that all of the components must be sets of consecutive vertices. Suppose C_p is a smallest component, so $\mid V(C_p) \mid = s$, and without loss of generality let $C_p = \{v_1, v_2, \cdots, v_s\}$. Suppose C_l is a largest component, so $\mid V(C_l) \mid = \tau(C_n^k - A) = m$, and $C_l = \{v_j, \cdots, v_{j+m}\}$. Let $C_1, C_2, \cdots, C_a$ be components with vertices between v_s and v_j, such that $\mid C_i \mid = n_i$ for $1 \leq i \leq a$ and $C_i = \{v_{i_1}, v_{i_2}, \cdots, v_{i_{n_i}}\}$. Now construct A' as follows, $A' = A - \{v_{s+1}, v_{1_{n_1}+1}, v_{2_{n_2}+1}, \cdots, v_{a_{n_a}+1}\} \cup \{v_{1_1}, v_{2_1}, \cdots, v_{a_1}, v_j\}$. Therefore $\mid A' \mid = \mid A \mid$, $\tau(C_n^k - A') \leq \tau(C_n^k - A)$ and $\omega(C_n^k - A') = \omega(C_n^k - A)$. So,

$\frac{|A'|+\tau(C_n^k-A')}{\omega(C_n^k-A')} \leq \frac{|A|+\tau(C_n^k-A)}{\omega(C_n^k-A)}$. Therefore $\tau(C_n^k - A') = \tau(C_n^k - A)$. But $C_n^k - A'$ has one less component of order s than $C_n^k - A$, and this is a contradiction. Thus all components of $C_n^k - A$ have order $\tau(C_n^k - A)$ or $\tau(C_n^k - A) - 1$. So $\tau(C_n^k) = \lceil \frac{n-k\omega}{\omega} \rceil$.

Theorem 2.1 : (Chvàtal [10]). For all graphs G, $\frac{\kappa(G)}{\alpha(G)} \leq t(G) \leq \frac{1}{2}\kappa(G)$.

Theorem 2.2 : (Woodal [50]). For all graphs G, $bind(G) \leq \frac{n+\kappa(G)}{n-\kappa(G)}$.

Theorem 2.3 : (Woodal [50]). For all graphs G, $bind(G) \leq t(G) + 1$.

The following four proposition were proved in [14].

Proposition 2.1 : If G is a spanning subgraph of H, then $T(G) \leq T(H)$.

Proposition 2.2 : For any graph G, $T(G) \geq \frac{\kappa(G)+1}{\alpha(G)}$.

Proposition 2.3 : If G is not complete, then $T(G) \leq \frac{n-\alpha(G)+1}{\alpha(G)}$.

Proposition 2.4 : If $k \leq n-k$, then $T(K_{k,n-k}) = \frac{k+1}{n-k}$.

Lemma 2.1 and lemma 2.2 allow us to determine precisely the tenacity of the power of cycles.

Theorem 2.4 : Let C_n^k be a power of cycles and $n = r(k+1) + s$, for $0 \leq s < k+1$. Then $T(C_n^k) = k + \frac{1+\lceil \frac{s}{r} \rceil}{r}$.

Proof : Let A be a minimal T-set of C_n^k. By lemma 1 and lemma 2, $|\ A\ | = k\omega$, and $\tau(C_n^k - A) = \lceil \frac{n-k\omega}{\omega} \rceil$. Thus, from the definition of tenacity we have

$$T = min\{\frac{k\omega + \lceil \frac{n-k\omega}{\omega} \rceil}{\omega} \mid 2 \leq \omega \leq r\}.$$

Now consider the function $f(\omega) = \frac{k\omega+\lceil \frac{n-k\omega}{\omega} \rceil}{\omega} = k + \frac{\lceil \frac{n}{\omega} - k \rceil}{\omega}$. Let ω_1 and ω_2 be any two integers in [2 , r] with $\omega_1 \leq \omega_2$, then $\lceil \frac{n}{\omega_2} \rceil \leq \lceil \frac{n}{\omega_1} \rceil$. Thus $f(\omega_2) = k + \frac{\lceil \frac{n}{\omega_2} - k \rceil}{\omega_2} \leq k + \frac{\lceil \frac{n}{\omega_1} - k \rceil}{\omega_1} = f(\omega_1)$. Hence the function $f(\omega)$ is a nonincreasing function and the minimum value occurs at the boundary. Thus $\omega = r$ and $\lceil \frac{n-k\omega}{\omega} \rceil = \lceil \frac{r(k+1)+s-kr}{r} \rceil = 1 + \lceil \frac{s}{r} \rceil$. Therefore, $T(C_n^k) = k + \frac{1+\lceil \frac{s}{r} \rceil}{r}$.

3 DISCUSION

Now consider the complete bipartite graph $K_{k,n-k}$. In [50], the binding number for a complete bipartite graph was calculated by Woodall, where he gives the result $bind(K_{a,b}) =$

$min\{\frac{a}{b},\frac{b}{a}\}$ for $a \geq 1$ and $b \geq 1$. Thus if $k \leq n-k$, then $bind(K_{k,n-k}) = \frac{k}{n-k}$. The connectivity of $K_{k,n-k}$ obviously is equal to k. From [10], we have $t(K_{k,n-k}) = \frac{k}{n-k}$. It is shown in [4] that $K_{k,n-k}$ has integrity equal to k+1. By proposition 2.4, $T(K_{k,n-k}) = \frac{k+1}{n-k}$. Thus we have the following results for $G = K_{k,n-k}$:
$\kappa(G) = k$; $t(G) = \frac{k}{n-k}$; $bind(G) = \frac{k}{n-k}$; $I(G) = k+1$; $T(G) = \frac{k+1}{n-k}$. The binding number implies that the neighborhood of subset $A \subseteq V(K_{k,n-k})$ has order k and $\mid A \mid= n-k$. The value of $\kappa(K_{k,n-k})$ shows us that at least k vertices must be destroyed in order to break a complete bipartite graph. But these two measures do not indicate how many components exists after removing the cutset from the graph. Since the toughness of $K_{n,n-k}$ is equal to $\frac{k}{n-k}$, then the cardinality of the cutset and the number of components are k and n-k respectively. The integrity of $K_{k,n-k}$ implies that $\mid A \mid= k$ and $\tau(K_{k,n-k} - A) = 1$. Hence both toughness and integrity attempt to describe the structure of the resulting graph after removing the cutset A from $K_{k,n-k}$. The tenacity of a bipartite graph shows us that $\mid A \mid= k$, $\tau(K_{k,n-k} - A) = 1$, $\omega(K_{k,n-k} - A) = n-k$. Hence we obtain the number of components, cardinality of cutset and, since $\tau(K_{k,n-k} - A) = 1$, all n-k components have order 1. Thus we have all of the necessary information for the repair and reconfiguration of the complete bipartite graphs. therefore, in this class, tenacity appears to be a better vulnerability measure.
In [4], the connectivity, binding number and toughness of C_n^k were determined. The integrity of C_n^k was calculated in [3]. By Theorem 2.4, we have the tenacity of C_n^k. Hence we have the following results for $G = C_n^k$:
$\kappa(G) = 2k, 2 \leq k \leq n-2$;

$$bindG = \begin{cases} 1 & k = 1, 2 \mid n \\ \frac{n}{2} - 1 & 2k = n-2 \\ \frac{n-1}{n-2k} & otherwise \end{cases};$$

$I(G) = k\lceil\sqrt{\frac{n}{k} - \frac{1}{4}} - \frac{3}{2}\rceil + \lceil\frac{n}{\sqrt{\frac{n}{k}+\frac{1}{4}} - \frac{1}{2}}\rceil$, where $1 \leq k \leq \frac{n}{2}$;
$t(G) = k$;
$T(G) = k + \frac{1+\lceil\frac{s}{r}\rceil}{r}$.

The value of $\kappa(C_n^k)$ shows us that it is necessary (and sufficient) to remove two disjoint nonadjacent subsets of k consecutive vertices each, along the circumference of the polygon. The toughness of C_n^k, uses the above fact and it was calculated and proved that the cardinality of the cutset is equal to 2k and the number of components is 2. But the enemy will selectively target more resources to break the network, since the resulting network with only two components is easily repaired. Also, toughness does not take into account the order of the components. Therefore, for breaking or reconstruction of C_n^k, tenacity and its minimal cutset seem to be better measures than connectivity and toughness in this class. If n is even and k = 1, the neighborhood of subset $A \subseteq V(C_n^k)$ and $\mid A \mid$, have the same order. When n is odd and k = 1, $bind(C_n^k) = 1 + min\frac{|N(A)|}{|A|}$. However, A is as large as possible when $\mid A \mid= n-2$ and this maximum value of $\mid A \mid$ coincides with the minimum value of N(A), namly, N(A) = 1. In both of the above cases and when

$k > 1$, the binding number does not show the order of the components, or number of components. By theorem 2.4 and lemma 2.2 tenacity gives us the number of components and the order of the largest component. If we compare connectivity, binding number and toughness with this class, integrity seems to be a better measure for the vulnerability of a network. But for repair and reconfiguration of C_n^k, we have a lack of information about the number of components. Thus in this class, for disruption and reconstruction of network, tenacity appears to be a better measure of the vulnerability of a graph.
We now turn our discussion to the vulnerability of $G_{n,k}$. For $G = G_{n,k}$:
$\kappa(G) = 1$;
$t(G) = \frac{1}{2}$;

$$bind(G) = \begin{cases} 1 & k = 1 \\ \frac{n-1}{n-2} & k = 2 \\ \frac{n-k}{n-k-1} & k \geq 3 \end{cases};$$

$I(G) = n - k$;
$T(G) = \frac{n-k}{2}$.

The graphs $G_{n,k}$ perhaps best illustrate the inability of connectivity to provide a realistic measure of the vulnerability of graphs. Certainly disabling a station located at vertex v is less damaging to the operation of the remaining system when k = 1 than when $k = \lfloor \frac{n-1}{2} \rfloor$. Yet neither $\kappa(G_{n,k})$ nor $t(G_{n,k})$ reflect this. Also, $bind(G_{n,k})$ is quite insensitive to the value of k. On the other hand, $T(G_{n,k})$ provides a significant indication of the change in the nature of the structure of the system for $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$. The integrity of $G_{n,k}$ implies that the cardinality of cutset $A \subseteq V(G_{n,k})$ is equal to 1 and $\tau(G_{n,k} - A) = n - k - 1$. Hence if we remove the vertex v, the integrity shows us the order of the largest component, but does not show the number of components. Therefore since $T(G_{n,k})$ has $\omega(G - A)$ in the denominator indicating the number of components, it then provides a more realistic measure of the vulnerability of the graphs. For instance, if a similar graph were constructed with three copies of K_m, the integrity would remain unchanged while tenacity would recognized this change.

4 CONCLUDING REMARKS

Deterministic measure tend to provide a worst case analysis of some aspects of the overall disconnection process. For example $\kappa(G)$, means that for a particular network, even if the enemy knows how the edges have been assigned to the vertices, at least $\kappa(G)$ vertices must be destroyed in order to break communications. Unfortunately, this measure does not indicate how many of these sets of vertices (called minimal cutset) actually exist in the network, nor does it attempt to describe of the resulting network.
For both of these reasons we would like to attempt to quantify connectivity as a relative, as well as an absolute parameter.
Vertex integrity provides some information about the network after disconnection has taken place but, once again, it does not seem to provide the fine resolution that is often

needed.
Deterministic measure are generally very difficult to compute. Eventhough $\kappa(G)$ can be computed quickly-linear in the number of vertices plus the number of edges - determination of vertex integrity of a graph is, in general, an NP-complete problem (see, for example [12]).
The effective design of a survivable communications network requires a means of accurately evaluating its structural vulnerability both as a whole and with respect to its individual resources. For a communications network operating in a tactical environment, this evaluation should be based on a worst-case assumption that the enemy will selectively target those resources most critical to its topological integrity. A critical concern of overall system survivability, therefore, must be the specific level of connectivity associated with the topological structure of the supporting communications network. In [23] Harary showed that in any graph or communications network, the connectivity of a graph with p vertices and q edges cannot exeed $\lfloor \frac{2q}{p} \rfloor$ if $q \geq n-1$ and is 0 otherwise. The power of cycles, C_n^k, is an example of a graph with maximum connectivity. We would like to show the maximum tenacity relative to the maximum connectivity. We found this relation in theorem 2.4. Since communication networks must be constructed to be as stable as possible, not only with respect to initial disruption, but also with respect to the possible reconstruction of the network, then C_n^k is a good example for network designers who are looking for a network with maximum connectivity relative to maximum tenacity.

5 Stability Measure of a Graph

Now we discuss about tenacity and its properties in stability calculation. We indicate relationships between tenacity and connectivity, tenacity and binding number, tenacity and toughness. We also give good lower and upper bounds for tenacity.

5.1 Tenacity and its Properties:

In this paper we will prove a number of basic results about tenacity. Without attempting to obtain the best possible result, we can prove the following relation between T(G) and t(G). This result gives us a number of corollaries.

Theorem 5.1: For any graph G, $T(G) \geq t(G) + \frac{1}{\alpha(G)}$.
Proof: Let $A \subseteq V(G)$ be a t-set and $B \subseteq G$ be a T-set. Then $\frac{|B|+\tau(G-B)}{\omega(G-B)} \geq \frac{|B|}{\omega(G-B)} + \frac{1}{\omega(G-B)} \geq \frac{|A|}{\omega(G-A)} + \frac{1}{\alpha(G)}$.

Corollary 5.1: For any graph G, $T(G^2) > \kappa(G)$.

Corollary 5.2: Let G be a non-empty graph and let m be the largest integer such that $K_{1,m}$ is an induced subgraph of G. Then $T(G) \geq \frac{\kappa(G)}{m} + \frac{1}{\alpha(G)}$.

Theorem 5.2: If G is connected and a noncomplete $K_{1,3}$-free graph then $T(G) > \frac{\kappa(G)}{2}$.

Theorem 5.3: For any nontrivial noncomplete graph G on n vertices and any vertex v, $T(G-v) \geq T(G) - \frac{1}{2}$.

Proof: Let $G' = G - v$. If $G' = K_{n-1}$, then $T(G') = \infty$, and the theorem holds. Hence, assume $G' \neq K_{n-1}$. Let A' be a T-set for G', and let $\mid A' \mid = m$, then $T(G') = \frac{m+\tau(G'-A')}{\omega(G'-A')}$. Now define $A = A' \cup \{v\}$. Clearly A is a disconnecting set for G and so $T(G) \leq \frac{|A|+\tau(G-A)}{\omega(G-A)}$. But $\mid A \mid = m+1$ and $G - A = G' - A'$, so $T(G) \leq \frac{m+1+\tau(G'-A')}{\omega(G'-A')} = \frac{m+\tau(G'-A')}{\omega(G'-A')} + \frac{1}{\omega(G'-A')} = T(G') + \frac{1}{\omega(G'-A')} \leq T(G') + \frac{1}{2}$, since $\omega(G' - A') \geq 2$. Hence $T(G) \leq T(G') + \frac{1}{2}$.

We next obtain some bounds on the tenacity of a graph.

Proposition 5.5: If G is connected, then $T(G) \geq \frac{1}{\Delta(G)}$.

Proof: K_n is a speacial case, otherwise the removal of any vertex of a connected graph G yields at most $\Delta(G)$ components. Similarly, the removal of any n verices yields at most $n\Delta(G)$ components. Then, from the definition we have $T(G) \geq \frac{n+1}{n\Delta(G)} \geq \frac{1}{\Delta(G)}$.

Lemma 5.1: If A is a minimal T-set for the graph G then, for each vertex v of A, the induced subgraph $< V(G) - A + v >$ has fewer components than does G-A.

Proof: Let $A' = A - v$. If G-A' has at least as many components as G-A, then $\mid A' \mid = \mid A \mid -1$ and $\tau(G - A') \leq \tau(G - A) + 1$. Therefore $\frac{|A'|+\tau(G-A')}{\omega(G-A)} = \frac{|A|-1+\tau(G-A')}{\omega(G-A)} \leq \frac{|A|-1+\tau(G-A)+1}{\omega(G-A)} = T(G)$, contrary to our choice of A.

Theorem 5.4: Let $G = G_1 + G_2$, where $\mid V(G) \mid = n$, $\mid V(G_i) \mid = p_i$, $T(G) = T$ and $T(G_i) = T_i$ for $i = 1, 2$. Then if $G \neq K_n$ we have

$$min\{\frac{[n+\tau(G_1-A_1)]T_1}{p_1+\tau(G_1-A_1}, \frac{[n+\tau(G_2-A_2)]T_2}{p_2+\tau(G_2-A_2)}\} < T \leq min\{\frac{n-\alpha_1+1}{\alpha_1}, \frac{n-\alpha_2+1}{\alpha 2}\},$$

where α_i is the independence number of G_i, and A_i is a disconnecting set of G_i for $i = 1, 2$.

Proof: Because of the structure of G, the graph cannot be disconnected without removing one of $V(G_1)$ or $V(G_2)$. Having removed the appropriate set, we can then disconnect the graph by disconnecting the remaining graph, either G_1 or G_2. Candidates for T are of the form $\frac{n_1+p_2+\tau(G_1-A_1)}{\omega(G_1-A_1)}$ or $\frac{n_2+p_1+\tau(G_2-A_2)}{\omega(G_2-A_2)}$ where $n_i = \mid A_i \mid$ for $i = 1, 2$. Then $T = min\{\frac{n_1+p_2+\tau(G_1-A_1)}{\omega(G_1-A_1)}, \frac{n_2+p_1+\tau(G_2-A_2)}{\omega(G_2-A_2)}\}$, where the minimum is taken over all A_1 and A_2 as defined. Also $T_1 \leq \frac{n_1+\tau(G_1-A_1)}{\omega(G_1-A_1)}$ which implies $\omega(G_1 - A_1) \leq \frac{n_1+\tau(G_1-A_1)}{T_1}$. Thus $\frac{n_1+p_2+\tau(G_1-A_1)}{\omega(G_1-A_1)} \geq \frac{[n_1+p_2+\tau(G_1-A_1)]T_1}{n_1+\tau(G_1-A_1)}$. Similary, $\frac{n_2+p_1+\tau(G_2-A_2)}{\omega(G_2-A_2)} \geq \frac{[n_2+p_1+\tau(G_2-A_2)]T_2}{n_2+\tau(G_2-A_2)}$. Thus

$T \geq min\{[1+\frac{p_2}{n_1+\tau(G_1-A_1)}]T_1, [1+\frac{p_1}{n_2+\tau(G_2-A_2)}]T_2\}$. Also we know that $n_1 < p_1$ and $n_2 < p_2$, therefore $T > min\{\frac{[n+\tau(G_1-A_1)]T_1}{p_1+\tau(G_1-A_1)}, \frac{[n+\tau(G_2-A_2)]T_2}{p_2+\tau(G_2-A_2)}\}$. From Proposition 5.3, we observe that two candidates for T are $\frac{(p_1-\alpha_1)+1+p_2}{\alpha_1}$ and $\frac{(p_2-\alpha_2)+p_1}{\alpha_2}$, which yield $T \leq min\{\frac{n-\alpha_1+1}{\alpha_1}\}$, $\frac{n-\alpha_2+1}{\alpha_2}\}$.

Theorem 5.5: Let G be a graph with n vertices and $G \neq K_n$, then $T(G) + T(\overline{G}) \geq \frac{1}{n-1}$.

Proof: We observe that at least one of G or $\overline{G}$ is connected. Suppose $\overline{G}$ is not connected. We proved (Proposition 5) that $T(G) \geq \frac{1}{\Delta(G)} \geq \frac{1}{n-1}$ for any graph G. Thus, $T(G) + T(\overline{G}) \geq \frac{1}{n-1}$. Now suppose G is not connected but $\overline{G}$ is connected. Again by Proposition 5, we have $T(\overline{G}) \geq \frac{1}{n-1}$. Therefore $T(G) + T(\overline{G}) \geq \frac{1}{n-1}$.

Theorem 5.6: Let G be a graph with $0 < T(G) < \infty$, and let $\lambda(G) = \lambda$, then $T(L(G)) > \frac{\lambda}{2}$.

Proof: Assume there exist vertex cutsets A for L(G) such that A is a t-set. By Theorem 1, $T(L(G)) > t(L(G))$. Let E be those edges of G which are incident to vertices of A. Then E is an edge-cutset of G. Thus we have $t(L(G)) = min\{\frac{|A|}{\omega(L(G)-A)}\} \geq min\{\frac{|E|}{\omega(G-E)}\} = t'(G)$, where A is a cutset and E is an edge cutset of G.
Using the result of Chvátal [13] we have $t'(G) = min\{\frac{|E|}{\omega(G-E)}\} = \frac{\lambda}{2}$. Therefore $T(L(G)) > \frac{\lambda}{2}$.

Theorem 5.7: For any graph G, $T(G) \geq bind(G) - 1$.

Proof: Let bind(G) = c. If $c < 1$, then $c - 1 < 0$ and the result follows since T(G) is nonnegative. Consider $c \geq 1$. Suppose that A is a subset of V(G) such that $\omega = \omega(G - A) \geq 2$. We want to prove that $\frac{|A|+1}{\omega} > (c - 1)$. If each of the ω components of G-A has at least two vertices, let S consist of the vertices in all the components except the smallest, so that

$$| S | \geq \frac{| V(G) - A | (\omega - 1)}{\omega} \geq \frac{2\omega(\omega - 1)}{\omega} = 2(\omega - 1) \geq \omega.$$

If, on the other hand, V(G)-A contains an isolated vertex, let $S = V(G) - A$. So that $| S | = | V(G) - A | \geq \omega$. In either case $N(S) \neq V(G)$, and since $bind(G) = c \geq 1$,

$$| S | + | A | +1 > | S | + | A | \geq | N(S) | \geq c | S | .$$

It follows that $| A | +1 > (c - 1) | S | \geq (c - 1)\omega$. Therefore $\frac{|A|+1}{\omega} > c - 1$, so $T > c - 1$.

In [14] we showed the Hamiltoinan Properties of tenacity. The results follows for a graph G:

1) $1 < \frac{\kappa(G)}{\alpha(G)} < \frac{\kappa(G)+1}{\alpha(G)} \leq T(G)$

2) $\frac{\kappa(G)+1}{\alpha(G)} \leq T(G) < 1.$

Graphs satisfying the second inequality are not Hamiltonian-connected. Graphs satisfying the first inequality are Hamiltonian-connected.

3) $1 + \frac{n+1}{\alpha(G)} \leq \frac{\kappa(G)+1}{\alpha(G)} \leq T(G)$

4) $\frac{\kappa(G)+1}{\alpha(G)} \leq T(G) < 1 + \frac{n+1}{\alpha(G)}$

If G satisfies the forth inequality it is not n-Hamiltonian.

If G satisfies the third inequality then G is n-Hamiltonian.

In [14], we also obtained some bounds on the tenacity of products of graphs. Note that the first inequality in the following theorem, is a corollary to Theorem 1

In [34], we compared integrity, connectivity, binding number, toughness and tenacity for several classes of graphs. The results suggest that tenacity is a most suitable measure of stability or vulnerability in that for many graphs it is best able to distinguish between graphs that intuitively should have different levels of vulnerability.

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