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RESEARCH PAPER

Analysis of a Two-Storage System for Advance Payment Policies with the Partial Backlogged Shortage

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Abstract

Nowadays, due to the highly competitive situation, every business organization faces many shortcomings for smoothly running of his/her own business. So, to survive in the competition, different types of business policies are required (all-unit discount, trade credit etc.). This type of problem is presented mathematically as an optimization problem and solved. In this work, the advance payment facility with *n* equal instalment before receiving the products is introduced to formulate a two-storage inventory model. This model is studied under the assumptions of price and advertisement's frequency-dependent customers' demand, constant deterioration and exponential backlogging rate. Deterioration is started in both warehouses at the same time. To solve the proposed model, MATHEMATICS and MATLAB software are used. The concavity of the objective function (average profit) is shown numerically as well as graphically by using MATHEMATICA and MATLAB software for supporting a numerical example. Also, it is shown that the objective function is negative definite by using an example. Finally, sensitivity analyses are carried out pictorially with the changes of various known parameters.

Keywords:
Two-Warehouse,
Advance Payment with Installment,
Deterioration,
Frequency of Advertisement,
Shortage

Introduction

Inventory of an item depends on several factors such as the price of the goods, customers demand for the items, the popularity of goods, decaying rate of goods etc. The selling price of goods is an important issue for the demand of an item. So, it cannot ignore for the analysis of inventory. From the economical point of view, it is observed that price has a huge impact on demand. The recent trend in the online market, visual of selling price and availability of an item has also a great effect on the demand of the item. Also, the effect of the advertisement of an item has a direct impact on demand and obviously it increases the demand rate of an item.

In the current business situation, it is a quite complicated task for the manufacturer/supplier to attract customers in order to buy their products. To avoid this difficulty, the manufacturer/supplier offers different types of facility to their potential customers in order to smooth running their business. The manufacturer/supplier provides different facilities such as credit facility, quantity discount, promotional discount, cash discount, advance payment facility, sessional discount, etc. A prepayment facility is the riskless (payment security) facility for manufacture/suppliers. In the current situation, the concept of advance payment is becoming very interesting to a lot of researchers and academicians.

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Due to the highly competitive business situation, it is quite difficult to find a large place in the popular market place. In this situation, retailers try to find another place nearby his/her shop in the popular market place. Generally, this type of situation is called a two warehouse system.

Literature review

The concept of an advertisement in inventory modelling was first introduced by Deighton et al. [1]. Bronnenberg [2] studied a problem related to inventory by introducing the impact of frequency of the advertisement under a budget constraint. Dutta and Pal [3] investigated a stock and selling price dependent inventory model. Bhunia and Shaikh [4] derived a two-warehouse model where the demand of an item considered as price and advertisement frequency. Fordyce et al. [5] proposed model related to inventory with a targeted advertisement to the audience. Razniewski [6] investigated the optimal frequency of advertisements for decaying information. Pervin et al. [7] introduced a selling price and stock dependent demand related two-echelon inventory model. Shaikh et al. [8] established a model related to inventory with the demand of an item dependent on price and frequency of advertisement. Rahman et al. [9] proposed a parametric approach of interval in the area of inventory control. Khan et al. [10] studied a perishable model for payment in advance scheme along with the demand of an item dependent on advertisement frequency and price.

Deterioration is the natural phenomenon and it is defined as spoilage, decay, damage and loses of utility of products. Hence, deterioration of an item plays a crucial role in the inventory management. In over the last few decades, years, many most of the researchers have developed deteriorating inventory models considering different types of demand. Cheng and Wang (2009) introduced trapezoidal demand related inventory model under deterioration. Mirzazadeh et al. [11] derived inflation related demand rate with deterioration in the area of inventory. Chang et al. [12] studied a stock dependent demand related problem with non-instantaneous decaying rate. Mandal [13] introduced ramp type demand and decaying inventory model. Then, Mishra [14] introduced a deteriorating inventory model with demand dependent on time and Das et al. [15] studied a deteriorated inventory model with shortages. Hung [16] studied an inventory model with generalized type demand, deterioration and backorder rates. Jolai et al. [17] proposed two-echelon supply chain for perishable item. Similarly, a researcher like Lee and Dye [18] introduced an inventory model with stock-dependent demand and deterioration and Sarkar [19] investigated an imperfect inventory model under reliability consideration. Again, Sarkar et al. [20] included variable demand and selling price in their inventory model with deterioration. Bhunia and Shaikh [21] investigated decaying inventory model with shortage. Silica et al. (2014) derived a model with time-varying demand and deterioration. Next, Taleizadeh et al. [22] developed a vendor managed inventory system with deteriorating items. Teimoury and Kazemi [23] investigated constant deteriorating inventory for pricing model with replacement. Gholami and Honarvar [24] proposed vendor managed inventory model for consideration of both amelioration and deterioration under three level supply chain. Again, Pallanivel and Uthayakumar [25] introduced a deteriorating model with variable production cost and time-dependent holding cost. Ghorieshi et al. [26] investigated a non-instantaneous decaying model with demand dependent on price. Duong et al. [27] proposed on a multi-criteria decision making inventory model for the perishable item. Alfares and Ghaithan [28] proposed pricing inventory model for holding cost considered as time-varying with all unit discount. Hasanpour Rodbaraki and Sharifi [29] derived deteriorating inventory model for imperfect quality item for destructive testing. Rabbaniv et al. [30] studied an integrated approach in the area of inventory under deterioration and pricing and advertisement. Zohoori et al. [31] studied stochastic demand related inventory model under close loop supply chain. Li and Teng [32] reported selling price and product freshness related inventory model for the deteriorating item.

Duong et al. [33] studied the effect of customer demand for perishable inventory system. Khan et al. [34] proposed pricing inventory model with expiration date related deterioration. Li et al. [35] investigated an inventory problem for expiration rate-dependent decaying with demand dependent on price. Das et al. [36] studied price dependent demand related inventory with backlogging under preservation facility. Rahman et al. [37] investigated a deteriorating model under preservation facility. Xu et al. [38] proposed non-perishable inventory model with warehouse selection mode under backlogging and trapezoidal demand. Xu et al. [39] reported carbon emission related inventory model with time dependent demand of the product.

To continue a business, the warehouse has a crucial role for controlling inventory. For insufficiency of space in an important market, additional space is required to storage the products. So, in a suitable market place, a rented warehouse (RW) in which products' deterioration rate is less than the owned warehouse (OW) is needed. Many researchers studied several two-warehouse inventory models. Among them, Liang and Zhou [40] proposed a two-storage credit policy inventory model for the deteriorating item. Sett et al. (2012) derived two-storage time varying deterioration inventory model for increasing demand. Liao et al. [41] investigated a two-storage supply chain model under trade credit financing. Bhunia and Shaikh [4] introduced a price dependent demand linked inventory model for deteriorating item. Bhunia et al. [42] proposed a two-storage deteriorating inventory model with partially backlogged situations. Das et al. [15] presented a two-storage inventory model taking delay payments for partial backlogged items. Ghoreishi et al. [26] proposed a non-instantaneous decaying inventory problem with partial backlogging under credit policy approach. Khanna et al. [43,44,45] studied an inventory model for deteriorating items with shortage policies. Shaikh et al. [46] studied a stock dependent inventory model under inflation and fully backlogged situation. Tiwari et al. [47] considered a non-instantaneous deteriorating item and inflation effect to develop a two-warehouse inventory model. Pervin et al. [48] investigated time dependent demand and holding costs related inventory model for deteriorating item. In this connection, one can refer some works such as Gautam and Khanna [49], Jaggi et al. [50], Khan et al. [34], Panda et al. [51] and Manna et al. [52,53] among others.

Advance payment and trade credit financing policies have extensive effects on profit. According to existing literature, the concept of advance payment was first proposed by Taleizadeh et al. [54]. After that, Maiti et al. [55] studied an inventory model related with advance payment where demand is taken on selling price and stochastic lead time. Gupta et al. [56] introduced pre-payment related inventory model and solved the said problem by using genetic algorithm. Tsao [57] and Thangam [58] studied an effect of delay in payment, discount on advance sales and pre-payment scheme in related with inventory model. Again, Thangam [59] established a two-echelon inventory problem with pre-payment and trade credit. Taleizadeh et al. [60] proposed multiple partial advance payment related inventory model with partial backlogging shortages. Taleizadeh [61,62] studied an evaporating item related inventory model with pre-payment facilities. Zia and Taleizadeh [63] introduced hybrid payment scheme and developed a lot-sizing model. Lashgari et al. [64] studied three phase supply chain problem with partly-up and down-stream payment scheme. Teng et al. [65] discussed an expiration date dependent deteriorating item under pre-payment facility. Tavakoli and Taleizadeh [66] proposed an EQO model with pre-payment facilities. Taleizadeh et al. [67] introduced planed backordering related inventory model under advance payment scheme. Shah and Naik [68] investigated price dependent demand related inventory model under advance payment policy. Manna et al. [52] studied a carbon emission related imperfect production model with pre-payment base free transportation partial transportation facilities. Chang et al. [69] formulated a manufacturing lot-sizing model for deteriorating products under analysis of cash flow. Li et al. [70] introduced an advance-cash-credit payment scheme and optimized the lot-sizing, pricing and backordering. Shaikh et al. [71] proposed interval cost related inventory model under advance payment scheme. Khan et al. [72] studied backlogging inventory model under pre-payment scheme for deteriorating item. Khan et al. [73,74] investigated deteriorating inventory model with pre-payment policy and shortages. Khakzad and Gholamian [75] studied an inventory model with effect of inspection on deterioration rate under advance payment scheme. Das et al. [76] derived a deteriorating inventory model under preservation facility and multiple credit base trade credit facility. Ghosh et al. [77] introduced a perishable inventory model with multiple advance and delayed payment policies. Ghosh et al. [78] proposed a supply chain model for different payment policies and solved by game theoretic approach.

Research gap and contribution

Two-warehouse has a significant role in inventory analysis. For inadequate storage facility in a popular business place, retailers are bound to engage a supplementary storeroom on a rental basis. This type of problem appeares almost everywhere in the popular market place. Here the impact of pre-payment with *n* instalment facilities is introduced. Combining these two concepts together, a two-storage inventory problem is proposed. Selling price and advertisement frequency dependent demand are considered to develop this model. The main contribution of this model summarised below

(i) Advance payment with *n* equal instalment facility in a two warehouse system.
(ii) Deterioration started both warehouses at the same time.
(iii) Frequency of advertisement is taken into the demand function.
(iv) Optimize with the help of eigenvalue of the objective function.

A numerical example is considered in order to illustrate and validate the proposed model and to solve the model MATHEMATICA software is employed. Also, to present the concavity graphically of the proposed model, MATLAB software is used.

Assumptions and nomenclature

To discuss the manuscript, following nomenclature and assumptions are used.

Nomenclature

Notation	Units	Description
O	$/order	Ordering cost
C_l	$/unit	Lost sale or opportunity cost/unit
C_s	$/unit	Shortage cost/ unit
C_{ho}	$/unit	holding cost/ unit in owned warehouse
C_{hr}	$/unit	holding cost/ unit in rented warehouse
W	unit	Capacity of rented warehouse
G	$/unit	Advertisement cost
C_p	unit	Purchase cost per unit

Notation	Units	Description
D	unit	Demand rate
a, b	constant	Demand parameters
A	constant	Frequency of advertisement
α	constant	Decaying rate at rented warehouse
β	constant	Decaying rate at owned warehouse
M	year	Lead time
n	constant	Equal number of instalment
C_d	$/unit	Deterioration cost per unit
$k\ (0 < k < 1)$	constant	Fraction of the purchasing cost need to be pay
$\delta\ (0 < \delta < 1)$	unit	Backlogging parameter
Dependent variable		
S	unit	Total inventory level
R	unit	Backlogging unit
Decision variable		
T	year	Cycle length
t_1	year	Time at which inventory level is empty
t_r	year	Time at which rented warehouse inventory level is empty

Assumptions

i. Demand of a product is dependent on price and advertisement frequency i.e., $D(p, A) = (a - pb)A^{\gamma}$ where A be an integer value and $A, a, b > 0$.
ii. The rate of deterioration are considered as non-instantaneous with the rates α(0<α<1) (decaying determination in RW) and β(0<β<1) (decaying determination in OW).
iii. Due to better facilities in RW the deterioration rates are considered as $\alpha > \beta$.
iv. Replacement and repair facility for deteriorated products are not available during the cycle length.
v. The retailers must pay k part amount of purchasing amount with n equal part of instalment during the lead time M. However, rest amount must be pay before receiving the item.
vi. Here it is assumed that rented warehouse holding cost per unit, C_{hr}, is higher than the own warehouse holding cost per unit C_{ho}.
vii. Planning horizon of inventory is infinite.
viii. Partially backlogged shortages are taken through the stock out situation with the backlogging rate $e^{-\delta(T-t)}$.

Problem description

Here it is assumed that an enterprise need to be purchase *(S+R)* units of goods after paying the fraction of *k* amount of total purchasing amount with *n* equal instalment and rest amount will pay at time *t=0*. Instantly retailers use *R* units to fulfil the backlogged amount of the product and the current available goods becomes *S*. *S-W* units of item are preserved in owned warehouse (OW) and the remaining part of the product *W* units are preserved into rented warehouse (RW). To the entire time interval $[0, t_r]$, the level of inventory reduces due to the effect of demand $D(p, A)$ and the constant decaying rate α .At the time $t = t_r$ inventory of RW becomes zero. Also, the amount of inventory in OW is depleted for accomplish of decaying rate β only the time interval $[t_d, t_r]$. After that, the amount of inventory in OW is reduced to the joint accomplish of customer demand $D(p, A)$ and decaying entire time interval $[t_r, t_1]$. At time $t = t_1$, shortage are appeared with rate $e^{-\delta(T-t)}$ over the time period $[t_1, T]$. The above described two-warehouse system can be presented in Fig. 1.

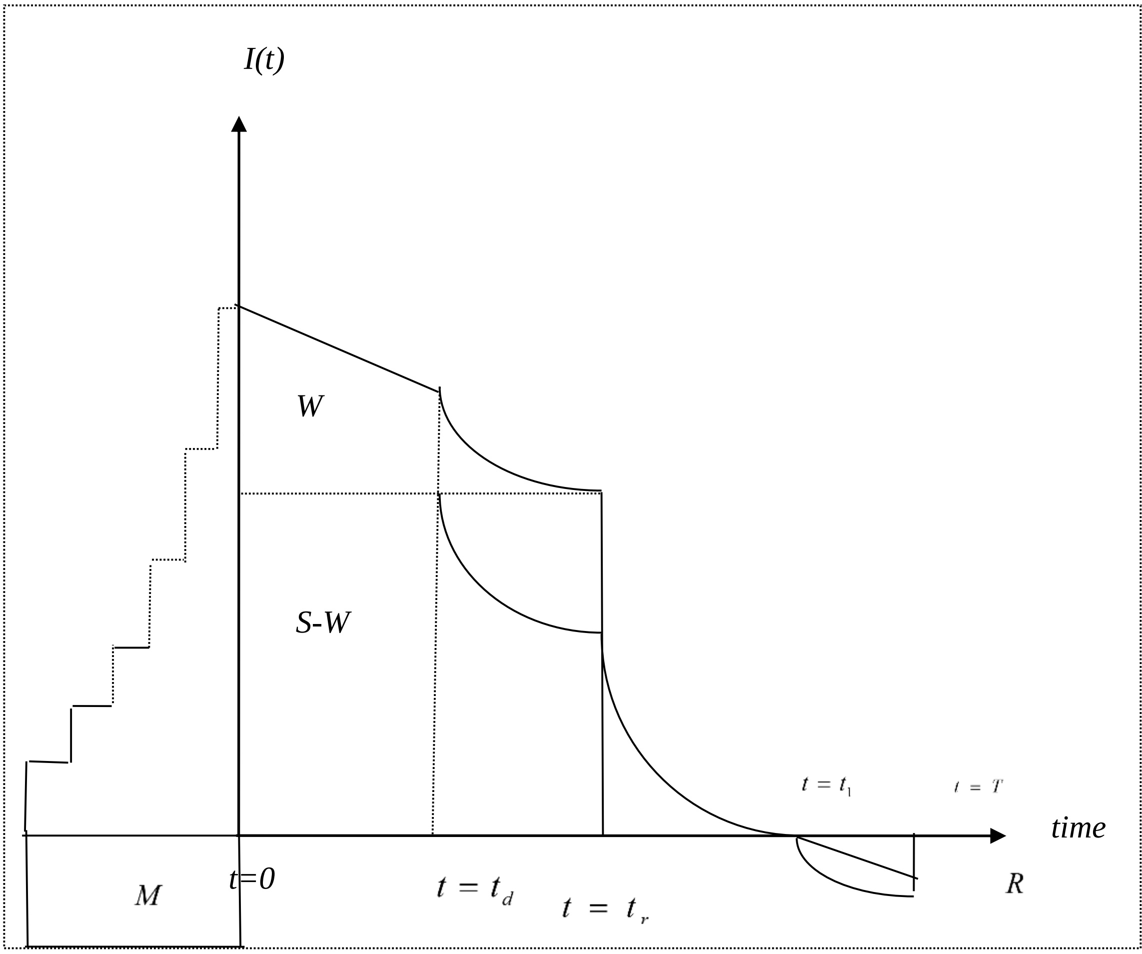

Fig. 1. Pictorial presentation of two-storage inventory system

Now the level of inventory $I_r(t)$ in the RW is satisfying the governing differential equations as follows:

$$\frac{dI_r(t)}{dt} = -D, 0 \leq t \leq t_d \tag{1}$$

$$\frac{dI_r(t)}{dt} + \alpha I_r(t) = -D, t_d < t \leq t_r \tag{2}$$

Subject to the conditions:

$I_r(t) = W$ at t=0 (3)
$I_r(t) = 0$ at t=t_r (4)

$I_r(t)$ is continuous at the point $t = t_d$.
Solving the Eqs. 1 and 2 with the boundary condition (3) and (4) respectively are given by:

$$I_r(t) = -Dt + W \quad , \; 0 \le t \le t_d. \tag{5}$$
$$I_r(t) = \frac{D}{\alpha}(e^{\alpha(t_r - t)} - 1) \; , \; t_d < t \le t_r. \tag{6}$$

Now from the continuity at the point $t = t_d$ from the Eq. 6 we get that:

$$W = Dt_d + \frac{D}{\alpha}(e^{\alpha(t_r - t_d)} - 1) \tag{7}$$

Again, the level of inventory $I_o(t)$ in the OW is satisfying the following differential equations:

$$\frac{dI_o(t)}{dt} = 0, 0 \le t \le t_d \tag{8}$$
$$\frac{dI_o(t)}{dt} + \beta I_o(t) = 0, t_d < t \le t_r \tag{9}$$
$$\frac{dI_o(t)}{dt} + \beta I_o(t) = -D, t_r < t \le t_1 \tag{10}$$
$$\frac{dI_o(t)}{dt} = -e^{-\delta(T-t)} D, t_1 < t \le T \tag{11}$$

Subject to the boundary conditions:

$I_o(t) = S - W$ at t=t_d (12)
$I_o(t) = 0$ at $t = t_1$ (13)
$I_o(t) = -R$ at $t = T$ (14)

$I_o(t)$ is continuous at the points $t = t_r$ and $t = t_1$.
The solution of the differential Eqs. 8, 9, 10, and 11 with the boundary condition (12), (13), (14) is given by:

$$I_o(t) = S - W \; , \; 0 \le t \le t_d \tag{15}$$
$$I_o(t) = (S - W)e^{\beta(t_d - t)}, \; t_d \le t \le t_r \tag{16}$$
$$I_o(t) = \frac{D}{\beta}(e^{\beta(t_1 - t)} - 1), \quad t_r \le t \le t_1 \tag{17}$$
$$I_o(t) = \frac{D}{\delta}(1 - e^{-\delta(T-t)}) - R, t_1 \le t \le T \tag{18}$$

Now from the continuity at the point $t = t_r$ from the Eqs. 16 and 17 we get that:

$$(S\text{-}W)e^{\beta(t_d - t_r)} = \frac{D}{\beta}(e^{\beta(t_1 - t_r)} - 1) \tag{19}$$

Now from the continuity at the point $t = t_1$ from the Eqs. 17 and 18 we get that:

$$R = \frac{D}{\delta}(1 - e^{-\delta(T-t_1)}) \tag{20}$$

RELATED COST:
Inventory related components are described below:

(a) Ordering cost: O
(b) Purchasing cost: $C_p(S+R) = C_p[Dt_d + W + \frac{D}{\alpha}(e^{\alpha(t_r-t_d)} - 1) + \frac{D}{\delta}(1 - e^{-\delta(T-t_1)})]$
(c) Holding cost: $C_{hr}\int_0^{t_d} I_r(t)dt + C_{hr}\int_{t_d}^{t_r} I_r(t)dt + C_{ho}\int_0^{t_d} I_o(t)dt + C_{ho}\int_{t_d}^{t_r} I_o(t)dt + C_{ho}\int_{t_r}^{t_1} I_o(t)dt = C_{hr}[(Wt_d - \frac{Dt_d^2}{2}) + \frac{D}{\alpha^2}(e^{\alpha(t_r-t_d)} - \alpha(t_r - t_d) - 1)] + C_{ho}[(S - W)t_d + \frac{(S-W)}{\beta}(1 - e^{\beta(t_d-t_r)}) + \frac{D}{\beta^2}(e^{\beta(t_1-t_r)} - \beta(t_1 - t_r) - 1)]$
(d) Shortage cost: $C_s\int_{t_1}^{T} -I_0(t)dt = -C_s[\frac{D}{\delta^2}(\delta(T-t_1) + e^{-\delta(T-t_1)} - 1) - R(T - t_1)]$
(e) Lost sale cost: $C_l D[(T - t_1) - \frac{(1-e^{-\delta(T-t_1)})}{\delta}]$
(f) Capital cost: $I_c(\frac{kC_p(S+R)}{n}.\frac{M}{n}(1+2+3+\ldots+n)) = \frac{(n+1)}{2n}I_c kC_p M(S+R) = \frac{(n+1)}{2n}I_c kC_p M[Dt_d + W + \frac{D}{\alpha}(e^{\alpha(t_r-t_d)} - 1) + \frac{D}{\delta}(1 - e^{-\delta(T-t_1)})]$
(g) Advertisement cost: $A*G$

Hence, total cost per unit time is given by:

TC=$\frac{1}{T}$[<Ordering cost>+<Purchasing cost>+<Holding cost>+<Shortage cost>+<Lost sale cost>+<Capital cost>+<advertisement cost>]

$$TC = \frac{1}{T}\left[\begin{array}{l} O + \left[C_p[Dt_d + W + \frac{D}{\alpha}(e^{\alpha(t_r-t_d)} - 1) + \frac{D}{\delta}(1 - e^{-\delta(T-t_1)})]\right] \\ + \left[\begin{array}{l} C_{hr}[(Wt_d - \frac{D}{2}t_d^2) + \frac{D}{\alpha^2}(e^{\alpha(t_r-t_d)} - \alpha(t_r - t_d) - 1)] \\ + C_{ho}[(S-W)t_d + \frac{(S-W)}{\beta}(1 - e^{\beta(t_d-t_r)}) + \frac{D}{\beta^2}(e^{\beta(t_1-t_r)} - \beta(t_1 - t_r) - 1)] \end{array}\right] \\ + \left[C_s[(R - \frac{D}{\delta})(T - t_1) + \frac{D}{\delta^2}(1 - e^{-\delta(T-t_1)})]\right] + \left[C_l D[(T - t_1) - \frac{\left(1 - e^{-\delta(T-t_1)}\right)}{\delta}]\right] \\ + \left[\frac{(n+1)}{2n}I_c kC_p M(S+R)\right] + A*G \end{array}\right]$$

Total sales revenue $(TE) = p\int_0^{t_1} D dt + pR$
$= pDt_1 + pR$
Total profit (X)=TE-T*TC
Profit per unit $Z = (TE - T * TC)/T$ (21)

Solution procedure

Here, the solution procedure of the proposed model is described in details.

Differentiate and simplifying the Eq. 21 by using Mathematica software with respect to t_r, we have

$$\frac{\partial Z}{\partial t_r}=-\frac{A^{\gamma}(a-bp)\left[2\beta c_{hr}\left(-1+e^{\alpha(-t_d+t_r)}\right)n+\alpha\left\{\begin{array}{l}2c_{ho}\left(-1+e^{\alpha(-t_d+t_r)}\right)n\\+\beta c_p\left(e^{\alpha(-t_d+t_r)}-e^{\beta(-t_d+t_r)}\right)\left(2n+I_c kM(1+n)\right)\\+2\beta\left(c_{hr}e^{\alpha(-t_d+t_r)}-c_{ho}e^{\beta(-t_d+t_r)}\right)nt_d\end{array}\right\}\right]}{2\alpha\beta nT} \tag{22}$$

Differentiate and simplifying the Eq. 21 by using Mathematica software with respect to t_1, we have

$$\frac{\partial Z}{\partial t_1}=\frac{A^{\gamma}(a-bp)e^{-\delta T-\beta t_d}\left[\begin{array}{l}-2\beta c_{ho}e^{\delta T}\left(e^{\beta t_1-\beta t_d}\right)n-\beta c_p\left(e^{\delta T+\beta t_1}-e^{\delta t_1+\beta t_d}\right)\left(2n+I_c kM(1+n)\right)\\+2\beta n\left\{\begin{array}{l}c_l e^{\beta t_d}\left(e^{\delta T}-e^{\delta t_1}\right)+e^{\delta T+\beta t_d}p-e^{\delta T+\beta t_d}\left(p-c_s T+c_s t_1\right)\\-c_{ho}e^{\delta T+\beta t_1}t_d\end{array}\right\}\end{array}\right]}{2\beta nT} \tag{23}$$

Again, differentiate the Eq. 21 with respect to T, we have

$$\frac{\partial Z}{\partial T}=\frac{1}{T}\frac{\partial TE}{\partial T}-\frac{TE}{T^2}-\frac{\partial TC}{\partial T} \tag{24}$$

The details calculation of Eq. 23 is provided in **Appendix**. Now equating the Eqs. 21, 22, and 23 with respect to zero, i.e.,

$$\frac{\partial Z}{\partial t_r}=0,\ \frac{\partial Z}{\partial t_1}=0\ and\ \frac{\partial Z}{\partial T}=0.$$

From the Eqs. 22, 23, and 24, the optimum value is obtained of the decision variable $t_r, t_1\ and\ T.$ Using these value, optimal results is obtained for the decision variable which is shown in numerical section.

Flowchart: Now, flowchart of the problem is provided in below:

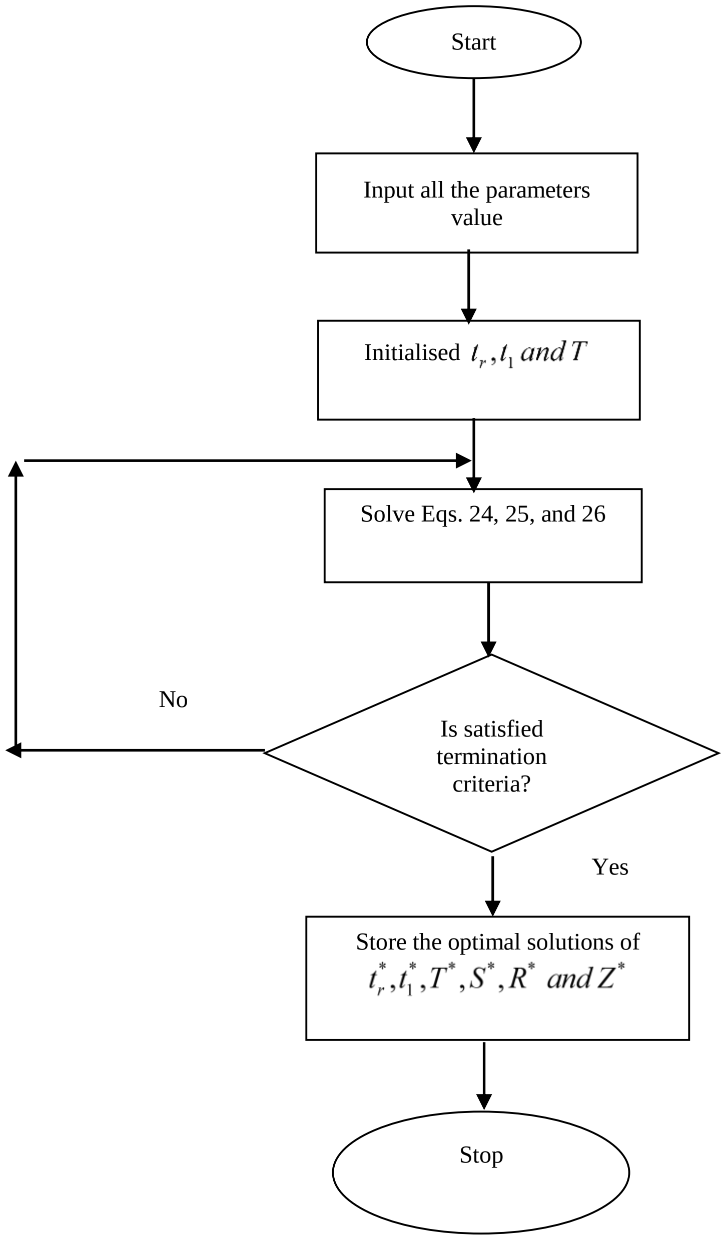

Fig. 2. Solution procedure shown in flowchart

Numerical illustration

To validate the model, a numerical example is taken into consideration and solved the problem by using MATHEMATICA software. However, MATLAB software is used to present the concavity of the problem (maybe anyone can use MATHEMATICA software). The values of the parameters are taken randomly and any case study cannot be considered. The values of different parameters are given below:

Example: *Model with shortages*

Let $O=\$300$ /order, $W=50$ unit, $t_d=0.1$ year, $a=200$, $b=0.5$, $C_p=\$16$ /unit, $p=\$20/unit$ $C_{hr}=\$1, C_{ho}=\$0.5$ /unit, $\alpha=0.05$, $\beta=0.07$, $\delta=1.5$, $C_s=\$20, C_l=\$25$ $n=20$, $k=0.4$, $I_c=0.12$ /dollar/year, $M=0.25$ year , $A=5$ unit, $\gamma=0.1$, $G=\$20$ /advertisement.

Using the above numerical example, optimal solution is obtained by using Lingo 18 software which is given below.

Hence, the optimal solutions are $t_r^*=.2236\,year$, $t_1^*=1.4277$ year, $T^*=1.4657$ year, $S^*=332.0109$ units, $R^*=8.2424$ units, $Q^*=340.2533$ units and $Z^{(\max)}(t_r,t_1,T)=\$352.7229$.

The eigenvalues of the Hessian matrix: $\begin{bmatrix} \frac{\partial^2 Z(.)}{\partial t_r^2} & \frac{\partial^2 Z(.)}{\partial t_r \partial t_1} & \frac{\partial^2 Z(.)}{\partial t_r \partial T} \\ \frac{\partial^2 Z(.)}{\partial t_r \partial t_1} & \frac{\partial^2 Z(.)}{\partial t_1^2} & \frac{\partial^2 Z(.)}{\partial T \partial t_1} \\ \frac{\partial^2 Z(.)}{\partial t_r \partial T} & \frac{\partial^2 Z(.)}{\partial t_1 \partial T} & \frac{\partial^2 Z(.)}{\partial T^2} \end{bmatrix}$ are -18044.8, -135.27 and -25.4974. So, from the above Hessian matrix, it is observed that all the eigen values are negative. It may conclude that the Hessian matrix is negative definite and from the above point of view, it may conclude that the obtained results are optimal. Moreover, the concavity of the objective function is presented from Fig. 3 (a)-(c) for a fixed frequency of advertisement.

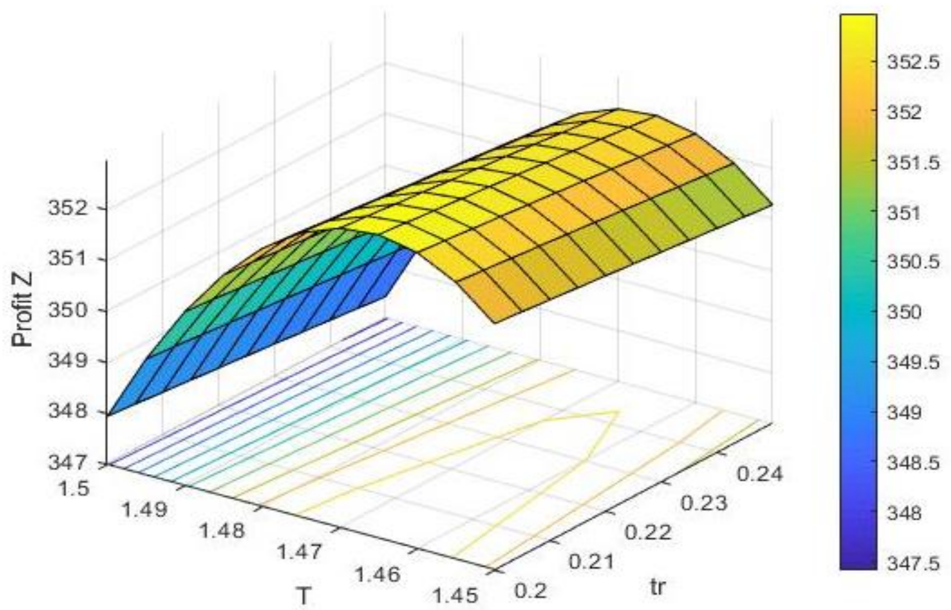

Fig. 3 (a). Concavity of the objective function w.r.t. T and t_r

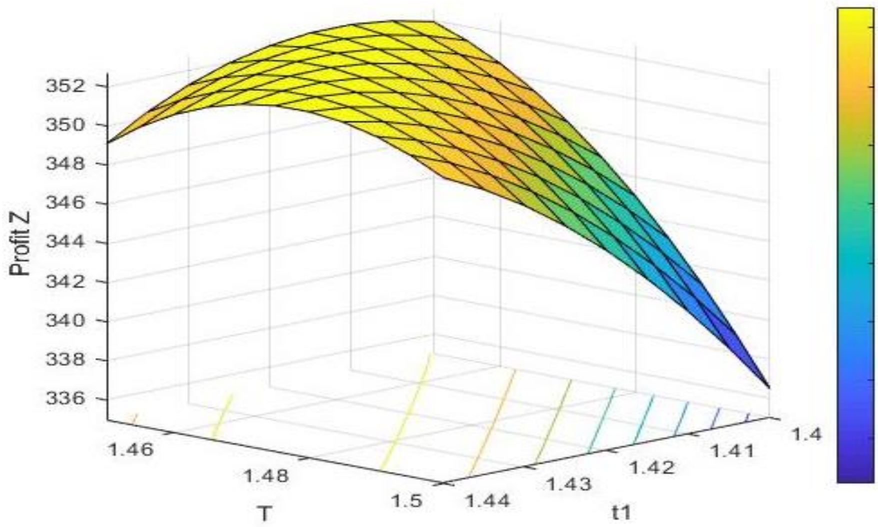

Fig. 3 (b). Concavity of the objective function w.r.t. *T* and t_1

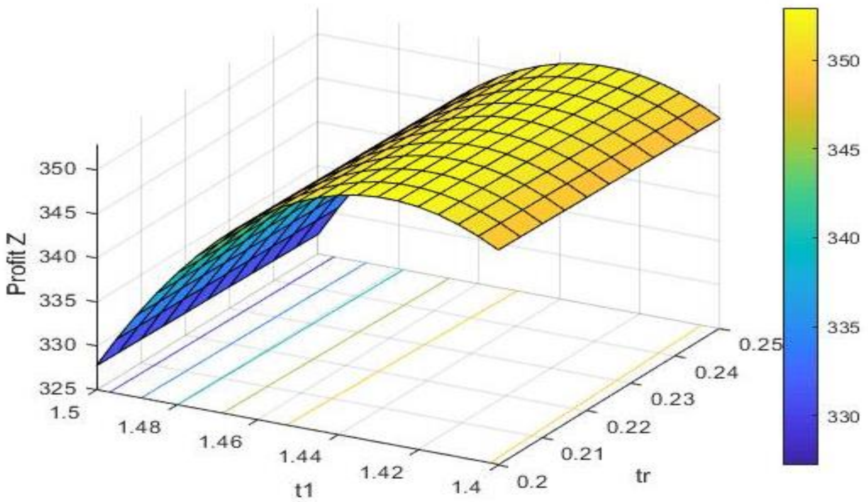

Fig. 3 (c). Concavity of the objective function w.r.t. t_1 and t_r

Post optimality studies

To observe the impact of the optimal solution of t_r, t_1, T, S, R and profit per unit time, a sensitivity analysis has been incorporated for changing one inventory parameters value and keeping fixed of the other parameters. These changes of one parameter are made from -20% to +20% and others parameters as the same, the results of the changes are shown from Figs. 4-12.

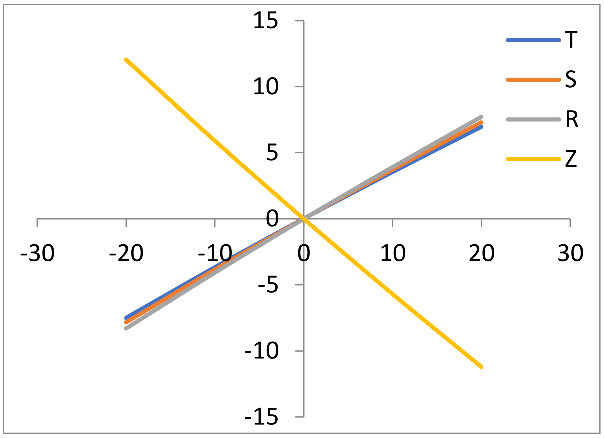

Fig. 4. Impact on optimal policy of 'O' on T, S, R and Z

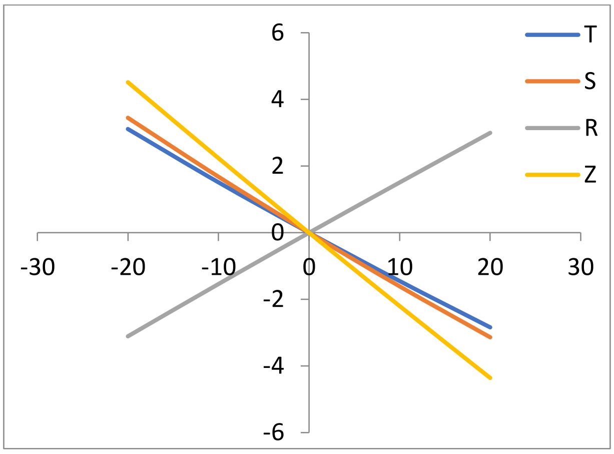

Fig. 5. Impact on optimal policy of 'C_{ho}' on T, S, R and Z

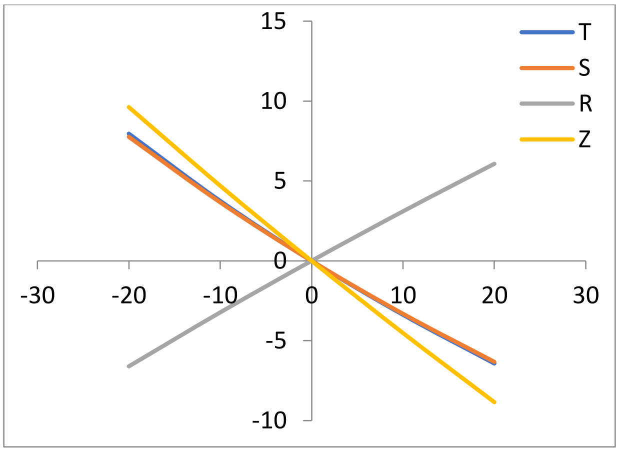

Fig. 6. Impact on optimal policy of 'β' on T, S, R and Z

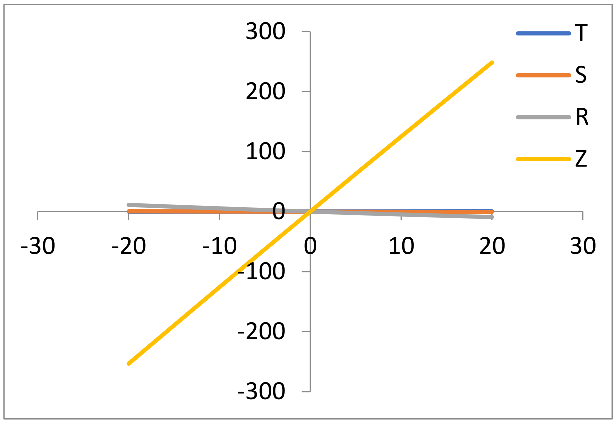

Fig. 7. Impact on optimal policy of ‘p’ on T, S, R and Z

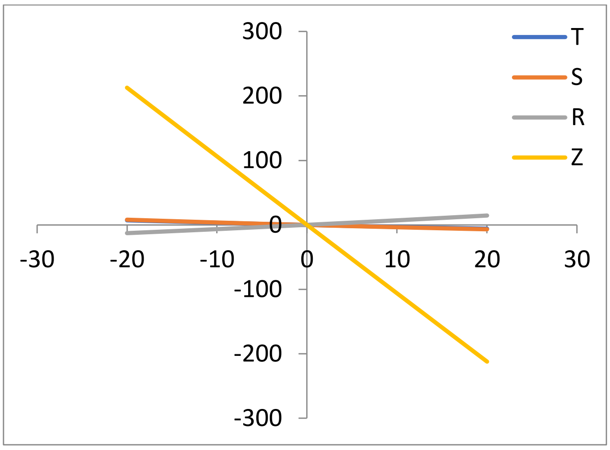

Fig. 8. Impact on optimal policy of ‘C_p’ on T, S, R and Z

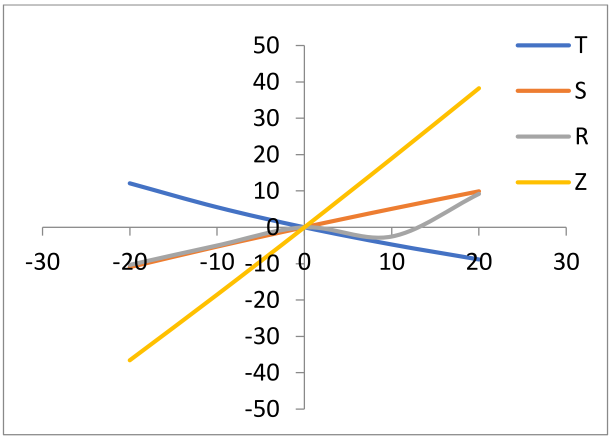

Fig. 9. Impact on optimal policy of ‘*a*’ on T, S, R and Z

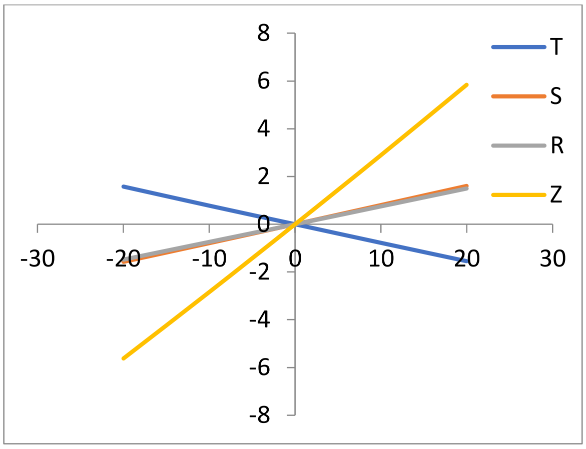

Fig. 10. Impact on optimal policy of ‘γ’ on T, S, R and Z

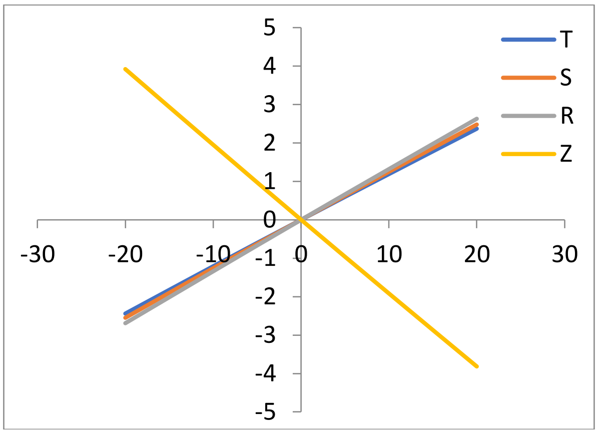

Fig. 11. Impact on optimal policy of ‘*G*’ on T, S, R and Z

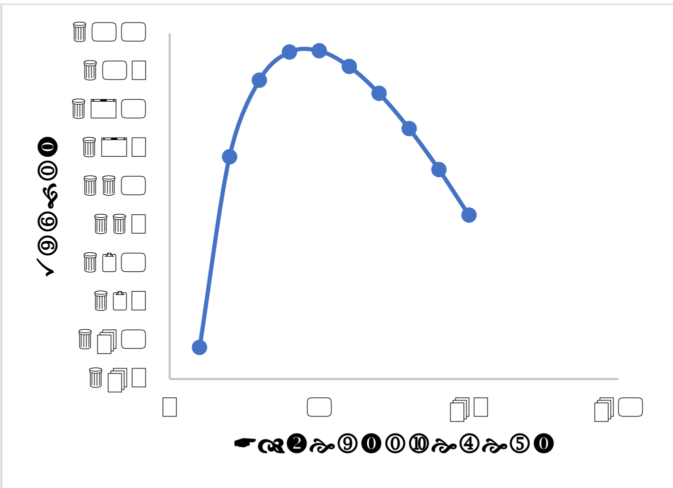

Fig. 12. Impact on optimal policy of ‘*A*’ on Profit

From Figs. 4-12, the following observation can be made:

- The business period of the system (*T*) is extremely sensible for the inventory parameters of ordering cost (*O*) and demand parameter (*a*). It is less sensitive for the parameters *n*, *M* and *K*. The inventory parameters of *n*, *M*, *K* hardly have any effect on the business period *T*. However, it is fairly sensible for the rest of the parameters.
- Highest initial capacity (*S*) is less sensitive for the related inventory parameters *n*, *M* and *K*. It has a huge impact on the parameters of ordering cost (*O*) and demand related parameter (*a*). Also, it is less effective on initial stock for the above-mentioned

parameters. The highest capacity (S) is moderately sensible for the rest of the others parameters.

- The total profit per unit (Z) is huge impact with for the inventory parameters of ordering cost (O), demand related parameter (a) and purchase cost (C_p). It is showing that if the above mentioned two parameters value is increased then total profit increases. The total profit of the system per unit time (Z) is slighter sensitive for instalment parameter n i.e., the instalment has less impact on the profit. It is to be mentioned that inventory systems are fairly sensible for the remaining parameters.
- From Fig. 12, it is clearly observed that after a certain level of advertisement profit increase but after that level if advertisement frequency increase then profit is decreased.

Managerial impact

The following remark and suggestions are concluded from post optimality analysis. If these suggestions are followed by the manager or the decision maker, then their business profit will increase.

- As the location parameter (a), ordering cost (O) and purchase cost (C_p) have a huge impact on the retailer's profit per unit time in both positive and negative directions respectively. So, the managers/decision-makers need to take decision cautiously in order to avoid loss run from their business and increase their profit.
- From the post optimality analysis, it is observed that the profit per unit time increase when the number of instalments increases throughout the lead time. Due to this reason, the decision maker must think about the selection of manufacturer or supplier. Those manufacturers or suppliers will provide a higher number of instalment facility they can select them for purchasing the goods.
- Advertisement cost has a great impact on the profit. Also, the advertisement of a product has huge impact on demand. Therefore, decision maker must think about the number of advertisement placed through popular media with reasonable cost in order to increase the profit.
- Advertisement for goods has a direct impact on the demand. However, it is remarkable that the manager cannot get unlimited advertisement of their product because the demand for the product may increase but after a certain level of advertisement profit will be decreased. So, the manager should select the optimal frequency of advertisement in order to avoid losses.

Conclusions

Here, a two-storage system for the non-instantaneous decaying item under an advance payment scheme with n equal instalments has been investigated. For the high complexity of the objective function, it cannot solve analytically. To avoid the complexity of the objective function, MATHEMATICA software is used in order to justify the applicability of the model by taking a numerical example. However, in concavity representation of the objective function, MATLAB software is used. The concavity of the profit function is shown pairwise (two decision variable at a time) along with the objective function. Also, it is remarkable that all the eigenvalues of the Hessian matrix are negative for the numerical example. So, the Hessian matrix is negative definite and the objective function is concave with respect to that particular example. It is also remarkable that the manufacturer/supplier cannot place an unlimited advertisement for the popularity of goods. Initially, the requirement of the goods is increased but after a certain level, the profit of the system is decreased. So, they need to think about how much advertisement will provide for increasing the total profit of the system reached a maximum level.

The suggested model can be extended in several directions by taking some other practical features via time-dependent demand, stock dependent demand, incorporate preservation technology, non-linear price variant demand, up-stream and down-stream payment policies, non-linear holding cost, time-varying holding cost, trade credit facilities, inflation, credit-based demand etc. Anyone can modify this model by taking the interval valued inventory costs or fuzzy valued inventory costs. Also, the parametric approach of an interval can be introduced in this model. There is a limitation of this model that the rented warehouse always not available nearby his/her shop. In that case, the smooth running of a business is a quite complicated task.

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Appendix

Total sales revenue (TE) $=P\int_0^{t_1} D dt + PR$

$=PDt_1 + PR=$ $pDt_1 + pD\left(1-e^{-\delta(T-t_1)}\right)/\delta$

Now $\frac{\partial TE}{\partial T} = pDe^{-\delta(T-t_1)}$

Again

$$\frac{\partial TC}{\partial T}=\frac{1}{T}\left[\begin{array}{l} C_s\left\{\frac{\partial R}{\partial T}(T-t_1)+(R-\frac{D}{\delta})+\frac{D}{\delta}e^{-\delta(T-t_1)}\right\} \\ +C_l D\left(1-e^{-\delta(T-t_1)}\right)+\frac{(n+1)}{2n}I_c k C_p M\frac{\partial R}{\partial T}\end{array}\right]$$

$$-\frac{1}{T^2}\left[\begin{array}{l} O+\left[C_p[Dt_d+W+\frac{D}{\alpha}(e^{\alpha(t_r-t_d)}-1)+\frac{D}{\delta}(1-e^{-\delta(T-t_1)})]\right] \\ +\left[\begin{array}{l} C_{hr}[(Wt_d-\frac{D}{2}t_d^2)+\frac{D}{\alpha^2}(e^{\alpha(t_r-t_d)}-\alpha(t_r-t_d)-1)] \\ +C_{ho}[(S-W)t_d+\frac{(S-W)}{\beta}(1-e^{\beta(t_d-t_r)})+\frac{D}{\beta^2}(e^{\beta(t_1-t_r)}-\beta(t_1-t_r)-1)]\end{array}\right] \\ +\left[C_s[(R-\frac{D}{\delta})(T-t_1)+\frac{D}{\delta^2}(1-e^{-\delta(T-t_1)})]\right]+\left[C_l D[(T-t_1)-\frac{\left(1-e^{-\delta(T-t_1)}\right)}{\delta}]\right] \\ +\left[\frac{(n+1)}{2n}I_c k C_p M(S+R)\right]+A*G\end{array}\right]$$

Where $\frac{\partial R}{\partial T}=De^{-\delta(T-t_1)}$