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RESEARCH PAPER

Closed form solutions of the Navier's Equations for axisymmetric elasticity problems of the elastic half-space

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Abstract

Closed form solutions are derived in this paper for Navier's equations for axisymmetric elastic half-space problems. They are solved assuming body forces are disregarded. The Boussinesq problem is considered. The displacements are used to obtain the stress fields. The shear stress-free boundary conditions on the boundary plane and the equilibrium of vertical stress and applied load are used to completely determine displacements and stresses. Other axisymmetric load problems considered are: (i) uniform (ii) conical (iii) inverted conical distributions. In each case, the Boussinesq solution is used as a Green function, yielding the vertical stress field as double integration problem. The vertical stress field for uniform load is obtained in terms of complete elliptic integrals of the second and third kinds. The vertical stress distribution under the center of a circular foundation under uniform load is obtained as a particularization of the solution for vertical stress at any point in the elastic half-space. The same result is derived by using the point load solution as an integral Kernel function. For conical distribution of load, the point load solution is used as a Green function, reducing the problem to double integration. The closed form expressions obtained for the vertical stress distributions under the center of the circular foundation for all the axisymmetrical load distributions considered are radially symmetrical functions; which agree with the symmetrical nature of the problem. The results obtained for all the load types considered are identical with previous results found in the literature.

Keywords: Navier's differential equations of equilibrium, Axisymmetric elasticity problem, Classical Boussinesq problem, Elastic half-space

Introduction

Background

Elasticity problems of the elastic half-space which involve the determination of the stress fields, and displacement fields within the half-space due to point and distributed loads acting on the boundary are problems of the classical mathematical theory of elasticity [1 – 21]. The elastic half-space material can be assumed to be isotropic or non-isotropic, or orthogonally isotropic (orthotropic); homogeneous or heterogeneous. Elastic half-space problems of heterogeneous, non-isotropic materials are usually very difficult to solve and in many of such problems, rigorous mathematical solutions are not available [1 – 14].

Elastic half-space problems are extensively encountered in the analysis and design of foundation structures, or structural footings, and road pavements. Axisymmetric elasticity problems of the elastic half-space are characterized by a circular symmetry of the state of the stress about a

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vertical axis, usually the vertical axis of symmetry of the applied point or distributed load. In general, in axially symmetric elasticity problems, the stress fields have a radial symmetry about a vertical axis of symmetry, which is usually the axis of application of the point load for the Boussinesq problem or the vertical axis of the centre of the circular foundation for the case of circular foundation areas subject to radially symmetric load distributions.

Typical cases of axisymmetric elasticity problems are:

(i) the Boussineq problem of finding stress and displacement fields due to a vertical point load applied at the origin of an elastic half-space considered homogeneous, isotropic and linear elastic

(ii) finding stress and displacement fields in elastic half-space due to circular foundation areas subject to radially symmetric load distributions such as (a) uniformly distributed loads, and (b) conical load distribution.

Axially symmetric elasticity problems are governed by the simultaneous requirements of the differential equations of equilibrium, the strain displacement relations and the constitutive equations that relate stresses to strains. These governing systems of equations are also required to satisfy the compatibility equations and the traction and deformation boundary conditions [1 – 21]. Closed form solutions of axisymmetric elasticity problems expectedly involve intensive analytical rigours, and are often unwieldly.

Three basic methods are used in the formulation of the governing equations of axially symmetric elasticity problems and they yield simplifications in the resulting governing equations to be solved. The three methods are the displacement method, the stress method and the mixed (hybrid) method [1 – 15]. The displacement methods are based on reformulation of the system of governing equations involving the differential equations of equilibrium, the kinematics equations and the material constitutive equations such that the stresses and strains components are eliminated, and the displacement components become the only unknown primary variables of the formulation. The governing equations are consequently reduced in number rendering the problem more amenable to solution. The displacement formulation was presented by Navier and Lamé and the resulting equations called the Navier displacement equations or the Lamé displacement equations.

In the stress-based methods, the system of governing equations that are expressed using stresses, strains and displacement components as the unknowns are reformulated such that the strains and displacements are eliminated and stresses are the only unknown primary variables. Consequently, the number of equations becomes reduced rendering the solution process to be easier.

Researchers such as Beltrami, Michell, Airy, Maxwell, Ike and Morera presented stress-based methods. In the mixed (hybrid) method, which is not commonly applied, the governing equations are formulated such that some components of displacement and some components of stress are the unknowns, and the rest of the displacement and stress components are eliminated.

The simplifications consequential to the reformulation of the general elasticity problem have inspired research on the development of stress and displacement functions that satisfy the stress and displacement formulation of the elasticity problems [22 – 23]. Such functions further simplify the solutions of elasticity problems to the search for suitable stress or displacement potential functions that satisfy the boundary conditions of the considered problem.

Airy, Morera, Maxwell, Love, Ike, Michell, Nowacki, Egorov and Boussinesq have derived stress functions of the space variables that satisfy the elasticity problem governing equations. Elasticity problems of the elastic half-space for various cases of boundary loads have been studied using stress-based and displacement-based methods. Nwoji et al [22] used the Green and Zerna displacement potential function method to obtain the solutions for stress and displacements for the three-dimensional small deformation elasticity problem of a point load acting at a point on the boundary of a linear elastic, homogeneous, isotropic medium of semi-infinite extent $(-\infty \le x \le \infty, -\infty \le y \le \infty, 0 \le z \le \infty)$. Their formulation reduced the problem to one of finding a suitable harmonic function satisfying the loading/stress boundary conditions. They used the harmonic function, the strain displacement relations, and the stress-strain relations to determine the stress fields and the displacement fields. Nwoji et al [23] used the Boussinesq displacement potential functions to solve the elastic half-space problem for a point load acting at the origin of the three dimensional Cartesian coordinate space. They found the displacement field components from the Boussineq potential functions using Love's expressions; and then used simultaneously the kinematics and material constitutive laws (relations) to obtain the stress fields from the displacement fields. They obtained solutions that were identical with solutions from the literature.

Ike *et al* [21] used the Trefftz potential function method to derive the solutions to the 3D elasticity problem of a point load acting at the origin of a linear elastic homogeneous, isotropic half-space. The Trefftz method simplified the problem to one of finding a suitable harmonic function of the space coordinates that is bounded and satisfies the loading and stress boundary conditions. The functions were used together with the kinematic and constitutive relations to obtain the stress and displacement fields. They obtained identical solutions as the Boussinesq solutions. Ike [14] used the Fourier-Bessel transformation method in a stress-based formulation to determine the vertical stress fields in axisymmetric elasticity problems of elastic half space involving circular foundation areas subject to uniformly distributed loads. The biharmonic stress compatibility equation was solved using the variable-separable technique to obtain a general solution for the bounded stress functions as Fourier-Bessel integrals. Egorov expressions for the vertical stress fields defined in terms of harmonic functions were used to obtain the vertical stress fields. The load distribution was similarly transformed by the Fourier-Bessel transformation. Enforcement of the boundary condition of the equilibrium of the internal vertical stress at the $z = 0$ plane and the applied load was used to obtain the unknown parameter of the bounded Fourier-Bessel transform integral, and hence the bounded stress function was completely found. The vertical stress fields were determined from the bounded stress function using Egorov expressions. Evaluation of the integration problem gave analytical expressions for the vertical stress fields in the elastic half-space. The vertical stresses at any point under the center and at any point at a radial distance r at a depth z were computed and tabulated. The mathematical expressions obtained were identical with those found in the literature, thus validating the study.

Ike [17] used the Hankel transform method to derive general solutions for the stress and displacement fields in semi-infinite, linear elastic, isotropic soil under axisymmetric load. Hankel transformation was applied to the governing equations in a stress-based formulation to obtain the Love stress function. Hankel transformation was similarly applied to the stress and displacement fields to obtain general solutions for the stresses and displacements. The general solutions obtained were used to solve the specific axisymmetric problem of Boussinesq, and it was found that the solutions agreed with the literature. Ike [18] also used the Hankel transform method to derive solutions for stresses and displacement fields in homogeneous, isotropic linear elastic half-space subject to uniformly distributed axisymmetric load over a circular area on the boundary (z

= 0 plane). Hankel transformation of the biharmonic stress compatibility equation was done to obtain bounded stress functions for the elastic half-space problem. Hankel transformations were similarly applied to the Love stress functions yielding the stresses and displacements in the Hankel transform space. Boundary conditions were used to obtain the unknown constants of the stress function. Inversion of the Hankel transform expressions for the stresses and displacements gave the corresponding expressions in the physical domain space, which were found to be identical with the results of previous research works which applied other different methodologies.

Onah *et al* [24] used the Boussinesq displacement potential functions to determine the vertical stress distribution and the vertical displacements in linear elastic, homogeneous, isotropic elastic half space due to uniformly distributed load applied over a rectangular area, and obtained results identical to those previously obtained by Newmark and Steinbrenner. More research work on the elasticity problems of the elastic half-plane and elastic half-space using Fourier transform methods, Mellin transform method, and exponential Fourier integral transform method can be found in Onah *et al* [25 – 26] and Ike [27 – 30]. Onah et al [31] derived from first principles displacement and stress functions for solving three-dimensional elasticity problems. Ike [32-34] solved two-and three-dimensional elasticity problems using such novel methods as Elzaki transform method, Fourier cosine transform method, and cosine integral transformation method. Ike et al [35] applied Trefftz displacement potential function method to solve elastic half-space problems. Ike[36] applied the Fourier integral transformation method to find solutions to two-dimensional elasticity problems for plane strain conditions by using Love stress functions.

In this work, the Navier's differential equations of equilibrium for axisymmetric problems of the elastic half-space are solved analytically to obtain solutions for the general axisymmetric load; and particular solution to the Boussinesq axisymmetric problem. The solutions for vertical stress distribution for the point load at the origin is then used as Green (Kernel) function to derive solutions for vertical stress for other axisymmetric load cases.

Research aim and objectives

The research aim is the determination of closed form solutions of the Navier's equations for axisymmetric elasticity problems of the elastic half-space, and the vertical stress distributions for axisymmetric elasticity problems for various axisymmetric load types. The objectives are as follows:

(i) to obtain the general mathematical solution to the Navier's differential equation for the elastic half space expressed in cylindrical coordinates system when body forces are disregarded.

(ii) to obtain the solution to the Navier's differential equations of equilibrium for the axisymmetric case of point load acting at the origin on the elastic half-space as shown in Figure 1.

(iii) to use the vertical stress distributions solutions obtained for the Boussinesq point load problem as Green functions to obtain the vertical stress distributions: (a) at any point (r, z) in the elastic half-space due to circular foundation areas subject to uniformly distributed load over the entire domain of the foundation which is shown in Figure 2; (b) at any point ($r = 0$, z) under the center of the circular foundation area subject to uniformly distributed load; (c) at any point under the center of a circular foundation area due to a conical load distribution (load intensity varies from zero at the center to a maximum at the perimeter) shown in Figure 3; (d) at any point under the center of a circular foundation area due to an inverted conical load distribution (load intensity varies linearly from a maximum at the center to zero at the perimeter or circumference) shown in Figure 4.

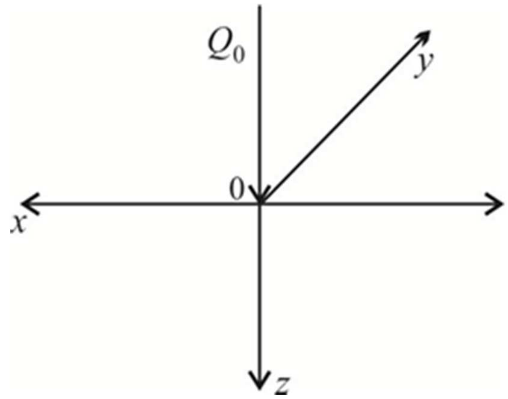

Figure 1: Axisymmetric problem of point load at the origin O on an elastic half-space ($-\infty \le x \le \infty$; $-\infty \le y \le \infty$; $0 \le z \le \infty$). ($0 \le r \le \infty$; $0 \le z \le \infty$; $0 \le \theta \le 2\pi$)

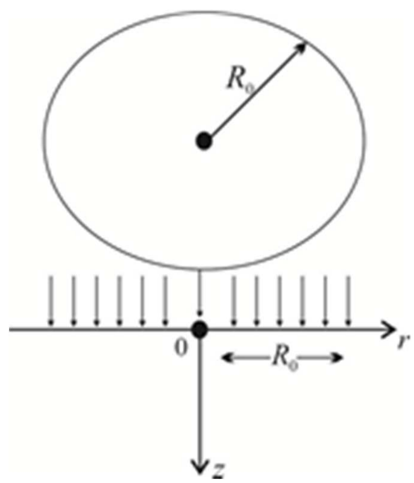

Figure 2: Axisymmetric problem of circular foundation of radius R_0 subject to uniformly distributed load $q(r) = q_0$

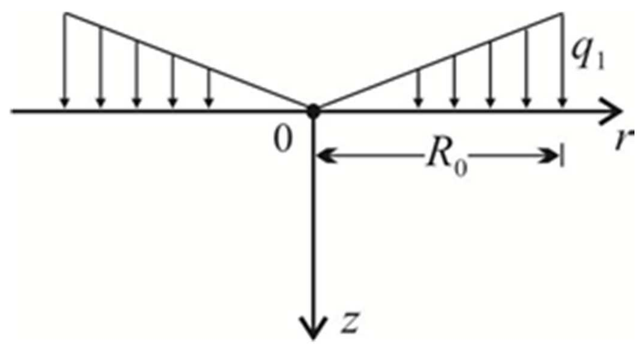

Figure 3: Axisymmetric problem of circular foundation of radius R_0 subject to a conical load distribution $q(r) = q_1 r/R_0$ (where $q(r = 0) = 0$; $q(r = R_0) = q_1$)

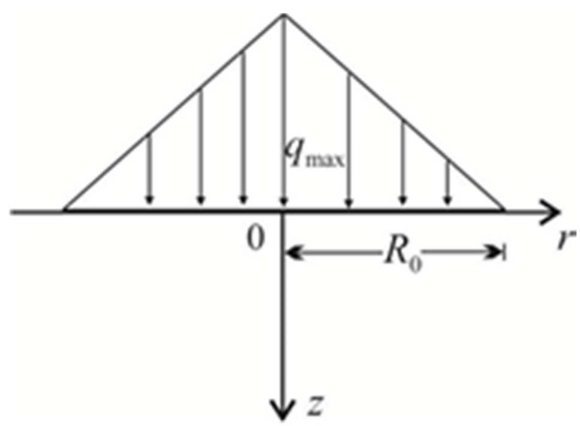

Figure 4: Axisymmetric problem of circular foundation of radius R_0 subject to an inverted conical load distribution given by $q(r) = q_{max}(1 - r/R_0)$ (where $q(r = 0) = q_{max}$; and $q(r = R_0) = 0$)

Theoretical framework

Governing equations

In axisymmetric elasticity problems of the elastic half-space ($-\infty \le x \le \infty$; $-\infty \le y \le \infty$; $0 \le z \le \infty$) or ($0 \le r \le \infty$; $0 \le z \le \infty$; $0 \le \theta \le 2\pi$) where x, y, and z are the three dimensional (3D) Cartesian coordinates, and r, θ, z are the cylindrical polar coordinates, the displacement field $\vec{v}$ is axially symmetric with respect to the z coordinate axis, yielding the displacement vector as:

$$\vec{v} = (u_r(r,z),\ 0,\ w(r,z)) = u_r(r,z)i_r + w(r,z)i_z \quad (1)$$

Since $u_\theta = 0$ (2)

$u_r(r, z)$ is the radial component, u_θ is the tangential component, and $w(r, z)$ is the z component of the displacement, i_r is the unit vector in the radial coordinate direction, i_z is the unit vector in the z coordinate direction, r is the radial coordinate, θ is the tangential coordinate while z is the depth (transverse) coordinate.

Navier's equations of equilibrium for axisymmetric elastostatic problems

The Navier's differential equation for the displacement formulation of 3D elastostatic problems is given in vector form as:

$$G\nabla^2 v + (\lambda + G)\nabla(\nabla \cdot \vec{v}) + \vec{F} = 0 \tag{3}$$

where G is the shear modulus or modulus of rigidity, λ is the Lamé constant, $\vec{F}$ is the body force vector, and

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \tag{4}$$

where i, j, k are the unit vectors of the 3D Cartesian coordinates system, while

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{5}$$

The elastic constants λ and G are expressed in terms of Young's modulus of elasticity E and Poisson's ratio, μ, as:

$$\lambda = \frac{E\mu}{(1+\mu)(1-2\mu)} \tag{6}$$

$$G = \frac{E}{2(1+\mu)} \tag{7}$$

For axisymmetric elastostatic problems, Navier's equations of equilibrium are given by the following system of partial differential equations (PDEs):

$$G\left(\nabla^2 u_r(r,z) - \frac{u_r(r,z)}{r^2}\right) + (\lambda + G)\frac{\partial}{\partial r}\left(\frac{\partial u_r(r,z)}{\partial r} + \frac{u_r(r,z)}{r} + \frac{\partial w(r,z)}{\partial z}\right) + F_r(r,z) = 0 \tag{8}$$

for equilibrium in the radial direction, and

$$G\nabla^2 w(r,z) + (\lambda + G)\frac{\partial}{\partial z}\left(\frac{\partial u_r(r,z)}{\partial r} + \frac{u_r(r,z)}{r} + \frac{\partial w(r,z)}{\partial z}\right) + F_z(r,z) = 0 \tag{9}$$

for equilibrium in the z coordinate direction.

$F_r(r, z)$ is the radial component of the body force vector while $F_z(r, z)$ is the z component of the body force vector.

Kinematic equations

For linear small (infinitesimal) displacement elasticity, the strain-displacement equations for axisymmetric elasticity problems are given by:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}(r,z) \tag{10}$$

$$\varepsilon_{\theta\theta} = \frac{u_r}{r}(r,z) \tag{11}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z}(r,z) \tag{12}$$

$$\gamma_{rz} = \frac{\partial w(r,z)}{\partial r} + \frac{\partial u_r(r,z)}{\partial z} \tag{13}$$

where ε_{rr} is the radial strain, $\varepsilon_{\theta\theta}$ is the circumferential strain, ε_{zz} is the strain in the z coordinate direction, γ_{rz} is the shear strain.

Constitutive equations

The Hooke’s stress-strain relations for axisymmetric elasticity problems expressed in terms of Lamé’s constants are given by:

$$\sigma_{rr} = \lambda(\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}) + 2G\varepsilon_{rr} \tag{14}$$

$$\sigma_{\theta\theta} = \lambda(\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}) + 2G\varepsilon_{\theta\theta} \tag{15}$$

$$\sigma_{zz} = \lambda(\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}) + 2G\varepsilon_{zz} \tag{16}$$

$$\sigma_{rz} = \tau_{rz} = G\gamma_{rz} \tag{17}$$

σ_{rr} is the radial stress, $\sigma_{\theta\theta}$ is the circumferential stress, σ_{zz} is the normal stress in the z coordinate direction, σ_{rz} (or τ_{rz}) is the shear stress.

Methodology

We denote the volumetric strain as φ and then have:

$$\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz} = \varphi = \nabla \cdot \vec{v} \tag{18}$$

The Navier’s equation can then be expressed in general as:

$$G\nabla^2 v + (\lambda + G)\nabla\varphi + F = 0 \tag{19}$$

From the Navier differential equation of equilibrium, if body forces are disregarded and harmonic displacement field $\vec{v}$, assumed we have:

$$\nabla^2\varphi = 0 \tag{20}$$

The most elementary potential function φ of the 3D space coordinates that is a potential function at every point in the elastic half-space geometry ($-\infty \le x \le \infty$; $-\infty \le y \le \infty$; $0 \le z \le \infty$) of the Boussinesq and other axisymmetric elasticity problems except at the origin ($x = 0$, $y = 0$, $z = 0$) of the coordinates is expressed by:

$$\varphi = c_1 \frac{\partial}{\partial z}\left(\frac{1}{R}\right) = \nabla \cdot \vec{v} \tag{21}$$

where $R^2 = x^2 + y^2 + z^2 = r^2 + z^2$ (22)

$$r^2 = x^2 + y^2 \tag{23}$$

and c_1 is a constant.

The Navier differential equation of equilibrium in the z direction, is given as:

$$G\nabla^2 w(r,z) + (\lambda + G)\frac{\partial}{\partial z}\varphi(r,z) + F_z(r,z) = 0 \tag{24}$$

In the absence of body forces or when the body force component in the F_z direction is disregarded, the Navier differential equation of equilibrium in the z direction simplifies to:

$$G\nabla^2 w(r,z) + (\lambda + G)\frac{\partial}{\partial z}\varphi(r,z) = 0 \tag{25}$$

Hence,

$$\nabla^2 w(r,z) = -\left(\frac{\lambda + G}{G}\right)\frac{\partial}{\partial z}\varphi(r,z) = -\left(\frac{\lambda + G}{G}\right)c_1\frac{\partial^2}{\partial z^2}\left(\frac{1}{R}\right) \tag{26}$$

$$\nabla^2 w(r,z) = -\left(\frac{\lambda+G}{G}\right)c_1\left(\frac{3z^2}{R^5}-\frac{1}{R^3}\right) \qquad (27)$$

$$\nabla^2 w(r,z) = \frac{1}{R^2}\left\{-\left(\frac{\lambda+G}{G}\right)c_1\frac{3z^2}{R^3}+\left(\frac{\lambda+G}{G}\right)\frac{c_1}{R}\right\} \qquad (28)$$

This suggests that a suitable solution for $w(r, z)$ would depend upon the functions $\frac{z^2}{R^3}$ and $\frac{1}{R}$.

We seek to find the general solution to the non-homogeneous partial differential equation (PDE) given as:

$$\nabla^2 w(r,z) = \frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(rw_r(r,z)\right)+\frac{\partial^2 w}{\partial z^2} = -\left(\frac{\lambda+G}{G}\right)\frac{\partial \varphi}{\partial z} \qquad (29)$$

The general solution to Equation (29) is obtained as the superposition of the homogeneous solution and the particular solution as:

$$w(r,z) = c_1\left(\frac{\lambda+G}{2G}\right)\frac{z^2}{R^3}+c_2\frac{1}{R} \qquad (30)$$

where c_1 and c_2 are the arbitrary constants of integration which can be obtained from the definition of the volumetric strain.

From

$$\varphi = \nabla\cdot v = \frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{\partial w}{\partial z} \qquad (31)$$

In order to avoid singularity in the radial displacement function $u_r(r, z)$, we seek to obtain $u_r(r, z)$ using the boundedness condition that as $r \to 0$, $u_r \to 0$ for any z. Thus, $u_r(r \to 0,\ z) \to 0$

By substitution of expressions for $w(r, z)$ and $\varphi(r, z)$ in Equation (31) we have:

$$c_1\frac{\partial}{\partial z}\left(\frac{1}{R}\right) = \frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{\partial}{\partial z}\left\{c_1\left(\frac{\lambda+G}{2G}\frac{z^2}{R^3}+c_2\frac{1}{R}\right)\right\} \qquad (32)$$

Equation (32) simplifies after evaluation of the partial derivatives to:

$$c_1\left(\frac{-z}{R^3}\right) = \frac{1}{r}\frac{\partial}{\partial r}(ru_r)+c_1\left(\frac{\lambda+G}{2G}\right)\left(\frac{2z}{R^3}-\frac{3z^3}{R^5}\right)+c_2\left(\frac{-z}{R^3}\right) \qquad (33)$$

Hence,

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) = c_1\left(\frac{-z}{R^3}\right)-c_2\left(\frac{-z}{R^3}\right)-c_1\left(\frac{\lambda+G}{2G}\right)\left(\frac{2z}{R^3}-\frac{3z^3}{R^5}\right) \qquad (34)$$

Simplification yields:

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) = (c_2-c_1)\frac{z}{R^3}-c_1\left(\frac{\lambda+G}{2G}\right)\left(\frac{2z}{R^3}-\frac{3z^3}{R^5}\right) \qquad (35)$$

Simplifying,

$$\frac{\partial}{\partial r}(ru_r) = (c_2-c_1)\frac{zr}{R^3}-c_1\left(\frac{\lambda+G}{2G}\right)\left(\frac{2zr}{R^3}-\frac{3z^3r}{R^5}\right) \qquad (36)$$

Integrating with respect to r,

$$\int\partial(ru_r) = (c_2-c_1)z\int\frac{r}{R^3}dr-c_1\left(\frac{\lambda+G}{2G}\right)2z\int\frac{r}{R^3}dr+c_1\left(\frac{\lambda+G}{2G}\right)3z^3\int\frac{r}{R^5}dr \qquad (37)$$

Hence,

$$ru_r = (c_2-c_1)z\left(-\frac{1}{R}\right)-c_1\left(\frac{\lambda+G}{2G}\right)2z\left(-\frac{1}{R}\right)+c_1\left(\frac{\lambda+G}{2G}\right)3z^3\left(-\frac{1}{3R^3}\right)+c_3 \qquad (38)$$

where c_3 is an integration constant.
Simplifying,

$$ru_r = (c_1 - c_2)\frac{z}{R} + c_1\left(\frac{\lambda+G}{2G}\right)\frac{2z}{R} - c_1\left(\frac{\lambda+G}{2G}\right)\frac{z^3}{R^3} + c_3 \quad (39)$$

Further simplification yields:

$$ru_r = c_1\frac{z}{R} - c_2\frac{z}{R} + c_1\left(\frac{2\lambda+G}{2G}\right)\frac{z}{R} - c_1\left(\frac{\lambda+G}{2G}\right)\frac{z^3}{R^3} + c_3 \quad (40)$$

$$ru_r = c_1\left(\frac{2\lambda+2G}{2G} + 1\right)\frac{z}{R} - c_2\frac{z}{R} - c_1\left(\frac{\lambda+G}{2G}\right)\frac{z^3}{R^3} + c_3 \quad (41)$$

$$ru_r = c_1\left(\frac{2\lambda+4G}{2G}\right)\frac{z}{R} - c_2\frac{z}{R} - c_1\left(\frac{\lambda+G}{2G}\right)\frac{z^3}{R^3} + c_3 =$$

$$c_1\left(\frac{\lambda+2G}{G}\right)\frac{z}{R} - c_2\frac{z}{R} - c_1\left(\frac{\lambda+G}{2G}\right)\frac{z^3}{R^3} + c_3 \quad (42)$$

As $r \to 0$, $u_r(r \to 0, z) \to 0$
Hence, $R(r \to 0) \to z$

$$0 = c_1\left(\frac{\lambda+2G}{G}\right) - c_2 - c_1\left(\frac{\lambda+G}{2G}\right) + c_3 \quad (43)$$

$$\therefore c_3 = c_1\left(\frac{\lambda+G}{2G}\right) - c_1\left(\frac{2G+\lambda}{G}\right) + c_2 = c_2 + c_1\left(\frac{\lambda+G}{2G} - \frac{2G+\lambda}{G}\right) =$$

$$c_2 + c_1\left(\frac{\lambda+G-4G-2\lambda}{2G}\right) = c_2 + c_1\left(\frac{-3G-\lambda}{2G}\right) = c_2 - c_1\left(\frac{3G+\lambda}{2G}\right) \quad (44)$$

Thus,

$$ru_r = c_2 - c_1\left(\frac{3G+\lambda}{2G}\right) + c_1\left(\frac{2G+\lambda}{G}\right)\frac{z}{R} - c_2\frac{z}{R} - c_1\left(\frac{\lambda+G}{2G}\right)\frac{z^3}{R^3} \quad (45)$$

The two displacement field components are now determined in terms of two unknown constants c_1 and c_2.

Results

Boussinesq problem of vertical point load on an elastic half space

For the Boussinesq problem of finding the stresses and displacements at any arbitrary point $A_p(x, y, z)$ in an elastic half-space due to a vertical point load Q_0 applied at the origin as shown in Figure 1, the two constants of integration c_1 and c_2 which are present in $u_r(r, z)$ and $w(r, z)$ are determined by using the shear stress free boundary conditions on the xy coordinate plane (i.e. $z = 0$ plane) and the requirement of equilibrium of the internal vertical stress and the applied vertical point load.

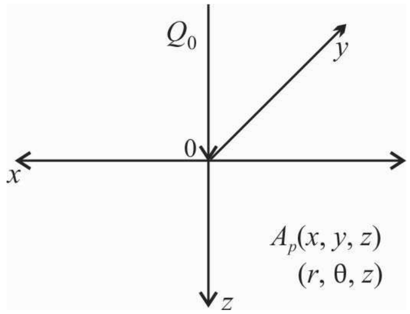

Figure 5: Boussinesq problem: stress and displacement at $A_p(x, y, z)$ due to vertical point load applied at the origin of an elastic half-space

Stress field components

The stress field components are obtained by using the stress-displacement equations given as follows:

$$\sigma_{rr} = \lambda\nabla\cdot\vec{v} + 2G\frac{\partial u_r}{\partial r} \quad (46)$$

$$\sigma_{\theta\theta} = \lambda\nabla\cdot\vec{v} + 2G\frac{u_r}{r} \quad (47)$$

$$\sigma_{zz} = \lambda\nabla\cdot\vec{v} + 2G\frac{\partial w}{\partial z} \quad (48)$$

$$\tau_{rz} = G\left(\frac{\partial w}{\partial r} + \frac{\partial u_r}{\partial z}\right) \quad (49)$$

From the expressions for w and u_r,

$$\frac{\partial w}{\partial z}(r,z) = \frac{\partial}{\partial z}\left(c_1\left(\frac{\lambda+G}{2G}\right)\frac{z^2}{R^3} + c_2\frac{1}{R}\right) \quad (50)$$

$$\frac{\partial w}{\partial z} = c_1\left(\frac{\lambda+G}{2G}\right)\frac{\partial}{\partial z}\left(\frac{z^2}{R^3}\right) + c_2\frac{\partial}{\partial z}\left(\frac{1}{R}\right) = c_1\left(\frac{\lambda+G}{2G}\right)\left(\frac{2z}{R^3} - \frac{3z^3}{R^5}\right) + c_2\left(\frac{-z}{R^3}\right) \quad (51)$$

$$\frac{\partial w}{\partial r} = \frac{\partial}{\partial r}\left(c_1\left(\frac{\lambda+G}{2G}\right)\frac{z^2}{R^3} + c_2\frac{1}{R}\right) \quad (52)$$

$$\frac{\partial w}{\partial r} = c_1\left(\frac{\lambda+G}{2G}\right)\frac{\partial}{\partial r}\left(\frac{z^2}{R^3}\right) + c_2\frac{\partial}{\partial r}\left(\frac{1}{R}\right) = c_1\left(\frac{\lambda+G}{2G}\right)z^2\frac{\partial}{\partial r}\left(\frac{1}{R^3}\right) + c_2\frac{\partial}{\partial r}\left(\frac{1}{R}\right) =$$

$$c_1\left(\frac{\lambda+G}{2G}\right)z^2\frac{\partial}{\partial r}\left(\frac{1}{R^3}\right) + c_2\left(\frac{-r}{R}\right) \quad (53)$$

$$\frac{\partial w}{\partial r}(r, z=0) = -c_2\left(\frac{1}{r^2}\right) \quad (54)$$

Since $R^3(z=0) = r^3$

$$\frac{\partial u_r}{\partial z} = \frac{\partial}{\partial z}\left(c_2 r^{-1} - c_1 r^{-1}\left(\frac{3G+\lambda}{2G}\right) - \left(c_2 - c_1\left(\frac{2G+\lambda}{G}\right)\right)\frac{1}{r}\frac{z}{R} - c_1\left(\frac{G+\lambda}{2G}\right)\frac{1}{r}\frac{z^3}{R^3}\right) \quad (55)$$

$$\frac{\partial u_r}{\partial z} = -\left(c_2 - c_1\left(\frac{2G+\lambda}{G}\right)\right)\frac{1}{r}\frac{\partial}{\partial z}\left(\frac{z}{R}\right) - c_1\left(\frac{G+\lambda}{2G}\right)\frac{1}{r}\frac{\partial}{\partial z}\frac{z^3}{R^3} \quad (56)$$

$$\frac{\partial u_r}{\partial z} = -\left(c_2 - c_1\left(\frac{2G+\lambda}{G}\right)\right)\frac{1}{r}\left(\frac{1}{R} - \frac{z^2}{R^3}\right) - c_1\left(\frac{G+\lambda}{2G}\right)\frac{1}{r}\left(\frac{3z^2}{R^3} - \frac{3z^3}{R^5}\right) \quad (57)$$

$$\frac{\partial u_r}{\partial z}(r, z=0) = -\left(c_2 - c_1\left(\frac{2G+\lambda}{G}\right)\right)\frac{1}{r}\frac{1}{R}(z=0) \tag{58}$$

$$\frac{\partial u_r}{\partial z}(r, z=0) = -\left(c_2 - c_1\left(\frac{2G+\lambda}{G}\right)\right)\frac{1}{r^2} \tag{59}$$

Since $R(z=0) = (r^2+z^2)^{1/2}\Big|_{z=0} = r$ (60)

$$\frac{\partial u_r}{\partial z}(r, z=0) = \left(c_1\left(\frac{2G+\lambda}{G}\right) - c_2\right)\frac{1}{r^2} \tag{61}$$

Enforcement of shear stress free boundary condition

The shear stress field on the $z = 0$ plane $\tau_{rz}(r,z)=0$ is obtained as:

$$\tau_{rz}(r, z=0) = G\left(\frac{\partial w}{\partial r}(r, z=0) + \frac{\partial u_r}{\partial z}(r, z=0)\right) \tag{62}$$

$$\tau_{rz}(r, z=0) = G\left\{-c_2\frac{1}{r^2} + \left(c_1\left(\frac{2G+\lambda}{G}\right) - c_2\right)\frac{1}{r^2}\right\} \tag{63}$$

Application of the shear stress free boundary condition yields:

$$\tau_{rz}(r, z=0) = G\left(-\frac{c_2}{r^2} + \left(c_1\left(\frac{2G+\lambda}{G}\right) - c_2\right)\frac{1}{r^2}\right) = 0 \tag{64}$$

Hence, $c_2 = c_1\left(\frac{2G+\lambda}{G}\right) - c_2$ (65)

$$c_2 = c_1\left(\frac{2G+\lambda}{2G}\right) \tag{66}$$

Vertical stress field

The vertical stress field is obtained from Equation (48). Using Equation (66),

$$\frac{\partial w}{\partial z} = c_1\left(\frac{\lambda+G}{2G}\right)\left(\frac{2z}{R^3} - \frac{3z^3}{R^5}\right) + c_1\left(\frac{2G+\lambda}{2G}\right)\left(-\frac{z}{R^3}\right) \tag{67}$$

$$\frac{\partial w}{\partial z} = c_1\left(\frac{\lambda+G}{2G}\right)\frac{2z}{R^3} - c_1\left(\frac{2G+\lambda}{2G}\right)\frac{z}{R^3} - c_1\left(\frac{\lambda+G}{2G}\right)\frac{3z^3}{R^5} \tag{68}$$

$$\frac{\partial w}{\partial z} = c_1\left(\frac{2\lambda+2G}{2G}\right)\frac{z}{R^3} - c_1\left(\frac{2G+\lambda}{2G}\right)\frac{z}{R^3} - c_1\left(\frac{\lambda+G}{2G}\right)\frac{3z^3}{R^5} \tag{69}$$

$$\frac{\partial w}{\partial z} = c_1\left(\frac{2\lambda+2G-2G-\lambda}{2G}\right)\frac{z}{R^3} - c_1\left(\frac{\lambda+G}{2G}\right)\frac{3z^3}{R^5} = c_1\frac{\lambda}{2G}\frac{z}{R^3} - c_1\left(\frac{\lambda+G}{2G}\right)\frac{3z^3}{R^5} \tag{70}$$

$$\sigma_{zz} = 2G\frac{\partial w}{\partial z} + \lambda c_1\frac{\partial}{\partial z}\frac{1}{R} \tag{71}$$

$$\sigma_{zz} = 2G\left(c_1\frac{\lambda}{2G}\frac{z}{R^3} - c_1\frac{(\lambda+G)}{2G}\frac{3z^3}{R^5}\right) + \lambda c_1\left(-\frac{z}{R^3}\right) \tag{72}$$

$$\sigma_{zz} = c_1\lambda\frac{z}{R^3} - c_1(\lambda+G)\frac{3z^3}{R^5} - c_1\frac{\lambda z}{R^3} \tag{73}$$

$$\sigma_{zz} = -3c_1(\lambda + G)\frac{z^3}{R^5} \tag{74}$$

Equilibrium of internal vertical stress and applied vertical point load yields:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma_{zz}(r, z)dxdy + Q_0 = 0 \tag{75}$$

We evaluate the integration problem by introduction of coordinate transformation from 3D Cartesian to cylindrical coordinates,

$$x = r\cos\theta \tag{76}$$

$$y = r\sin\theta \tag{77}$$

$$z = z \tag{78}$$

$0 \le z \le \infty \qquad 0 \le r \le \infty \qquad 0 \le \theta \le 2\pi$

The equation of equilibrium becomes:

$$\int_0^{2\pi}\int_0^{\infty} \sigma_{zz}(r, z)|\,J\,|\; dr\, d\theta + Q_0 = 0 \tag{79}$$

where $|J|$ is the Jacobian of the coordinate transformation from Cartesian to polar coordinates, defined as:

$$|\,J\,| = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\sin\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = r \tag{80}$$

Hence

$$Q_0 + \int_0^{2\pi}\int_0^{\infty} \frac{-3c_1(\lambda + G)z^3}{R^5} r\, dr\, d\theta = 0 \tag{81}$$

$$Q_0 = \int_0^{2\pi}\int_0^{\infty} \frac{3c_1(\lambda + G)z^3}{(r^2 + z^2)^{5/2}} r\, dr\, d\theta \tag{82}$$

$$Q_0 = 3c_1(\lambda + G)\int_0^{2\pi} d\theta \int_0^{\infty} \frac{z^3 r\, dr}{(r^2 + z^2)^{5/2}} \tag{83}$$

$$Q_0 = 3c_1(\lambda + G)[\theta]_0^{2\pi} \int_0^{\infty} \frac{z^3 r\, dr}{(r^2 + z^2)^{5/2}} \tag{84}$$

$$Q_0 = 3c_1(\lambda + G)2\pi z^3 \int_0^{\infty} \frac{r\, dr}{(r^2 + z^2)^{5/2}} \tag{85}$$

$$Q_0 = 6\pi c_1(\lambda + G)z^3 \int_0^{\infty} \frac{r\, dr}{(r^2 + z^2)^{5/2}} \tag{86}$$

Let $r^2 + z^2 = b(r)$ (87)

$$\frac{db(r)}{dr} = 2r \tag{88}$$

$$rdr = \frac{db(r)}{2} \tag{89}$$

$$Q_0 = 6\pi c_1(\lambda + G)z^3 \int_0^{\infty} \frac{db(r)}{2\left(b(r)\right)^{5/2}} \tag{90}$$

$$Q_0 = 3\pi c_1(\lambda+G)z^3\int_0^\infty (b(r))^{-5/2}\, db(r) \tag{91}$$

$$Q_0 = 3\pi c_1(\lambda+G)z^3\left[-\tfrac{2}{3}(b(r))^{-3/2}\right]_0^\infty \tag{92}$$

$$-2\pi(\lambda+G)c_1 z^3\left[(r^2+z^2)^{-3/2}\right]_{r=0}^{r=\infty} = Q_0 \tag{93}$$

$$2\pi(\lambda+G)c_1 = Q_0 \tag{94}$$

$$c_1 = \frac{Q_0}{2\pi(\lambda+G)} \tag{95}$$

$$\sigma_{zz} = -3(\lambda+G)\frac{z^3}{R^5}\frac{Q_0}{2\pi(\lambda+G)} = -\frac{3Q_0}{2\pi}\frac{z^3}{R^5} = -\frac{Q_0}{z^2}\frac{3}{2\pi}\left(1+\frac{r^2}{z^2}\right)^{-5/2} = K\left(\frac{r}{z}\right)\left(-\frac{Q_0}{z^2}\right) \tag{96}$$

where $K(r/z)$ is the Boussinesq vertical stress influence coefficient which is presented in Table 1 and Figure 11.

Table 1: Boussinesq vertical stress influence coefficients $K(r/z)$ for vertical point load at the origin on an elastic half-space.

r/z	***K*(r/z)**	**r/z**	***K*(r/z)**
0	0.4775	1.10	0.0658
0.05	0.4745	1.20	0.0513
0.10	0.4657	1.30	0.0402
0.15	0.4516	1.40	0.0317
0.20	0.4329	1.50	0.0251
0.25	0.4103	1.6	0.0200
0.30	0.3849	1.7	0.0160
0.35	0.3577	1.8	0.0129
0.40	0.3294	1.9	0.0105
0.45	0.3011	2.0	0.0085
0.50	0.2733	2.5	0.0034
0.55	0.2466	3	0.0015
0.60	0.2214	4	0.0004
0.65	0.1978	∞	0
0.70	0.1762		
0.75	0.1565		
0.80	0.1386		
0.85	0.1226		
0.90	0.1083		
0.95	0.0956		
1.00	0.0844		

$$c_2 = c_1\left(\frac{\lambda+2G}{2G}\right) = \frac{Q_0}{2\pi(\lambda+G)}\left(\frac{\lambda+2G}{2G}\right) \tag{97}$$

The stresses are obtained from the stress-displacement relations as:

$$\sigma_{rr} = \frac{Q_0}{2\pi}\left\{(1-2\mu)\left(\frac{1}{r^2} - \frac{z}{Rr^2}\right) - \frac{3zr^2}{R^5}\right\} \tag{98}$$

$$\sigma_{\theta\theta} = -\frac{Q_0(1-2\mu)}{2\pi}\left\{\frac{1}{r^2} - \frac{z}{Rr^2} - \frac{z}{R^3}\right\} \tag{99}$$

$$\sigma_{rz} = -\frac{3Q_0 rz^2}{2\pi R^5} \tag{100}$$

The displacements are:

$$u_r = \frac{Q_0}{4\pi G}\left(\frac{rz}{R^3} - \frac{(1-2\mu)(R-z)}{Rr}\right) \tag{101}$$

$$w = \frac{Q_0}{4\pi G}\left(\frac{z^2}{R^3} - \frac{2(1-\mu)}{R}\right) \tag{102}$$

On the surface of the soil, $z = 0$,

$$u_r(r, z=0) = -\frac{(1-2\mu)Q_0}{4\pi Gr} \tag{103}$$

$$w(r, z=0) = \frac{(1-\mu)Q_0}{2\pi Gr} \tag{104}$$

Vertical stress due to uniformly loaded circular foundation areas

The vertical stress at any point, *B*, in an elastic half-space due to uniformly loaded circular foundation is obtained by using the Boussinesq point load solution for vertical stress as a Green function as follows:

$$\sigma_z = \int_0^{2\pi}\int_0^{R_0} \frac{3z^3 q_0\, r\, dr\, d\theta}{2\pi(r^2 + e^2 + z^2 - 2er\cos\theta)^{5/2}} \tag{105}$$

where from the cosine law (cosine rule),

$$R^2 = r^2 + e^2 + z^2 - 2er\cos\theta \tag{106}$$

R_0 is the radius of the circular foundation.

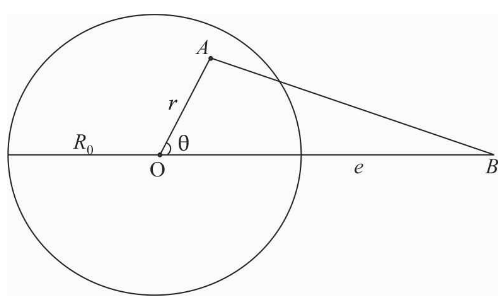

Figure 6: Vertical stress at point *B* due to a circular foundation carrying uniformly distributed load

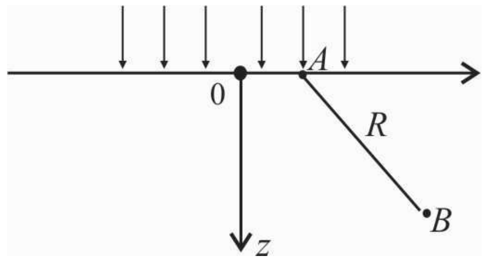

Figure 7: Arbitrary point *B* in an elastic half-space for circular foundation subject to uniformly distributed load

For uniformly distributed loads

$$\sigma_v = q_0 \cdot \frac{3z^3}{2\pi} \int_0^R \int_0^{2\pi} \frac{r\,dr\,d\theta}{(r^2 + e^2 + z^2 - 2er\cos\theta)^{5/2}} \quad (107)$$

Egorov and Serebjanyi [37] and Egorov [38] evaluated the complicated integration problem in Equation (107). Using Egorov's results for the double integration problem, the vertical stress is obtained as follows:

$$\sigma_r(r,z) = q_0 \left\{ F_1 - \frac{n}{\pi\sqrt{n^2 + (1+\alpha)^2}} \left[\frac{n^2 - 1 + \alpha^2}{n^2 + (1-\alpha)^2} E(k) + \frac{1-\alpha}{1+\alpha} \Pi_0(k,m) \right] \right\} \quad (108)$$

$$\sigma_r(r,z) = q_0 I\left(r/R_0 ,\ z/R_0 \right) \quad (108)$$

in which, $I\left(r/R_0 ,\ z/R_0 \right)$ is presented in Table 2 and,

$$n = z/R_0 \quad (109)$$

$$\alpha = r/R_0 \quad (110)$$

$$k = \frac{4\pi}{n^2 + (1+\alpha)^2} \quad (111)$$

$$m = \frac{-4\alpha}{(1+\alpha)^2} \quad (112)$$

$$F_1 = \begin{cases} 1 & r < R_0 \\ \frac{1}{2} & r = R_0 \\ 0 & r > R_0 \end{cases} \quad (113)$$

$E(k)$ is the complete elliptic integral of the second kind with a modulus of k and parameter, m. $\Pi_0(k_1 m)$ is the complete elliptic integral of the third kind with a modulus of k and parameter, m. The result obtained for vertical stress field $\sigma_{zz}(r, z)$ are similar to results obtained by Harr [39]. Generally, the vertical stress at any arbitrary point (r, z) in an elastic half-space due to uniformly distributed load of intensity q_0 applied over a circular foundation area of radius R_0 is expressed in terms of non-dimensional vertical stress influence factors $I\left(r/R_0 ,\ z/R_0 \right)$ which depend upon the dimensionless ratios r/R_0 and z/R_0.

A table of values for $I\left(r/R_0 ,\ z/R_0 \right)$ for various of r/R_0 and z/R_0 is presented in Table 2.

Vertical stress distribution under the center of circular foundation carrying uniformly distributed load

The vertical stress distribution under the center of circular foundation subjected to uniformly distributed load can be obtained as a special case of the general solution presented in Equation (108) when $r = 0$, since $r = 0$ at all points under the center of the circular foundation. Then,

$$\sigma_{zz}(r=0,z) = q_0\left\{F_1 - \frac{n}{\pi\sqrt{1+n^2}}\left[\frac{n^2-1}{n^2+1}E(0) + \Pi_0(0,0)\right]\right\} = q_0 I\left(\frac{r}{R_0}=0,\ \frac{z}{R_0}\right) \qquad (114)$$

$$\sigma_{zz}(r=0,z) = q_0\left\{1 - \frac{n^3}{\sqrt{(1+n^2)^3}}\right\} = q_0\left\{1-(1+n^{-2})^{-3/2}\right\} =$$

$$q_0\left\{1-\left(1+\left(\frac{R_0}{z}\right)^2\right)^{-3/2}\right\} = q_0 I_{c_1}\left(\frac{R_0}{z}\right) \qquad (115)$$

Table of values for $I_{c_1}\left({R_0}/{z}\right)$ is given in Table 3. The same result for vertical stress distribution under the center of circular foundations subject to uniformly distributed loads is obtained by using the Boussinesq point load solution as the integral Kernel function.

Table 2: Vertical stress influence coefficients $I\left({r}/{R_0},\ {z}/{R_0}\right)$ for circular foundation areas subjected to uniformly distributed loads

z/R_0 \ r/R_0	0	0.2	0.4	0.5	0.6	0.8	1.0	1.5	1.8
0	1.000	1.000	1.000	1.0	1.000	1.000	1.0	0	1.000
0.1	0.999	0.999	0.998		0.996	0.976			0.484
0.2	0.992	0.991	0.987		0.970	0.890			0.468
0.25	0.990			0.96			0.50	0.03	
0.3	0.976	0.973	0.963		0.922	0.793			0.451
0.4	0.949	0.943	0.920		0.860	0.712			0.435
0.5	0.911	0.902	0.869	0.83	0.796	0.646	0.41	0.07	0.417
0.6	0.864	0.852	0.814		0.732	0.591			0.400
0.7	0.811	0.782	0.756		0.674	0.545			0.367
0.8	0.756	0.743	0.699		0.619	0.504			0.366
0.9	0.701	0.688	0.644		0.570	0.467			0.348
1.0	0.646	0.633	0.591	0.56	0.525	0.434	0.34	0.11	0.332
1.2	0.546	0.535	0.501		0.447	0.377			0.30
1.5	0.424	0.416	0.392	0.37	0.355	0.308	0.24	0.13	0.256
2	0.286	0.286	0.268	0.26	0.248	0.224	0.91	0.13	0.196
2.5	0.200	0.197	0.191		0.180	0.167			0.151
3	0.146	0.145	0.141		0.135	0.127			0.118
4	0.087	0.086	0.085		0.082	0.080			0.075

Table 3: Vertical stress influence coefficients for points in the elastic half space under the center of circular foundation areas under uniformly distributed loads

R/z	$I_{c1}(R/z)$	R/z	$I_{c1}(R/z)$	R/z	$I_{c1}(R/z)$
0	0	1.05	0.67198	3	0.96836
0.05	0.00374	1.10	0.69562	4	0.98573
0.10	0.01481	1.15	0.71747	5	0.99246
0.15	0.03283	1.20	0.73763	6	0.99556
0.20	0.05713	1.25	0.75622	7	0.99717
0.25	0.08692	1.30	0.77334	8	0.99809
0.30	0.12126	1.35	0.78911	9	0.99865
0.35	0.15915	1.40	0.80364	10	0.99901
0.40	0.19959	1.45	0.81701	20	0.99988
0.45	0.24165	1.50	0.82932	30	0.99996
0.50	0.28446	1.55	0.84067	40	0.99998
0.55	0.32728	1.60	0.85112	50	0.99999
0.60	0.36949	1.65	0.86077	100	1.00000
0.65	0.41058	1.70	0.86966	∞	1.00000
0.70	0.45018	1.75	0.87787		
0.75	0.48800	1.80	0.88546		
0.80	0.52386	1.85	0.89248		
0.85	0.55766	1.90	0.89897		
0.90	0.58934	1.95	0.90498		
0.95	0.61892	2	0.91056		
1.00	0.64645	2.5	0.94877		

The use of the Boussinesq point load solution as the Green function gives the solution for vertical stress at points under the center of circular foundation areas subject to uniformly distributed loads as:

$$\sigma_r = \sigma_v = \int_0^{2\pi}\int_0^{R_0} \frac{3q_0 z^3 r\,dr\,d\theta}{2\pi(r^2+z^2)^{5/2}} = \frac{3q_0}{2\pi}\int_0^{2\pi} d\theta \int_0^{R_0} \frac{z^3 r\,dr}{(r^2+z^2)^{5/2}} = \frac{3q_0 z^3}{2\pi}[\theta]_0^{2\pi}\int_0^{R_0}\frac{r\,dr}{(r^2+z^2)^{5/2}} \quad (116)$$

$$\sigma_{zz} = \frac{3q_0 z^3}{2\pi} 2\pi \int_0^{R_0} \frac{r\,dr}{(r^2+z^2)^{5/2}} = 3q_0 z^3 \int_0^{R_0} \frac{r\,dr}{(r^2+z^2)^{5/2}} \quad (117)$$

Let, $r^2 + z^2 = a^2(r)$ (118)

Then, $2r\,dr = 2a(r)\,da(r)$

$$\therefore \sigma_v = 3q_0 z^3 \int_{a(r)=z}^{(R^2+z^2)^{1/2}} \frac{a(r)da(r)}{a(r)^5} \quad (119)$$

$a(r=0) = z$ (120)

$a(r=R) = (R^2+z^2)^{1/2}$ (121)

$$\sigma_r = 3q_0 z^3 \int_z^{\sqrt{R^2+z^2}} \frac{da(r)}{a(r)^4} = 3q_0 z^3 \left[\frac{a(r)^{-4+1}}{-4+1}\right]_z^{(R^2+z^2)^{1/2}} \quad (122)$$

$$\sigma_r = 3q_0 z^3 \left[-\frac{a(r)^{-3}}{3} \right]_z^{\sqrt{R^2+z^2}} = -q_0 z^3 \left[\frac{1}{(a(r))^3} \right]_z^{\sqrt{R^2+z^2}} \tag{123}$$

$$\sigma_r = -q_0 z^3 \left(\frac{1}{(R^2+z^2)^{3/2}} - \frac{1}{z^3} \right) = -q_0 \left(\frac{z^3}{(R^2+z^2)^{3/2}} - \frac{z^3}{z^3} \right) = -q_0 \left(\frac{(z^2)^{3/2}}{(R^2+z^2)^{3/2}} - 1 \right) \tag{124}$$

$$\sigma_r = q_0 \left(1 - \left(\frac{z^2}{(R^2+z^2)} \right)^{3/2} \right) = q_0 \left(1 - \left(\frac{1}{\left(1 + R^2/z^2\right)} \right)^{3/2} \right) \tag{125}$$

$$\sigma_r = q_0 \left(1 - \left(1 + \frac{R^2}{z^2} \right)^{-3/2} \right) \tag{126}$$

Vertical stress fields (distribution) under the center of a circular foundation area due to conical load distribution

A conical load distribution shown in the Figure 8 such that $q(t) = q_1$ if $t = R_0$, $q_0(t) = 0$, if $t = 0$ is considered to act on the circular foundation of radius R, t is a dummy radial coordinate.

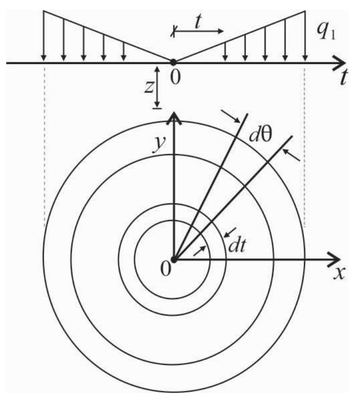

Figure 8: Circular foundation area subject to conical load distribution ($q(t) = q_1 t/R_0$, where $q(t = 0) = 0$, and $q(t = R_0) = q_1$)

$$dQ_0 = \frac{q_1 t}{R_0} t\, dt\, d\theta \tag{127}$$

The vertical normal stress distribution at any arbitrary depth z under the center of the circular foundation is obtained as:

$$\sigma_{zz}(r = t = 0, z) = \int_0^{2\pi} \int_0^{R_0} \frac{3 dQ_0 z^3}{2\pi R^3} t\, dt\, d\theta \tag{128}$$

where $R^5 = (t^2 + z^2)^{5/2}$ (129)

$$\sigma_{zz}(r = t = 0, z) = \int_{\theta=0}^{\theta=2\pi} \int_{t=0}^{R_0} \frac{3q_1 z^3}{2\pi R_0} \frac{t^2 dt\, d\theta}{(t^2+z^2)^{5/2}} \tag{130}$$

$$\sigma_{zz}(r = t = 0, z) = \frac{3q_1 z^3}{2\pi R_0} \int_0^{2\pi} d\theta \int_0^{R_0} \frac{t^2 dt}{(t^2+z^2)^{5/2}} = \frac{3q_1 z^3}{2\pi R_0} [\theta]_0^{2\pi} \int_0^{R_0} \frac{t^2 dt}{(t^2+z^2)^{5/2}} \tag{131}$$

$$\sigma_{zz}(r=t=0,z)=\frac{3q_1z^3}{R_0}\int_0^{R_0}t^2(t^2+z^2)^{-5/2}dt=\frac{3q_1z^3}{R_0}\left[\frac{t^3}{3z^2(t^2+z^2)^{3/2}}\right]_0^{R_0} \tag{132}$$

$$\sigma_{zz}(r=t=0,z)=\frac{q_1z^3}{R_0}\left[\frac{t^3}{z^2(t^2+z^2)^{3/2}}\right]_0^{R_0} \tag{133}$$

$$\sigma_{zz}=\frac{q_1z^3}{R_0}\left[\frac{R_0^3}{(R_0^2+z^2)^{3/2}}-\frac{0^3}{(z^2)^{3/2}}\right]=\frac{q_1z^3}{R_0}\left(\frac{R_0^3}{z^2(R_0^2+z^2)^{3/2}}\right) \tag{134}$$

$$\sigma_{zz}=q_1z^3\left(\frac{R_0^2}{z^2(R_0^2+z^2)^{3/2}}\right)=q_1z^3\frac{R_0^2/z^2}{(R_0^2+z^2)^{3/2}} \tag{135}$$

$$\sigma_{zz}=q_1\frac{R_0^2}{z^2}\cdot\frac{z^3}{(R_0^2+z^2)^{3/2}}=q_1\frac{R_0^2}{z^2}\frac{(z^2)^{3/2}}{(R_0^2+z^2)^{3/2}} \tag{136}$$

$$\sigma_{zz}=q_1\left(\frac{R_0}{z}\right)^2\left(\frac{z^2}{(R_0^2+z^2)}\right)^{3/2}=q_1\left(\frac{R_0}{z}\right)^2\left(\frac{1}{R_0^2/z^2+1}\right)^{3/2} \tag{137}$$

$$\sigma_{zz}=q_1\left(\frac{R_0}{z}\right)^2\left(1+\left(\frac{R_0}{z}\right)^2\right)^{-3/2} \tag{138}$$

$$\sigma_{zz}(r=0,z)=q_1\left\{\left(\frac{R_0}{z}\right)^2\left(\left(\frac{R_0}{z}\right)^2+1\right)^{-3/2}\right\}=q_1I_{c_2}\left(\frac{R_0}{z}\right) \tag{139}$$

$$I_{c_2}\left(\frac{R_0}{z}\right)=\left(\frac{R_0}{z}\right)^2\left(1+\left(\frac{R_0}{z}\right)^2\right)^{-3/2} \tag{140}$$

Table of values for $I_{c2}(R_0/z)$ is presented in Table 4. The results agree with solutions presented by Harr and Lovell [34].

Circular foundation under inverted conical load distribution

Vertical stress (under the center) of circular foundation due to a circular foundation area subject to load intensity that varies linearly from a maximum at the center to zero at the perimeter as shown in Figure 4 was also considered and mathematical solutions sought.

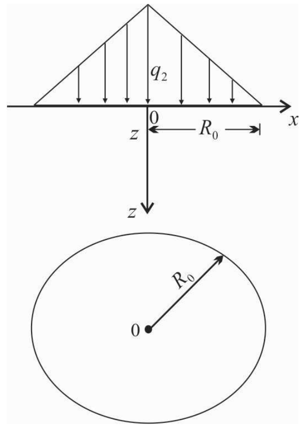

Figure 9: Vertical stress field in an elastic half space due to circular foundation under inverted conical load distribution (load varies linearly from a maximum intensity q_2 at the center of the circular foundation where $r = 0$ to zero intensity at the circumference (perimeter) where $r = R_0$)

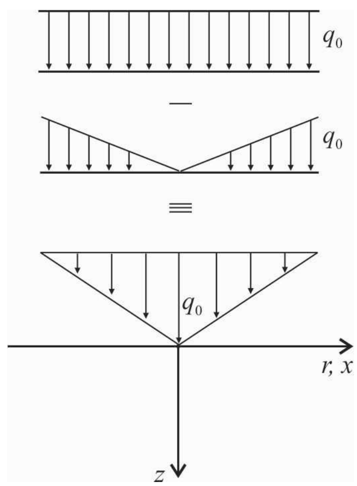

Figure 10: Application of the superposition principle in the computation of stress fields in elastic half-space due to inverted conical load distribution from the stress fields for uniformly distributed loads, and stress fields due to conical load distribution.

$$\sigma_{zz}(r=0,z) = \sigma_{zz}(r=0,z)_1 - \sigma_{zz}(r=0,z)_2 \tag{141}$$

where $\sigma_{zz}(r=0,z)$ is for the center of a circular foundation under uniformly distributed load (q_0) and $\sigma_{zz}(r=0,z)_2$ is for the center of a circular foundation under conical distribution of load $\left(q_0 r / R_0\right)$

Hence,

$$\sigma_{zz}(r=0,z) = q_0\left(1-\left(1+\left(R_0/z\right)^2\right)^{-3/2}\right) - q_0\left(\left(R_0/z\right)^2\left(\left(R_0/z\right)^2+1\right)^{-3/2}\right) \tag{142}$$

$$\sigma_{zz}(r=0,z) = q_0\left\{1-\left(1+\left(R_0/z\right)^2\right)^{-3/2} - \left(R_0/z\right)^2\left(1+\left(R_0/z\right)^2\right)^{-3/2}\right\} \tag{143}$$

$$\sigma_{zz}(r=0,z) = q_0 \left\{ 1 - \left(\frac{1}{\left(1+\left(R_0/z\right)^2\right)^{3/2}} + \frac{\left(R_0/z\right)^2}{\left(1+\left(R_0/z\right)^2\right)^{3/2}} \right) \right\} \tag{144}$$

$$\sigma_{zz}(r=0,z) = q_0 \left\{ 1 - \left(\frac{1+\left(R_0/z\right)^2}{\left(1+\left(R_0/z\right)^2\right)^{3/2}} \right) \right\} = q_0 \left\{ 1 - \left(\frac{1}{\left(1+\left(R_0/z\right)^2\right)^{1/2}} \right) \right\} \tag{145}$$

$$\sigma_{zz}(r=0,z) = q_0 \left\{ 1 - \left(1+\left(R_0/z\right)^2 \right)^{-1/2} \right\} = q_0 \, I_{c_3}\left(R_0/z\right) \tag{146}$$

$I_c\left(R_0/z\right)$ is calculated and presented for various values of R_0/z in Table 5. The results for $\sigma_{zz}(r=0,z)$ given in Equation (146) are identical with solutions presented by Harr and Lovell [40].

The distributed load is expressed as:

$$q(r) = q_{\max}\left(1-\frac{r}{R_0}\right) \tag{147}$$

The Boussineq point load solution is used as a Green function to express the vertical stress under the center of the circular foundation subject to the considered load in this section as follows:

$$\sigma_{zz} = \int_0^{2\pi}\int_0^{R_0} \frac{3}{2\pi} q_{\max}\left(1-\frac{r}{R_0}\right)\frac{z^3}{(r^2+z^2)^{5/2}}\, r\, dr\, d\theta = 3\int_0^{R_0} q_{\max}\left(1-\frac{r}{R_0}\right)\frac{z^3 r\, dr}{(r^2+z^2)^{5/2}} \tag{148}$$

$$\sigma_{zz} = q_{\max}\cdot 3\int_0^{R_0}\left(1-\frac{r}{R_0}\right)\frac{z^3 r\, dr}{(r^2+z^2)^{5/2}} \tag{149}$$

$$\sigma_{zz} = q_{\max}\cdot 3\left\{ \int_0^{R_0}\frac{z^3 r\, dr}{(r^2+z^2)^{5/2}} - \int_0^{R_0}\frac{z^3 r^2\, dr}{(r^2+z^2)^{5/2}} \right\} \tag{150}$$

$$\sigma_{zz} = q_{\max}\cdot 3\left\{ \left[-\frac{z^3}{3}(R^2+z^2)^{-3/2}\right]_0^{R_0} - \frac{z^3}{R}\left[\frac{r^3}{3z^2(r^2+z^2)^{3/2}}\right]_0^{R_0} \right\} \tag{151}$$

$$\sigma_{zz} = q_{\max}\left\{ -\left[\frac{z^3}{(R_0^2+z^2)^{3/2}}-1\right] - \frac{z^3}{R_0}\left[\frac{R_0^3}{z^2(R_0^2+z^2)^{3/2}}\right] \right\} \tag{152}$$

$$\sigma_{zz} = q_{\max}\left\{ -\frac{z^3}{(R_0^2+z^2)^{3/2}} + 1 - \frac{R_0^2/z^2}{\left(\frac{R_0^2+z^2}{z^2}\right)^{3/2}} \right\} \tag{153}$$

$$\sigma_{zz} = q_{\max}\left\{1-\left(\frac{1}{\left(1+\left(R_0/z\right)^2\right)^{3/2}}+\frac{R_0^2/z^2}{\left(1+\left(R_0/z\right)^2\right)^{3/2}}\right)\right\} \quad (154)$$

$$\sigma_{zz} = q_{\max}\left(1-\left(1+\left(R_0/z\right)^2\right)^{-1/2}\right) = q_{\max} I_{c_3}\left(R_0/z\right) \quad (155)$$

$$I_{c_3}\left(R_0/z\right) = 1-\left(1+\left(R_0/z\right)^2\right)^{-1/2} = \frac{\sigma_{zz}}{q_{\max}} \quad (156)$$

The results presented in Equation (155) are identical with solutions presented by Harr and Lovell [40], thus validating the research work presented in this paper.

Discussion

In this work, the Navier's differential equations of equilibrium, presented in terms of the axisymmetric coordinates as a system of two differential equations were solved in closed form to obtain the general expressions for the displacement components. The solutions were sought for the case when body forces were disregarded and for harmonic displacement field. The solution for the Navier's differential equation of equilibrium in the z coordinate direction was obtained in general as Equation (30). In Equation (24), the two unknown constants were obtained from the equation for the volumetric strain given as Equation (31). The radial component of the displacement $u_r(r, z)$ was obtained by solving Equation (31) in closed form such that singularities of $u_r(r, z)$ were avoided by implementation of the boundedness condition that requires that as $r \to 0$, $u_r(r \to 0, z) \to 0$ for any z. By substitution of the expressions for the volumetric strain φ and the vertical z component of the displacement $w(r, z)$, the equation to be solved for u_r simplified to Equation (36). Integration of Equation (36) gave Equation (38) which contains another integration constant c_3 which was evaluated from the condition for nonsingular solution for u_r as $r \to 0$.

The enforcement of nonsingular solution for $u_r(r, z)$ as $r \to 0$, gave the expressions for c_3 as Equation (44). The radial displacement component was thus found in terms of the two unknown integration constants c_1 and c_2 as Equation (45). The general solution to the Navier's displacement equations of equilibrium were thus found as Equation (30) for $w(r, z)$ and Equation (45) for $u_r(r, z)$.

The specific classical Boussinesq problem of vertical point load applied at the origin on an elastic half space assumed linear elastic, homogeneous and isotropic was considered. The obtained solutions for the displacements were used in the stress displacement equations expressed as Equations (46 – 49) to obtain the shear stress field on the $z = 0$ plane as Equation (63). The enforcement of the shear stress free boundary condition gave the expression for the unknown integration constant c_2 in terms of c_1 as Equation (66). The vertical stress field, obtained by use of Equation (48) – vertical stress – displacement equation – was thus obtained in terms of only one unknown integration constant as Equation (74).

The requirement of equilibrium of the resultant internal vertical stress and the applied vertical point load (at the origin)-expressed as Equation (75)-was used to derive the unknown integration constant c_1. This yielded a full determination of the two constants, and hence, the vertical stress distribution and the radial and vertical components of the displacement. The integration problem expressed in Equation (75) was solved by change or transformation of coordinates to the cylindrical coordinate system using the transformation equations – Equations (76 – 78); yielding the transformed problem as Equation (79). The requirement of equilibrium of resultant internal vertical stresses and the applied vertical point load is explicitly given by Equation (81) in the cylindrical polar coordinate system. The integration problem in Equation (81) was solved using the change of variables of integration given by Equation (87) to obtain the solutions for the unknown constants c_1 as Equation (95) and c_2 as Equation (97). Thus, the vertical stress field was determined as Equation (96). The stresses were determined using the stress displacement relations as Equation (98 – 100). The displacements determined by substitution of the obtained values for c_1 and c_2 are Equations (101) and (102). The displacement components on the surfaces $z = 0$ were obtained as Equations (103) and (104). The vertical stress at any point in the elastic half-space material due to a uniformly loaded circular foundation was obtained by using the Boussinesq point load solution for vertical stress as a Green function, thus yielding the vertical stress field as the complicated double integration problem given by Equation (105).

For uniformly distributed loads, the stress field was found to simplify to the integration problem given after factoring out the constants to obtain Equation (107). The vertical stress field was obtained as Equation (108) using Egorov's results in terms of complete elliptic integrals of the second kind and complete elliptic integrals of the third kind with a modulus dependent on the radius of the circular foundation and the radial distance of the arbitrary point in the elastic half-space. The vertical stress field was presented in Table 2 in terms of non-dimensional vertical stress influence factors (coefficients) which depend upon the ratios r/R_0 and z/R_0.

The vertical stress distribution under the center of circular foundation subjected to uniformly distributed load was obtained as a special case of the general solution for vertical stress at any point (r, z) in the elastic half-space. This is observed to be the case by setting $r = 0$ in the expression to obtain Equation (114), which upon evaluation of $E(0)$ and $\Pi_0(0, 0)$ and algebraic simplification yielded Equation (115). The vertical stress distribution under the center of circular foundation subject to uniformly distributed load was alternatively derived in the study by using the point load solution obtained as an integral Kernel function to obtain Equation (116). Evaluation of the double integration problem, and algebraic simplifications gave the result as Equation (126).

The work also considered the vertical stress field (at any depth under the center) in an elastic half space due to a circular foundation area subject to a conical load distribution which varies linearly from an intensity $q_0(r = 0) = 0$ at the center to $q(r = R_0) = q_0$ at the radius R_0. The integration problem derived from the use of the point load solution obtained in this work was shown in Equation (128). The evaluation of the double integration problem over the two-dimensional domain of the circular plate yields, after algebraic simplifications, the solution for $\sigma_{zz}(r = 0, z)$ as Equation (139), which is expressed in terms of dimensionless influence coefficients for vertical stress $I_{c2}(R_0/z)$ given by Equation (140) and presented in Table 4 and Figure 12.

The vertical stress field under the center of circular foundation of radius R_0 subject to an inverted conical load distribution (in Figure 9) was obtained by the application of superposition from the vertical stress field results for the uniformly distributed load and the conical load distribution. They are presented in terms of non-dimensional influence factors (coefficients) of vertical stress

$I_{c3}(R_0/z)$ as Equation (146). The same results were alternatively derived by the application of the principles of superposition from the point load solution by solving the double integration problem given in Equation (149) over the two dimensional (2D) region of the circular plate, to obtain Equation (155).

Table 4: $I_{c_2}\left(\frac{R_0}{z}\right)=\left(\frac{R_0}{z}\right)^2\left(1+\left(\frac{R_0}{z}\right)^2\right)^{-3/2}$

Vertical stress influence coefficients $I_{c2}(R_0/z)$ for vertical stress at any depth z under the center of a circular foundation of radius R_0 carrying a conical load distribution $q(r)=q_1 r/R_0$

R_0/z	$I_{c2}(R_0/z)$	R_0/z	$I_{c2}(R_0/z)$
0.20	0.037715	5	0.18857
0.40	0.128066	6	0.159956
0.50	0.178885	8	0.12213
0.75	0.288	10	0.09852
1.0	0.353553	20	0.04981
1.2	0.377814	50	0.019988
1.50	0.384023	100	0.0099985
2	0.357771	∞	0
2.5	0.3201644		
3	0.284605		
3.5	0.2539875		
4	0.2282688		

Table 5: $I_{c_3}\left(\frac{R_0}{z}\right)=1-\left(1+\left(\frac{R_0}{z}\right)^2\right)^{-1/2}$

Vertical stress influence coefficients for vertical stress at any depth z under the center of a circular foundation of radius R_0 subject to an inverted conical load distribution

$$q(r)=q_{\max}\left(1-r/R_0\right)$$

R_0/z	$I_{c2}(R_0/z)$	R_0/z	$I_{c2}(R_0/z)$
0.20	0.01942	5	0.803884
0.40	0.071523	6	0.83560
0.50	0.105573	7	0.858579
0.75	0.20	8	0.875965
1.00	0.29289	9	0.88957
1.2	0.3598156	10	0.900496
1.5	0.44530	12.5	0.920255
2	0.552786	15	0.933481
2.5	0.628609	20	0.95006
3	0.68377	50	0.98000
3.5	0.72528	∞	1.00000
4	0.75746		

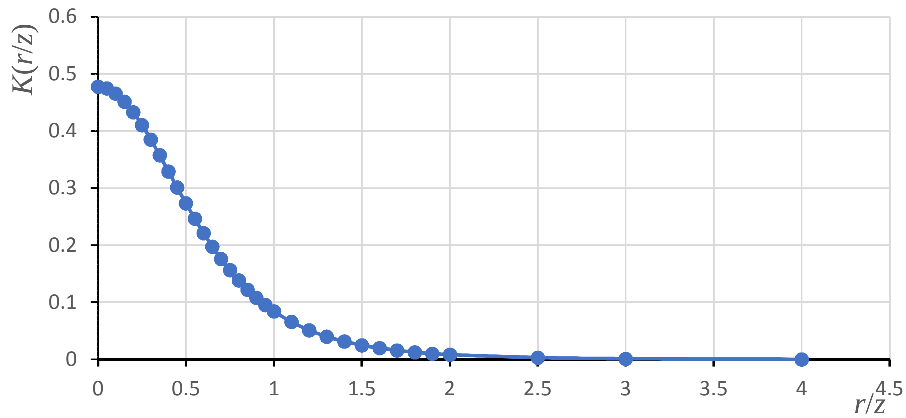

Figure 11: Variation of $K(r/z)$ with (r/z) for the Boussinesq problem

$$K\left(r/z\right)=\frac{3}{2\pi}\left(1+\left(r/z\right)^2\right)^{-5/2}$$

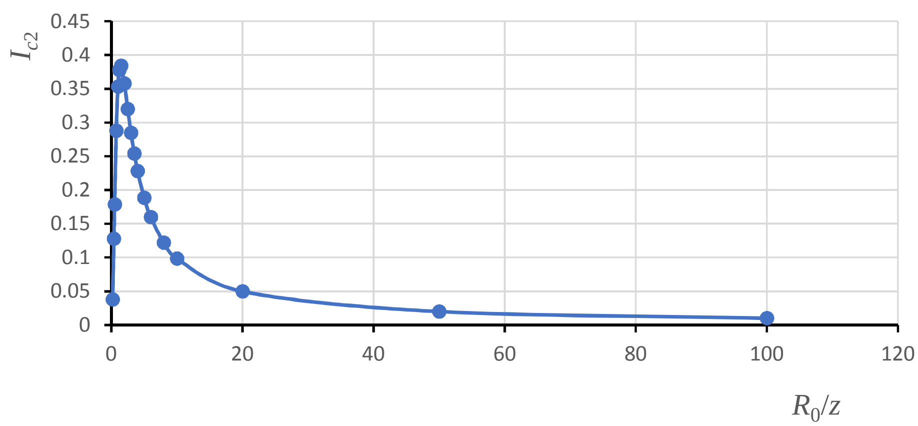

Figure 12: Variation of $I_{c2}(R_0/z)$ with (R_0/z)

$$I_{c_2}\left(R_0/z\right)=\left(\frac{R_0}{z}\right)^2\left(1+\left(R_0/z\right)^2\right)^{-3/2}$$

Conclusion

The following conclusions are made from the study:

(i) Navier's differential equations of equilibrium for elastostatic axisymmetric problems of the half-space which are two partial differential equations in terms of the radial and vertical displacement components have been integrated (solved) in general. The general solution gave the displacement components in terms of two unknown constants of integration c_1 and c_2, with the third integration constant obtained (found) by the appropriation of the boundedness condition for $u_r(r \to 0, z)$ as $r \to 0$.

(ii) Closed form expressions were obtained for the specific classical elasticity problem of a vertical point load applied at the origin of the half-space by the enforcement of the shear stress free boundary conditions. This helped to determine the one unknown integration constant c_2 in terms of the other (c_1). The requirement of the equilibrium of the resultant internal vertical stresses and the applied vertical point load was used to determine the unknown constant c_1, thus leading to the full determination of the vertical displacement and radial displacement components as well as the normal and shear stress fields.

(iii) Vertical stress distribution for the point load solution, being the most useful of the stress field expressions consequent to its significant use in elastic settlement analysis, was used as integral Kernels to determine the vertical stress distributions for distributed loads that produce axisymmetric stresses, strains and displacements.

(iv) Expressions for vertical stress fields in the elastic half- space due to circular foundation areas subject to uniformly distributed load were obtained by considering the point load solutions as integral kernels and performing appropriate integrations over the circular foundation domain. Expressions obtained by the evaluation of the resulting complicated double integration problems in general involve elliptic integrals of the second and third kind, and were presented as tables.

(v) Closed form expressions were obtained for the case of points in the elastic half-space under the center of circular foundation areas subject to uniform loads by putting $r = 0$ in the general solution for $\sigma_{zz}(r, z)$. Consequently, simpler expression for the vertical stress influence coefficients were found in terms of the ratio of the radius of the foundation R_0 and the depth, z (R_0/z).

(vi) The vertical stress fields (distributions) under the center of circular foundation areas due to conical distribution of loads were similarly derived by considering the point load solution obtained as a Green function, resulting in the evaluation of double integration problem over the two dimensional domain of the circular foundation.

(vii) The vertical stress distribution under the center of circular foundation areas due to inverted conical distribution of load were also derived by considering the point load solution obtained as a Green function in a double integration problem over the two dimensional domain of the circular foundation.

(viii) The closed form expressions obtained for the vertical stress distributions under the center of the circular foundation for all the axisymmetrical load distributions considered were radially symmetrical functions with respect to the vertical axis of symmetry ($r = 0$) of the problem which is the vertical axis directly under the center of the circular foundation. This agrees with the symmetrical character of the elastostatic half-space problem considered and the symmetrical nature of the applied distributed loads about the vertical axis of symmetry ($r = 0$).

Nomenclature

x, y, z three dimensional Cartesian coordinates

r, z, θ cylindrical polar coordinates

$2D$ two-dimensional

$3D$	three-dimensional
∞	infinity
$\vec{v}$	displacement field
$u_r(r, z)$	radial component of displacement field
$u_\theta(r, z)$	tangential component of displacement field
i_r	unit vector in the radial coordinate direction
i_z	unit vector in the z coordinate direction
r	radial coordinate
θ	tangential coordinate
z	depth (transverse) coordinate
E	Poisson's ratio
G	shear modulus or modulus of rigidity
λ	Lamé constant
$\vec{F}$	body force vector
i, j, k	unit vectors of the 3D Cartesian coordinate system
∇	vector differential operator
∇^2	Laplacian operator
μ	Poisson's ratio
PDE	Partial Differential Equation
ε_{rr}	radial strain
$\varepsilon_{\theta\theta}$	circumferential strain
ε_{zz}	strain in the z coordinate direction
γ_{xz}	shear strain
σ_{rr}	radial stress
$\sigma_{\theta\theta}$	circumferential stress
σ_{zz}	normal stress in the z coordinate direction
σ_{rz} (or τ_{rz})	shear stress
F_z	body force component in the z direction
$F_r(r, z)$	radial component of body force vector
$F_z(r, z)$	z-component of body force vector
c_1, c_2, c_3	constants
$\rightarrow$	tend to

$\lvert J \rvert$	Jacobian of the coordinate transformation from Cartesian to polar coordinates
Q_0	point load acting at the origin of an elastic half-space
q_0	intensity of uniformly distributed load acting over a circular area on the surface of an elastic half-space
R_0	radius of circular foundation
$\frac{\partial}{\partial z}$	partial derivative with respect to z
$\int$	integration sign or integral sign
$\iint$	double integration
$\lvert\ \rvert$	determinant
$k\left(r/z\right)$	Boussinesq vertical stress influence coefficient expressed in terms of r/z
$E(k)$	complete elliptic integral of the second kind with a modulus of k, and parameter, m
$\Pi_0(k,m)$	complete elliptical integral of the third kind with a modulus of k, and parameter, m
α	dimensionless parameter defined in terms of r and R_0
n	dimensionless parameter defined in terms of z and R_0
k	parameter defined in terms of n and α
m	parameter defined in terms of α
F_1	parameter defined in terms of r and R_0
$I\left(r/R_0, z/R_0\right)$	dimensionless vertical stress influence factors that depend upon r/R_0 and z/R_0

References

[1] K.T. Chau, Analytic methods in geomechanics, CRC Press Taylor and Francis Group, New York, 2013

[2] M.H. Sadd, Elasticity theory, applications and numerics, Third Edition, Elsevier Academic Press, Amsterdam, 2014.

[3] P. Padio-Guidugli, A. Favata, Elasticity for geotechnicians: A modern exposition of Kelvin, Boussinesq, Flammant, Cerrutti, Melan and Mindlin problems, Solids Mechanics and its Applications, Springer, 2014.

[4] D. Palaniappan, A general solution of equations of equilibrium in linear elasticity, *Applied Mathematical Modelling*, Vol 35, pp 5494-5499, 2011.

[5] J.R. Barber, Elasticity, 3rd Revised Edition, Springer Science and Business Media, Dordrecht, the Netherlands, 2010.

[6] M. L. Kachanov, B. Shafiro, I. Tsukrov I. Handbook of elasticity solutions, Springer Science and Business Media, Kluwer Academic Publishers, Dordrecht, the Netherlands, 2003.

[7] Hazel Andrew, MATH 35021 Elasticity, University of Manchester, Manchester. www.maths.manchester.ac.uk/ahazel/MATH35021/MATH35021.html, 2015.

[8] T.G. Sitharam, L. Govinda-RejuL. Applied Elasticity for Engineers Module: Elastic Solutions and Applications in Geomechanics, 14.139.172.204/nptel/1/CSE/web/ 105108070/module8/lecture17.pdf

[9] R. Abeyaratne, Continuum Mechanics, Volume II of Lecture notes on the Mechanics of Elastic Solids, Cambridge, http://web.mit.edu/abeyartne/ lectures_notes.html, 11th May 2012 updated 6th May 2015, ISSN - 13:978-0-9791865-0-9, ISBN – 10-0-9791865-0-1, 2012.

[10] C. C. Ike, Principles of Soil Mechanics, De-Adroit Innovation, Enugu, Nigeria, 2006.

[11] H.K.Teymoueri, A. Khojasteh, M. Rahimian, R.Y.S. Pak, An analytical solution of wave motion in a transversely isotropic poroelastic half-space underlying a liquid layer. *Civil Engineering Infrastructures Journal*, 54(1), pp.169–179. DOI: 10.22059/ceij.2020.28370, 2021.

[12] M.Raoofian Naeni, M. Eskandari-Ghadi, A potential method for body and surface wave propagation in transversely isotropic half- and full-spaces. *Civil Engineering Infrastructures Journal*, Vol.49, No. 2, pp. 263 – 288. DOI: 10.7508/CEIJ.2016.02.006, 2014.

[13] Shariati Mojtaba, Shishehsaz Mohammad, Sahbafar Hossein, Pourabdy Mortaza, Hosseini Mohammad, A review on stress driven nonlocal elasticity theory, *Journal of Computational Applied Mechanics*,Vol. 52, No.3, pp. 535–552, DOI: 10.22059/jcamech.2021.331410.653, 2021.

[14] C. C. Ike, Fourier-Bessel transform method for finding vertical stress fields in axisymmetrical elasticity problems of elastic half-space involving circular foundation areas. *Advances in Modelling and Analysis A,* Vol. 55, No. 4, December 2018, pp. 207-216. https://doi.org/10.18208/ama_a.550405, 2018.

[15] C.C. Ike, H.N. Onah, C.U. Nwoji, Bessel functions for axisymmetric elasticity problems of the elastic half-space soil: A potential function method. *Nigerian Journal of Technology,* Vol. 36, No. 3, pp. 773-781. http//dx.doi.org/w4314/nijtv36i3.16, 2017.

[16] C.C. Ike, First principles derivation of a stress function for axially symmetric elasticity problems, and application to Boussinesq problem. *Nigerian Journal of Technology,* Vol. 36, No. 3, pp 767-772, http//dx.doi.org/10.4314/nijtv36i3.15, 2017.

[17] C.C. Ike, General solutions for axisymmetric elasticity problems of elastic half space using Hankel transform method, *International Journal of Engineering and Technology,* Vol 10, No 2, pp 565-580, https://10.21817/ijet2018/v10i2/181002112, 2018.

[18] C.C. Ike, Hankel transform method for solving axisymmetric elasticity problems of circular foundation on semi-infinite soils. *International Journal of Engineering and Technology,* Vol 10, No. 2, pp. 549-564, https://10.21817/ijet2018/v10i2/181002111, 2018.

[19] C.C. Ike, Love stress function method for solving axisymmetric elasticity problems of the elastic half space, *Electronic Journal of Geotechnical Engineering,* Vol 24, No 3, pp 663-706, 2019.

[20] C.C. Ike, Hankel transform method for solving the Westergaard problem for point, line and distributed loads on elastic half-space, *Latin American Journal of Solids and Structures,* Vol 16, No 1, 2019, pp. 1-19, http://dx.doi.org/10.1590/1679-78255313, 2019.

[21] C. C. Ike, B.O. Mama, H.N. Onah ., C.U. Nwoji, Trefftz harmonic function method for solving Boussinesq problem, *Electronic Journal of Geotechnical Engineering,* Vol, 22, Bundle 12 (22.12), pp 4589**-4601, 2017.**

[22] C.U. Nwoji, H.N. Onah, B.O. Mama, C.C. Ike, Solution of the Boussinesq problem of half space using Green and Zerna displacement potential function method, *Electronic Journal of Geotechnical Engineering,* Vol 22, Bundle 11, pp 4305-4314, 2017.

[23] C.U. Nwoji, H.N. Onah, B.O. Mama, C.C. Ike, Solution of elastic half-space problem using Boussinesq displacement potential functions. *Asian Journal of Applied Sciences,* Vol 5, Issue 5, October 2017, pp 1087-1093, www.ajouronline.com, 2017.

[24] H.N. Onah, B.O. Mama, C.U. Nwoji, C.C. Ike, Boussinesq displacement potential function method for finding vertical stresses and displacement fields due to distributed load on elastic half

space, *Electronic Journal of Geotechnical Engineering,* Vol.22, Bundle 15, pp. 5687-5709, available at www.ejge.com, 2017.

[25] H.N. Onah, N.N. Osadebe, C.C. Ike C.C., C.U. Nwoji, Determination of stresses caused by infinitely long line loads on semi-infinite elastic soils using Fourier transform method, *Nigerian Journal of Technology,* Vol 35 No 4, October 2016, pp. 726-731. http://doi.org/10.4314/nijt.v35i4.7 [Cross 2016.

[26] H.N. Onah, C.C. Ike, C.U. Nwoji, B.O. Mama, Theory of elasticity solutions for stress fields in semi-infinite linear elastic soil due to distributed load on the boundary using the Fourier transform method, *Electronic Journal of Geotechnical Engineering,* Vol. 22, Bundle 13, pp. 4945-4962, 2017.

[27] C.C. Ike, Exponential Fourier integral transform method for stress analysis of boundary load on soil, *Mathematical Modelling of Engineering Problems,* Vol 5 No 1, March 2018, pp 33-39. https://doi.org/10.18280/mmep.050105, 2018.

[28] C.C. Ike, Fourier sine transform method for solving the Cerrutti problem of elastic half-plane in plane strain, *Mathematical Modelling in Civil Engineering,* Vol 14 No 1, pp 1-11, doi:10/2479/mmce-2018-0001, 2018.

[29] C.C. Ike, On Maxwell's stress functions for solving three dimensional elasticity problems in the theory of elasticity, *Journal of Computational Applied Mechanics,* Vol 49 Issue 2, December 2018, pp 342-350, doi:10.22059/jcamech.2018.266787.330, 2018.

[30] C.C. Ike, Solution of elasticity problems in two dimensional polar coordinates using Mellin transform, *Journal of Computational Applied Mechanics,* Vol 50, Issue 1, June 2019, pp 174-181, doi.1022059/jcamech.2019.278288.370, 2019.

[31] H.N. Onah, M.E. Onyia, B.O.Mama, C.U. Nwoji, C.C. Ike, First principles derivation of displacement and stress function for three-dimensional elastostatic problems, and application to the flexural analysis of thick circular plates. *Journal of Computational Applied Mechanics,* Vol.51, Issue 1, June 2020, pp.184-198. DOI:10.22059/jcamech.2020.295989.471, 2020.

[32] C.C. Ike, Elzaki transform method for finding solutions to two-dimensional elasticity problems in polar coordinates formulated using Airy stress functions, *Journal of Computational Applied Mechanics,* Vol. 51, Issue 2, December 2020, pp. 302-310, DOI: 10.22059/jcamech.2020.296012.472. 2020.

[33] C.C. Ike, Fourier cosine transform method for solving the elasticity problem of point load on an elastic half plane, *International Journal of Scientific and Technology Research,* Vol. 9, Issue 4, April 2020, pp 1850-1856, 2020.

[34] C.C. Ike, Cosine integral transformation method for solving the Westergaard problem in elasticity of the half-space, *Civil Engineering Infrastructures Journal,* Vol.53, Issue 2, pp. 313-339, DOI: 10.22059/ceij.2020.285125.1596, 2020.

[35] C.C. Ike, B.O. Mama, H.N. Onah, C.U. Nwoji, Trefftz displacement potential function method for solving elastic half-space problems, *Civil Engineering and Architecture,* Vol 9, No 3, pp 559-583. https://doi.org/10.13189/cea.2021.090301, 2021.

[36] C.C. Ike, Fourier integral transformation method for solving two dimensional elasticity problems in plane strain using Love stress functions, *Mathematical Modelling of Engineering Problems,* Vol.8, No.3, June 2021,pp 333-346. https://doi.org/10.18280/mmep.080302, 2021.

[37] K.E. Egorov, R.V. Serebrjanyi, Determination of stresses in a rigid circular foundation. 2nd Asian Regional Conference on Soil Mechanics and Foundation Engineering, Tokyo Japan, Vol. 1, pp 246-250, 1963.

[38] M.E. Harr, Foundations of Theoretical Soil Mechanics, McGraw Hill, New York, 1966.

[39] K.E. Egorov, Concerning the question of computing stresses under foundation with footings in the shape of rings, Mekanika Gruntov (Soil Mechanics), pp. 34-57, Grosstroiizdat Moscow, 1958.

[40] M.E. Harr, C.W. Lovell Jr., Vertical stresses under certain axisymmetrical loadings, Highway Research Board Rec. 39, Onlinepubs.trb.org>onlinepubs>hrr>1963, Accessed on 7th September 2019,1963.