Markowitz-Based Cardinality Constrained Portfolio Selection Using Asexual Reproduction Optimization (ARO)

Mohammad Reza Sadeghi Moghadam[1][*], Taha Mansouri[2], Morteza Sheykhizadeh[3]

1. Department of Production and Operation Management, Faculty of Management, University of Tehran, Tehran, Iran
2. Department of Computing, Science and Engineering, University of Salford, Greater Manchester, UK
3. M.Sc. in Industrial Management, Department of Industrial Management, Faculty of Management, University of Tehran, Tehran, Iran

Abstract
The Markowitz-based portfolio selection turns to an NP-hard problem when considering cardinality constraints. In this case, existing exact solutions like quadratic programming may not be efficient to solve the problem. Many researchers, therefore, used heuristic and metaheuristic approaches in order to deal with the problem. This work presents Asexual Reproduction Optimization (ARO), a model-free metaheuristic algorithm inspired by the asexual reproduction, in order to solve the portfolio optimization problem including cardinality constraint to ensure the investment in a given number of different assets and bounding constraint to limit the proportions of fund invested in each asset. This is the first time that this relatively new metaheuristic is applied in the field of portfolio optimization, and we show that ARO results in better quality solutions in comparison with some of the well-known metaheuristics stated in the literature. To validate our proposed algorithm, we measured the deviation of the obtained results from the standard efficient frontier. We report our computational results on a set of publicly available benchmark test problems relating to five main market indices containing 31, 85, 89, 98, and 225 assets. These results are used in order to test the efficiency of our proposed method in comparison to other existing metaheuristic solutions. The experimental results indicate that ARO outperforms Genetic Algorithm (GA), Tabu Search (TS), Simulated Annealing (SA), and Particle Swarm Optimization (PSO) in most of test problems. In terms of the obtained error, by using ARO, the average error of the aforementioned test problems is reduced by approximately 20 percent of the minimum average error calculated for the above-mentioned algorithms.

Keywords: portfolio optimization, cardinality constraints, Markowitz mean-variance model, asexual reproduction optimization, efficient frontier.

1. Introduction

Portfolio optimization, which is the problem of allocating the initial amount of capital among a given number of assets or securities, has attracted a lot of attention in the field of quantitative finance (Moral-Escudero et al., 2006). In order to help investors with optimally forming their portfolio of assets, Markowitz (1952, 1959) has proposed a quantitative framework. Markowitz mean-variance model of portfolio selection, which caused the development of Modern Portfolio Theory (MPT), formulates the problem as a multi-objective optimization problem with two conflicting objectives: maximizing the expected return and minimizing the risk (measured by variance) of a portfolio (Esfahani, hossein Sobhiyah, &

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[*] **Corresponding Author, Email:** rezasadeghi@ut.ac.ir

Yousefi, 2016). Considering these two competing criteria simultaneously, there is no single optimal portfolio, but a set of portfolios forming the efficient frontier (EF). In other words, efficient frontier, also called Pareto-optimal front, is the collection of portfolios that result in the minimum risk for a given level of return or equivalently, the maximum return for a given level of risk.

The standard Markowitz model assumes a perfect market without any transaction costs and taxes, where short selling is forbidden, and assets are tradable in any non-negative fractions. This basic model belongs to the category of Quadratic Programming (QP) problems (Fernández & Gómez, 2007); thus, the efficient frontier could be found using standard QP solvers that are easily available and can guarantee finding the optimal solution, and be modified to include linear constraints (Chang et al., 2000). However, in the absence of these unrealistic assumptions or presence of some nonlinear real-world constraints, QP is not necessarily feasible for finding efficient portfolios any more.

Many researchers have tried to extend Markowitz's model (Markowitz, 1952) in order to capture more realistic market conditions by introducing some additional constraints. These include cardinality constraint, which limits the number of assets held in the portfolio; bounding (quantity or floor-ceiling) constraint, also known as buy-in thresholds, imposing lower and/or upper bounds on funds invested in each asset; pre-assignment constraint, reflecting the investor's preferences by requiring some specific assets to be held in the portfolio; round lot (minimum lots) constraint, which forces the amount invested in each asset to be a multiple of minimum transaction lot; class constraint which limits the total weight assigned to a class (assets with common characteristics); and turnover and trading constraints that impose upper and lower bounds respectively on the variation of the assets weight from one period to another, which are particularly useful in multi period investments (Crama & Schyns, 2003; Lwin et al., 2014; Ponsich & Antonio, 2012; Tollo & Roli, 2008).

According to Metaxiotis and Liagkouras (2012), cardinality and bounding constraints occupy the main focus of the researchers. In practice, many investors prefer to hold a certain number of assets in their portfolio so as to facilitate its management, decrease transaction costs, and assure a minimum degree of diversification. Moreover, they prefer to avoid holding very small and large proportion of assets in order to reduce administrative costs and risk, respectively (Anagnostopoulos & Mamanis, 2011; Lwin et al., 2014).

In this paper, we tackle the problem with regard to the extended Markowitz mean-variance model, which includes cardinality and bounding constraints. In this case, we refer to the EF as cardinality constrained efficient frontier (CCEF). By introducing the cardinality constraint into the classic quadratic programming model, this problem turns to a mixed integer quadratic programming one that is NP-hard (Bienstock, 1996; Moral-Escuderoet al., 2006; Shaw et al., 2008). In this case, exact optimization methods are not efficient for large problem sizes (Kalayci et al., 2017). Many researchers, therefore, take advantage of heuristic and metaheuristic approaches in order to deal with the problem (Maringer, 2006; Tollo & Roli, 2008). Although these approaches do not guarantee finding the optimal solution, they are efficient for finding near-optimal solutions.

Asexual Reproduction Optimization (ARO), proposed relatively recently in Farasat et al. (2010) and Mansouri et al. (2011), is an evolutionary individual-based metaheuristic algorithm inspired by the budding mechanism of asexual reproduction and has been used in very few studies (e.g., Khanteymoori et al., 2011; Kazemi et al., 2012; Noormohammadi Asl et al., 2014; Ahmadian & Khanteymoori, 2015; Yazdanparast et al., 2015). None of these studies deals with the portfolio selection problem (PSP).

The ARO has advantages that make it completely different from other metaphors. First, it is an individual-based algorithm. Thus, unlike population-based algorithms that require a

large amount of computational resources to convert, ARO consumes small amounts. Hence, it converges faster. The second case is mathematical convergence, so it has good exploration and exploitation rates. Third, ARO does not require parameter settings, so you are unlikely to have trouble setting parameters that are a common meta-cognitive problem such as genetic algorithms (GA), annealing simulation (SA), Tabu search (TS), and Particle Swarm Optimization (PSO). In addition, the ARO does not use any selective mechanism such as a roulette wheel. The inappropriate selection of selection mechanisms may lead to problems such as premature convergence due to excessive selection pressure. Fourth providers in many benchmark issues have shown the computational power of this algorithm. Fifth, ARO is a free model algorithm that can be applied to various types of optimization. (Mansouri et al., 2011) Finally, the ARO used in this paper presents better results than the algorithms used in other papers. For these reasons, we take advantage of ARO to tackle the Markowitz-based cardinality constrained portfolio selection problem. The main contribution of this study is to solve this problem more efficiently using a new approach. We apply a method that uses ARO to confront the portfolio selection problem. Our proposed method results in better quality solutions compared to some of the well-known metaheuristics that have been used in this field. Computational results are reported for five analyses of weekly price data with regard to the following indices for the period between March 1992 and September 1997: Hang Seng 31 in Hong Kong, DAX 100 in Germany, FTSE 100 in the UK, S&P 100 in the USA, and Nikkei 225 in Japan.

The rest of this paper is organized as follows. A literature survey of exact, heuristic, and hybrid approaches to the problem is presented in Section 2. Section 3 describes the generic mean-variance portfolio selection problem, followed by the specific model in the presence of cardinality and bounding constraints. Section 4 introduces the proposed ARO algorithm along with its application to the problem under consideration. Computational experiments and results are discussed in Section 5. Conclusion and future work are presented in Section 6.

2. Literature Review

According to Woodside-Oriakhi et al. (2011), researchers have dealt with the cardinality constrained portfolio optimization using either exact or heuristic approaches. In this paper, we consider a third category to structure our literature survey: hybrid methods that combine an exact method with a heuristic one.

2.1. Exact Approaches

As stated earlier, when considering the cardinality constraint into the model, exact methods may not be efficient to solve portfolio optimization problem for large problem samples. However, some researchers have tried to deal with the problem using a relaxed version of cardinality constraint that imposes an upper bound on the number of assets present in the portfolio. This approach, in which Equation (14) is an inequality rather than equality, has a significantly less computational complexity (Woodside-Oriakhi et al., 2011). Moreover, the results show that researchers were able to handle this version of problem for limited problem sizes (Lwin et al., 2014). Table 1 summarizes the exact approaches used in the literature to solve the PSP.

Table 1. Exact Approaches for Portfolio Selection Problem

Authors	Constraints	Method	Datasets	Risk measure
Bienstock (1996)	Upper bound on the cardinality	Branch-and-cut	3897 assets	Variance
Lee & Mitchell (1997)	Upper bound on the cardinality	Interior point nonlinear branch-and-bound solver	5 datasets involving up to 150 assets	Variance
Young (1998)	Transaction costs	Minimax	No computational results	Minimum return
Li et al. (2006)	Upper bound on the cardinality, round lots	Convergent Lagrangian and contour-domain cut	30 assets from the Hong Kong stock market	Variance
Shaw et al. (2008)	Upper bound on the cardinality	Lagrangian relaxation	8 datasets involving up to 500 assets	Variance
Vielma et al. (2008)	Upper bound on the cardinality	LP based branch-and-bound	Problem sets involving up to 200 assets	Variance
Bertsimas & Shioda (2009)	Upper bound on the cardinality, bounding	Tailored branch-and-bound	Problem sets involving up to 500 assets	Variance
Gulpinar et al. (2010)	Cardinality, bounding, round lots	DC functions programming	98 assets	Variance

2.2. Heuristic Approaches

With regard to the above-mentioned approach, using other risk measures than variance, Mansini and Speranza (1999) considered PSP with minimum transaction lots and showed that in this case, the problem of finding a feasible solution is NP-complete no matter what the risk measure is. In their work, they used Mean Semi-absolute Deviation as a measure of risk and presented three heuristics based on solving the linear programming relaxation to tackle the problem. Likewise, Kellerer, et al. (2000) considered the same risk measure in their paper. They added fixed transaction costs to the previous model and employed two of the three Mixed Integer Linear Programming-based (MILP-based) heuristics that were had been in the previous study. In a more recent work, Chang et al. (2009) considered different risk criteria other than variance; semi-variance, mean absolute deviation (MAD), and variance with skewness. They employed Genetic Algorithm (GA) and showed its efficiency for solving these problems in different risk measures. Considering the importance of both heuristic and hybrid approaches, this section provides a comprehensive literature review on both of these approaches in the context of portfolio selection problem, which are summarized in Table 2.

Table 2. Heuristic and Hybrid Approaches to Portfolio Selection Problem

Heuristic approaches				
Authors	**Constraints**	**Method**	**Datasets**	**Risk measure**
Anagnostopoulos & Mamanis (2010)	Class, bounding	NSGA-II, PESA, SPEA2	Two randomly generated data sets containing 200 and 300 assets	Variance
Anagnostopoulos & Mamanis (2011)	Cardinality, bounding	NPGA2, NSGA-II, PESA, SPEA2, e-MOEA	Seven test problems involving up to 2196 assets	Variance
Woodside-Oriakhi et al. (2011)	Cardinality, bounding	GA, SA, TS	Seven test problems involving up to 1318 assets	Variance
Deng et al., 2012)	Cardinality, bounding	PSO	Test problems provided by Chang et al. (2000)	Variance
Lwin & Qu (2013)	Cardinality, bounding	PBILDE	Test problems provided by Chang et al. (2000)	Variance
Lwin et al. (2014)	Cardinality, bounding, pre-assignment, round lots	MODEwAwL	Seven test problems involving up to 1318 assets	Variance
Ni et al. (2017)	Cardinality, bounding	DRTCPSO-AD, DRTCPSO-D, DRTCPSO-LIAD, DRTCPSO-LID	Test problems provided by Chang et al. (2000)	Variance
Kalayci et al. (2017)	Cardinality, bounding	ABC	Hang Seng, DAX, FTSE, S&P100, Nikkei, XU030, XU100	Variance
Meghwani & Thakur (2017)	Cardinality, bounding, pre-assignment, self-financing Liquidity	NSGA-II, MOEA/D, GWAS- FGA	FF38, FF48	VaR, CVaR
Liagkouras & Metaxiotis (2018)	Upper and lower limit of the cardinality constraint Lower and upper bound constraints, transaction costs	Proposed new Multi-period Fuzzy Portfolio Optimization Algorithm (MFPOA)	92 assets from FTSE-100 index in London	Possibilistic variance of the portfolio return at each period
Babazadeh & Esfahanipour (2019)	Cardinality Constraint Budget Constraint at each period	New design of NSGA-II	20 selected assets from S&P100	VaR
Ling et al. (2019)	Stochastic constraint Budget Constraint	A computationally tractable approximation approach based on second-order cone optimization	10Ind, 12Ind, iShare and DJIA, as in Rujeerapaiboon et al. (2016), with all data from the French data library, Yahoo finance, and the Wind database	Lower partial moment (LPM).
Ahmadi-Javid & Fallah-Tafti (2019)	Cardinality, bounding	Primal-dual interior-point algorithm EVaR- PD algorithm	50 assets 100 assets 200 assets 750 assets 1000 assets	Entropic value-at-risk (EVaR)
Gupta et al. (2019)	Budget Constraint, Cardinality No short selling constraint Contingent constraint	Polynomial goal programming approach	A sample of twenty assets (n = 20) from National Stock Exchange (NSE), India	Variance entropy
Mohammadi & Nazemi (2020)	Budget constraint	Neural network	10 securities	VaR

Table 2.

	Hybrid Approaches			
Research	**Constraints**	**Method**	**Datasets**	**risk measure**
(Moral-Escudero et al. (2006)	Cardinality, bounding	Combination of GA and QP	Test problems provided by Chang et al. (2000)	Variance
Streichert & Tanaka-yamawaki (2006)	Upper bound on the cardinality, maximum limits with one exception	Combination of a MOEA and a QP local search	Hang Seng 31, DAX 100	Variance
Branke et al. (2009)	Cardinality, bounding, 5-10-40 rule from German investment law	Envelope-based MOEA	Hang Seng 31, S&P 100, Nikkei 225, and a benchmark with 500 assets	Variance
Ruiz-torrubiano & Suárez (2010)	Upper bound on the cardinality, bounding	Preprocessing (pruning), SA, GAs, EDAs	Test problems provided by Chang et al. (2000)	Variance
Baykasoğlu et al. (2015)	Cardinality, bounding	GRASP-QUAD	Test problems provided by Chang et al. (2000)	Variance
Li et al. (2018)	Bankruptcy Budget Constraint transaction costs Total wealth at each period	GA combined with penalty function	8 stocks	Variance
Zhang & Li (2019)	Liquidity risk-free asset constrained in each period lower and upper limits No short selling of assets Cardinality	Hybrid DA-GA (Dragonfly Algorithm-Genetic Algorithm)	10 risky assets	Semi-entropy
Chen et al. 2020)	no-shorting constraint equality and inequality constraints	Radial basis function neural network and GA	Shanghai Stock Exchange	Variance
(Corazza, di Tollo, Fasano, & Pesenti, 2021)	Minimum desired expected return of the portfolio and budget constraints	PSO with a dynamic penalty approach	The daily closing prices of 38 of the 40 assets composing the Italian stock index	Variance

2.3. Research Gaps and Our Contributions

According to the literature reviewed above, since the pioneering work of Harry Markowitz (Markowitz, 1952), the mean-variance model of portfolio optimization has been the main framework for choosing optimal portfolios. Some extensions have been proposed for this model, out of which MVCCPO (including cardinality and bounding constraints) has attracted the most of researchers' attention. Using metaheuristic algorithms became the main trend for dealing with this model after the research conducted by Chang et al. (2000), and hybrid methods that take advantage of both heuristic and exact solutions have been used more recently in the literature.

The contribution of our paper is to complement the reviewed literature by proposing a new approach for portfolio selection based on the Markowitz mean-variance-model, which results in better quality solutions in comparison with some of the well-known metaheuristics stated in the literature.

3. Methodology

Let us start with the basic (unconstrained) Markowitz model. In its multi-objective form, it can be formulated as follows:

$$Minimize \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij} \tag{1}$$

$$Maximize \sum_{i=1}^{N} w_i \mu_i \tag{2}$$

$$subject\ to \sum_{i=1}^{N} w_i = 1 \tag{3}$$

$$0 \le w_i \le 1, \quad i = 1,\ldots,N \tag{4}$$

where N is the number of available assets, μ_i is the expected return of asset i, σ_{ij} is the covariance between asset i and j, and w_i is the decision variable representing the proportion of money invested in asset i. Equation (1) minimizes the risk of the portfolio (measured by variance) while Equation (2) maximizes the expected return of the portfolio. Equation (3) defines the budget constraint which forces the investment of all the money in hand, i.e., asset weights must sum up to one. Finally, Equation (4) states that all weights should be nonnegative.

By solving the above model, a set of efficient portfolios can be found. These Pareto-optimal (non-dominated) solutions form the unconstrained efficient frontier (UEF), i.e., a continuous curve representing the best possible tradeoff between risk and return.

This bi-objective model can be also represented as a single objective optimization problem; therefore, it could be solved by applying single objective solution techniques. The famous single objective representation of the basic Markowitz model is as follows:

$$Minimize \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij} \tag{5}$$

$$subject\ to \sum_{i=1}^{N} w_i \mu_i = R^* \tag{6}$$

$$\sum_{i=1}^{N} w_i = 1 \tag{7}$$

$$0 \le w_i \le 1, \quad i = 1,\ldots,N \tag{8}$$

This model attempts to minimize risk by considering the expected return as a constraint. Hence, solving the above single objective problem for different levels of expected return results in tracing the unconstrained efficient frontier.

According to Chang et al. (2000), designing a heuristic based on the above formulation is difficult in that it requires the expected return of the portfolio to be exactly R^*.

In practice, for tracing the UEF, a popular approach is to introduce a weighting parameter λ ($0 \le \lambda \le 1$); thus, the objective function could be represented in a Lagrangian relaxation form (Anagnostopoulos & Mamanis, 2011; Chang et al., 2000):

$$Minimize\ \lambda \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij} - (1-\lambda) \sum_{i=1}^{N} w_i \mu_i \tag{9}$$

$$subject\,to \sum_{i=1}^{N} w_i = 1 \tag{10}$$

$$0 \leq w_i \leq 1, \quad i = 1,\ldots,N \tag{11}$$

By solving this QP problem for various values of λ, the UEF can be traced from the portfolio with maximum return (λ = 0) to the portfolio with minimum level of risk (λ = 1). Chang et al. (2000) showed that when considering the unconstrained problem, by changing λ in Equation (9), we can obtain exactly the same efficient frontier as we would get by solving Equations (5) - (8) for varying values of R*.

In order to find the cardinality constrained efficient frontier (CCEF), many researchers extended the above-mentioned model by adding cardinality and bounding constraints (e.g., Baykasoğlu et al., 2015; Chang et al., 2000; Cura, 2009; Deng et al., 2012; Fernández & Gómez, 2007; Woodside-Oriakhi et al., 2011):

$$Minimize\,\lambda \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij} - (1-\lambda)\sum_{i=1}^{N} w_i \mu_i \tag{12}$$

$$subject\,to \sum_{i=1}^{N} w_i = 1 \tag{13}$$

$$\sum_{i=1}^{N} z_i = K \tag{14}$$

$$\varepsilon_i z_i \leq w_i \leq \delta_i z_i, \quad i = 1,\ldots,N \tag{15}$$

$$z_i \in \{0,1\}, \quad i = 1,\ldots,N \tag{16}$$

where z_i is the decision variable indicating the existence of each asset in the portfolio, hence it is equal to 1, if asset i is included in the portfolio and zero otherwise. Equation (14) defines the cardinality constraint (portfolio consists of exactly K assets) and Equation (15) defines the bounding constraint, which imposes lower and upper limits on the weight of each asset. In this work, we will consider the same MVCCPO model (Equations (12) – (16)).

4. ARO for Portfolio Selection Problem

In this section, we present our proposed algorithm for solving the cardinality constrained portfolio selection problem. First, we give a brief overview of the general ARO. Then, the particular implementation of this proposed algorithm that is customized for finding the CCEF will be presented.

4.1. Asexual Reproduction Optimization

ARO, which is an individual-based evolutionary algorithm modeling the budding mechanism of asexual reproduction, was first described by Farasat et al. (2010) and Mansouri et al. (2011). In ARO, each individual produces a bud via a reproduction mechanism; afterward, the bud and its parent compete with respect to their fitness, which is obtained from the objective function of the underlying optimization problem. Through competition for limited resources, the fitter one will survive, while the other will be discarded. This reproduction cycle is repeated until the stopping criteria are met. Consider the following optimization problem:

$$Maximize\,f(X) \qquad subject\,to\,X \in S \tag{17}$$

where $X = (x_1, x_2, \ldots, x_n); x_i \in \mathbb{R} \ (i = 1, 2, \ldots, N)$ are the decision .variables, $f(X)$ is the objective function, and S defines the search space.

The pseudo code of ARO is illustrated in Figure 1.

```
begin
t = 1;
P = Initialize (S);        // parent initialization
Fitness_P = f (P);         // calculating fitness of the parent
while (stopping conditions are not met)      // stopping criteria
{
  Bud (t) = Reproduce (P);    // bud reproduction from the parent
  Fitness_Bud (t) = f (Bud (t));       // calculating fitness of the bud
  if (Fitness_Bud (t) is better than Fitness_P)
       P = Bud (t);      // parent is replaced by bud
  else
       clear Bud(t);     // discard bud
  t = t +1;
}
end
```

Figure 1. Pseudo code for ARO.

4.2. The Proposed ARO for Finding the CCEF

Here, we introduce the customized version of ARO to deal with the cardinality constrained portfolio optimization problem.

4.2.1. Notation

Table 3 introduces the notations used to describe the proposed version of ARO to solve the problem.

Table 3. Notations for the Proposed ARO

Symbol	Description
N	The whole number of available assets
K	The number of assets present in the portfolio
w_i	The proportion of capital invested in asset i
ε_i	The lower bound on the proportion invested in asset i
δ_i	The upper bound on the proportion invested in asset i
T_{max}	The number of running iterations for ARO
$R[x, y]$	A random integer number in $[x, y]$
$r[0, 1]$	A random real number in $[x, y]$
R	The set of i whose proportions are fixed at δ_i
Q	The set of K distinct assets in the current solution
g	The length of selected substring from parent's *chromosome*
b	The number of buds reproduced from the current parent

4.2.2. Solution Representation and Encoding

In our solution representation, a vector of size 2K is used to represent a portfolio. This vector consists of two distinct parts, the first part indicates the asset indices present in the portfolio, and the second part determines the proportion of capital to be invested in each asset in the portfolio. Therefore, the first part would be an integer vector of size K with its elements belonging to $\{1, 2, \ldots, N\}$, and the second part consists of K real numbers from [0,1] (Figure 2).

| x_1 | x_2 | ... | x_k | w_1 | w_2 | ... | w_k |

Fig. 2. Solution representation.

where x_i is an integer variable that belongs to $\{1, 2, ..., N\}$ and represents the index of the ith asset in our portfolio.

As mentioned before, we will have K distinct assets in our portfolio, so, $(i = 1, 2, ..., K)$. In the second part of the solution representation, $w_i \in [0,1]$ denotes the value of investment in the asset x_i. For instance, if K= 4, which means that we are constrained to have 4 distinct assets in our portfolio, and the whole number of assets in the market is N=100, we should pick 4 distinct integers from $\{1, 2, ..., 100\}$ to represent the asset indices we are going to hold in the portfolio. Assume that we pick asset number 1, 7, 34, and 87, and put the equal weights for them in the portfolio. Our solution representation will be like this:

| 1 | 7 | 34 | 87 | 0.25 | 0.25 | 0.25 | 0.25 |

Fig. 3. A sample of our solution representation for K= 4.

It is important to note that each number in $\{1, 2, ..., N\}$ cannot appear more than once in the integer part of the chromosome.

4.2.3. Constraints Satisfaction

To meet the bounding constraint and ensuring that the sum of the proportions invested in assets equals one (Equation (15) and Equation (13)), the following approach is applied based on Chang et al. (2000):

Let Q be the set of K distinct assets in the current solution. The lower limits constraint can be satisfied if all weights in the candidate solution are adjusted by setting $w_i' = \varepsilon_i + w_i(1 - \sum_{i \in Q} \varepsilon_i) / \sum_{i \in Q} w_i$. Note here that $\sum_{i \in Q} w_i' = 1$. Therefore, the above formula satisfies both the lower proportion limits and sum to one.

Let R be the set of i whose proportions are fixed at δ_i. In order to satisfy upper limits constraint, an iterative algorithm can be applied as shown in Figure 4.

begin
$R = \emptyset$;   *// let R be the set of i whose weights are fixed at δ_i*
while (there exists an $i \in Q - R$ with $w_i > \delta_i$)   *// iterate untill feasible*
{
  for $(i \in Q - R)$
      if $(w_i > \delta_i)$
          $R = R \cup \{i\}$;   *// add i to R if w_i is greater than δ_i*

  $w_i = \varepsilon_i + \frac{w_i(1 - \sum \varepsilon_i)}{\sum w_i}$; $\forall i \in Q - R$ *// adjust weiths for each i in Q - R*
  $w_i = \delta_i$; $\forall i \in R$ *// adjust weiths for each i in R*
}
end

Figure 4. Pseudo code for satisfying upper limits constraint

It is also noteworthy that there is no need for any mechanism to handle the cardinality constraint (Equation (14)), since our solution representation presented in Section 4.2.2 requires each solution to contain exactly K distinct assets.

4.2.4. Mutation

In order for our proposed ARO to maximize diversity, we present two types of mutation.

4.2.4.1. Mutation of Shares

In this type of mutation, a bud is reproduced by altering some genes from the integer part of its parent's chromosome. In optimization terms, a new solution is generated by changing some asset indices present in the portfolio while weights remain unchanged.

In order to reproduce the bud, a substring of length g is randomly selected from the integer part of the parent's chromosome. Thereafter, the genes presented in this substring are replaced with g integers which are absent in the remaining string.

To clarify more, suppose that we have $N = 10$, $K = 5$, and the integer part of the parent's chromosome is shown in Figure 5. In order to select a substring, we randomly generate two distinct integers in $[1, K]$, i.e., $r_1 = [1, K]$ and $r_2 = R[r_1, K]$. so, $g = r_2 - r_1 + 1$. Assume that $r_1 = 2, r_2 = 3$. Hence, $\{8, 5\}$ should be eliminated from parent's chromosome, and a string of length 2 composing of two distinct elements from $\{1, 3, 5, 6, 8, 9, 10\}$ (e.g., $3, 8$) will be substituted (see Figure 5).
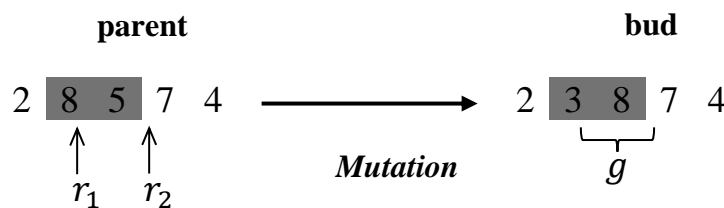

Figure 5. Mutation of shares.

4.2.4.2. Mutation of Weights

We use two kinds of variation here to expand the search space, namely stochastic and chaotic ones. The former is used when we are stuck in local optimum; it helps us to exit from it. By using the latter, we try to visit maximum number of points near the solution. To select the kind of variation, we use the following function:

$$f(i,b) = \sin\left(\max(1 - \varphi^{\ln(i)} / b, 0) * \frac{\pi}{2}\right)$$ (18)

where i shows the number of iteration, b is a variable representing the number of iterations we searched in local − i.e., the number of buds reproduced from the current parent − and φ is the golden number, which is approximately equal to 1.618. The value of $f(i, b)$ is decreasing in i and increasing in b. Equation (18) tries to produce a probable measure to determine the extent of being stuck in local optimum. We examined the above-mentioned function to produce this measure and found it suitable for our purpose.

Then, based on the value of $f(i, b)$, we use the following procedure to select the variation: A random real number is generated in $[0, 1]$, i.e., $r_3 = r[0, 1]$. If r_3 is lower than $f(i, b)$, stochastic variation will be selected. Otherwise, we apply the chaotic variation.

In stochastic variation, we select a random substring of length g from the real part of the parent's chromosome, and then define p as follows:

$$p = 1 / (1 + \ln(g))$$ (19)

For each gene from the selected substring, let $r_4, r_5 = r[0,1]$. If $r_4 \leq p$ and $r_5 \leq 0.3$, the value of the gene will be replaced with p multiplied by a random number in $[0,1]$. If $r_4 \leq p$ and $r_5 > 0.3$, the new value of the gene takes a random real number in $[0,1]$. Otherwise, the value of the gene remains unchanged.

In chaotic variation, we apply these steps for every gene in the real part of the parent's chromosome: let $r_6 = r[0,1]$. If $r_6 \leq 0.2$, the value of the gene should be multiplied by $0.2 * f(i,b)$. If r_6 belongs to $[0.3, 0.7]$, then let $r_7 = r[0,1]$, and the value of the gene will be multiplied by $r_7 + 0.2 * f(i,b)$. Otherwise, the value of the gene remains unchanged.

4.2.4.3. Termination

Our proposed ARO terminates after running for a predefined number of iterations, T_{max}.

5. Computational Results

In this section, our proposed ARO is evaluated and compared to other well-known existing heuristics used for tackling the cardinality constrained portfolio optimization based on the standard test problems. The heuristics that are used for comparison are Genetic Algorithm (GA), Simulated Annealing (SA), Tabu Search (TS), and Particle Swarm Optimization (PSO). We use the results reported by Chang et al. (2000) for GA, SA, and TS, while PSO results were those reported by Deng et al. (2012). We report the computational results for finding 50 different portfolios on the CCEF for each data set using the values $\varepsilon_i = 0.01, \delta_i = 1 \ (i = 1, 2, ..., N)$, and $K = 10$. We set the number of iterations, which is the termination condition for ARO, to 20000. The proposed algorithm is implemented in MATLAB language.

5.1. Datasets

The performance of our proposed algorithm is evaluated on the benchmark data related to five well-known major market indices from the publicly available OR-Library (Beasley, 1990). These test problems were built based on weekly prices between March 1992 and September 1997 for the following market indices: Hang Seng 31 in Hong Kong, DAX 100 in Germany, FTSE 100 in UK, S&P 100 in USA, and Nikkei 225 in Japan. The number of assets, N, related to each dataset is 31, 85, 89, 98, and 225, respectively. These data files contain mean return of each stock, covariance between these stocks, and the unconstrained efficient frontier composing of 2000 points (i.e., standard efficient frontier) , which are accessible at http://people.brunel.ac.uk/~mastjjb/jeb/orlib/portinfo.html. They were first provided by Chang et al. (2000) and have used in many other studies since then (e.g., Baykasoğlu et al., 2015; Chiam et al., 2008; Cura, 2009; Deng et al., 2012; Fernández & Gómez, 2007; Lwin & Qu, 2013; Moral-Escudero et al., 2006; Ruiz-torrubiano & Suárez, 2010; Schaerf, 2002).

5.2. Performance Indicator

To evaluate the performance of a heuristic, the quality of results could be measured in terms of the deviation of obtained results from the optimal solution (Woodside-Oriakhi et al., 2011). In the case of finding the cardinality-constrained efficient frontier, because of unavailability of the optimal CCEF, the quality of results could be measured according to their deviation from UEF, which can be found simply by QP. Thus, we used exactly the same approach previously proposed by Chang et al. (2000) as the most commonly used approach in the literature. Examples of the other studies that have used the same approach include Woodside-Oriakhi et al. (2011), Deng et al. (2012), Lwin and Qu (2013), and Baykasoğlu et al. (2015).

Let (s_p, R_p) be the standard deviation and return corresponding to a portfolio p found by ARO heuristic. By using linear interpolation, we can find s_p^*, which is the standard deviation associated with R_p in standard efficient frontier. Hence, the standard deviation error of portfolio p is defined as follows:

$$\text{Standard deviation error } (p) = 100 \left| (s_p - s_p^*) / s_p^* \right| \tag{20}$$

Similarly, let R_p^* be the return associated with s_p using linear interpolation in standard efficient frontier. Then, the return error of portfolio p would be:

$$\text{Return error } (p) = 100 \left| (s_p - s_p^*) / s_p^* \right| \tag{21}$$

Furthermore, the minimum between two above-mentioned errors for portfolio p is defined as percentage error, and by averaging this for all obtained portfolios, we can define mean percentage error.

$$\text{Percentage error } (p) = min \left\{ 100 \left| (s_p - s_p^*) / s_p^* \right|, 100 \left| (R_p - R_p^*) / R_p^* \right| \right\} \tag{22}$$

$$\text{Mean percentage error} = \sum_{p=1}^{50} percentage\, error (p) / 50. \tag{23}$$

In other words, after obtaining all portfolios based on the targeted portfolios, we measure the vertical and horizontal distances to the standard UEF for each portfolio and compute the minimum of these two numbers. Then, our performance indicator, mean percentage error, is the average of these derived minimums for all obtained portfolios. For more details about this approach, see Chang et al. (2000).

5.3. Experiments

As mentioned before, the standard efficient frontier is the set of 2000 optimal portfolios on the UEF that is available from OR-Library (Beasley, 1990) for each of the five tested data sets. Figure 6 shows the heuristic frontier that is formed by the 50 portfolios found by ARO as well as the standard efficient frontier for each data set.
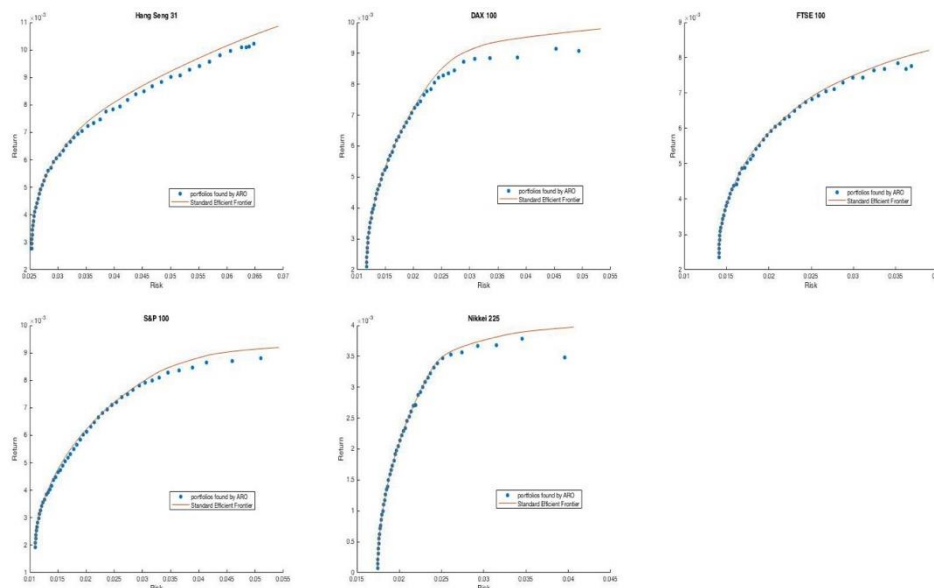
Figure 6. Standard efficient frontier and portfolios found by ARO for Hang Seng 31, DAX 100, FTSE 100, S&P 100, and Nikkei 225.

This figure clearly illustrates that our proposed algorithm performs really well when dealing with portfolios requiring lower levels of risk and expected return. However, when finding portfolios that have much higher risk and expected return, the distance between standard efficient frontier and ARO heuristic frontier increases. The reason is that when we want to constitute portfolios with higher expected return, we should choose fewer assets that have higher mean return. In the unconstrained problem, it would be possible to choose even one asset with the highest level of mean return. However, when dealing with the constrained problem, our algorithm is forced to select exactly 10 assets because of the cardinality constraint. Hence, the portfolios that are close to top right corner of CCEF have much significant percentage error.

Table 4 compares our results with those obtained by Chang et al. (2000) and Deng et al. (2012) based on the mean percentage error for each data set. The best mean percentage error among them for each problem is written in boldface. These results show that our proposed method outperforms other heuristics in four out of five test problems. Therefore, the superiority of it is clear from the experimental results.

Table 4. Comparison of Proposed ARO With Other Heuristics for Finding the CCEF

Index	N	GA Chang et al. (2000)	SA Chang et al. (2000)	TS Chang et al. (2000)	PSO Deng et al. (2012)	ARO
Hang Seng	31	1.0974	1.0957	1.1217	1.0953	1.4181
DAX 100	85	2.5424	2.9297	3.3049	2.5417	1.3190
FTSE 100	89	1.1076	1.4623	1.1217	1.06283	0.8151
S&P 100	98	1.9328	3.0696	3.3092	1.6890	1.4468
Nikkei	225	0.7961	0.6732	0.8975	0.6870	0.6179
Average	-	1.4953	1.8461	2.0483	1.4152	1.1234

6. Conclusion

In this paper, we considered the portfolio selection problem under cardinality constraint, which requires a predetermined number of assets to be present in the portfolio as well as bounding constraint that impose upper and lower limits on the proportions of capital invested in each asset. These real world constraints turn the problem to an NP-hard one. Consequently, the classical methods may not be efficient to find the optimal solution for large problem sizes.

In our work, a version of ARO is proposed to find the cardinality constrained efficient frontier. Our algorithm uses two types of mutation that modify share indices and weights of them separately. We also took advantage of both stochastic and chaotic variations for mutation of shares. We evaluated the performance of the proposed approach using standard data sets considered previously in the literature that are related to five major market indices containing up to 225 assets. We also compared the results with those related to some well-known heuristics proposed previously to tackle the problem. The comparison showed that our proposed ARO outperforms GA, SA, and TS applied by Chang et al. (2000) to the problem as well as the PSO proposed by Deng et al. (2012) in most cases. Numerical results showed that by using ARO, the average error of the aforementioned test problems is reduced by approximately 20 percent of the minimum average error calculated for the above-mentioned algorithms (see Table 4). It is recommended that future efforts might focus on using the competitive co-evolutionary genetic algorithm to deal with the problem that is currently underway and test the model from different perspective. In addition, improvement and modification in constrains and objects of proposed model in context of portfolio selection problem are suggested for future efforts.

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