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RESEARCH PAPER

Buckling analysis of three-dimensional functionally graded Euler-Bernoulli nanobeams based on the nonlocal strain gradient theory

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Abstract

This paper presents a nonlocal strain gradient theory for capturing size effects in buckling analysis of Euler-Bernoulli nanobeams made of three-dimensional functionally graded materials. The material properties vary according to any function. These models can degenerate to the classical models if the material length-scale parameters is assumed to be zero. The Hamilton's principle applied to drive the governing equation and boundary conditions. Generalized differential quadrature method used to solve the governing equation. The effects of some parameters, such as small-scale parameters and constant material parameters are studied.

Keywords: Buckling analysis, Strain gradient elasticity theory, Nano beam, Three-directional functionally graded materials (TDFGMs), Generalized differential quadrature method (GDQM).

1. Introduction

Micro- and nanotechnology have recently found a special place in various sciences such as medicine and engineering. So the attention of scientists has focused on this science. With that in mind, mechanical engineers have done a lot of research to unravel the ambiguities of nanotechnology [1, 2]. One of the first problems was that classical mechanics did not have the ability to examine issues in the nano-scale. In order to solve this problem, reinforcement continuum mechanics theories have been proposed, which consider inherent characteristics of materials at the nano-scale [3, 4]. In recent decades, with the development of various engineering fields related to micro- and nano-systems, much attention has been given to size effects on material behaviors. Some advanced methods to address the weaknesses of the conventional theory are; Cosserat continuum mechanics [5], nonlocal

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elasticity [6-8], strain gradient elasticity [9], couple stress theory [10] and nonlocal strain gradient theory [11]. Among the non-classical continuum mechanics theories, nonlocal elasticity and strain gradient theory have been widely used to analyze the nanostructures [12-41]. According to the nonlocal elasticity theory, in contrast to classical elasticity, the stress tensor at an arbitrary point x in the domain of material depends not only on the strain tensor at x but also on strain tensor at all other points in the domain. According to this theory, a stress–strain relationship for a homogeneous elastic solid is:

$$\sigma_{ij}^{nl} = \int_{v} \alpha\left(\left|x' - x, \tau\right|\right) \sigma_{ij}^{l} dv\left(x'\right)$$

where α is the nonlocal modulus or kernel function. It contains the small-scale effects incorporating into constitutive equations the nonlocal effects at the reference point x produced by local strain at the source x'. This function depends on two variables $|x - x'|$ and α. $|x - x'|$ represents the distance in Euclidean form and $\tau = e_0 \bar{a}/L$ is a material constant that depends on internal and external characteristic length (such as the lattice spacing and wavelength). The parameter e_0 is vital for the validity of nonlocal models. This parameter was determined by matching the dispersion curves based on atomistic models. Also, σ^{nl} is the nonlocal stress tensor at the reference point and σ^{l} is the classical stress tensor at local point. In addition, the classical stress tensor is defined as follows:

$$\sigma^{l} = C : \varepsilon$$

here C is the fourth order elasticity tensor and ':' denotes the double dot product. Eringen determined the functional form of the kernel numerically. By appropriate choice of the kernel function, Eringen showed that the nonlocal constitutive equation given in integral form can be represented, for unbounded domains, in an equivalent differential form as:

$$\left(1 - \mu \nabla^2\right) \sigma^{nl} = C : \varepsilon, \quad \mu = \left(e_0 a\right)^2$$

On the contrary, for bounded structural domains, Eq.(1) is equivalent to a differential problem with suitable boundary constitutive conditions which are in contrast with equilibrium [42]. Accordingly, the elastostatic problem of a continuous nano-structure, formulated with Eringen's integral model, admits no solution and hence such a theory cannot be adopted in nano-mechanics [43, 44]. The differential form of Eringen's integral law is not adequate to analyze size effects in nano-beams [45]. A mathematically and mechanically well-posed model to investigate devices at nanoscale was proposed in [46] by a stress-driven integral methodology. Such an approach was successfully applied in [47] to study size-dependent dynamical behavior of a Bernoulli-Euler nano-beam. The stress-driven integral law exhibits, like strain gradient models, a hardening structural response for increasing values of the nonlocal parameter. Recently, Lim et al. [48] pointed out the nonlocal strain gradient theory can predict both increase and decrease in the structural stiffness, a result confirmed by the experimental data [48]. These methods are widely used in modeling of nano structure problems. For example, Li and Hu [49] studied buckling behavior of a nonlinear Euler-Bernoulli simply supported beam of nonlocal strain gradient theory. Hamilton's principle was used to drive the governing equation and associated boundary conditions. Results show that post-buckling deflection has a positive relation to nonlocal parameter and reverse relation to material characteristic parameter. Besides, size dependent parameters have a significant effect on the higher-order buckling deflection. Li et al. [50] offered a model to investigate flexural wave propagation in the functionally graded beams based on nonlocal strain gradient theory. The effect of some parameters such as nonlocal parameter and material characteristic parameter were investigated. Farajpour et al. [51] used a higher-order nonlocal strain gradient in order to study thermoelastic buckling behavior of orthotropic size-dependent plate which is resting on the elastic foundation. Differential quadrature method was employed to solve the higher-order governing differential equation. They studied the effect of scale parameter, temperature and aspects of the plate on the buckling behavior of nanoplate. Ebrahimi and Barati [52] analyzed vibration through the thickness functionally graded smart nanobeams by using of a new nonlocal higher-order refined magneto-electro-viscoelastic model. They considered various boundary conditions for nanobeams. The effects of some parameters such as boundary conditions, damping coefficient, magnetic field and electric voltage on the natural frequency of nanobeams were studied. Nejad et al. [16-18] studied the bending, buckling and vibration behavior of nano-beams in the framework of nonlocal elasticity theory. They used Euler-Bernoulli beam model to analyze the nano-beams and considered that the mechanical property in nano-beam changed in the thickness and length directions according to arbitrary function. In these studies, the effects of length scale parameter and inhomogeneity parameter were explored. Tuns and Kirca [53, 54] presented an exact solution for integral form of Eeingen's nonlocal theory in order to analyze bending and buckling of Euler-Bernoulli and Timoshenko nanobeams. Ebrahimi and Barati [55] studied wave propagation in the functionally graded nanobeams using of nonlocal strain gradient theory. Nanobeams were resting on an elastic foundation and was subjected to axial magnetic field. Results show that length scale parameter, material inhomogeneity parameter, elastic foundation and magnetic field have significant effect on the wave propagation behavior. Li et al. [56] investigated free vibration behavior of functionally graded size-dependent Timoshenko beams on the basis of nonlocal strain gradient theory. They used Hamilton's principle to drive governing equation and boundary conditions. Li and Hu [57] used nonlinear Euler-Bernoulli and Timoshenko beam models along nonlocal strain gradient theory in order to study bending behavior of through the thickness functionally graded size-dependent beams. They show that the effect of small-scale parameter and material inhomogeneity parameter on the bending deflection and vibration frequency of size-dependent beams. Romanoff et al. [58] employed nonlocal sandwich Timoshenko beam theory in order to explain the macro- and micro-structural responses. Ebrahimi and Barati [59] investigate the influences of surface and thermal effects on the vibration analysis of viscoelastic Euler-Bernoulli nanobeams by using of nonlocal strain gradient theory. The nanobeams made of functionally graded material and resting on the viscoelastic foundation. Hamilton's principle was used to drive the governing equation and boundary conditions. They described the effects of damping coefficient on the vibration frequency of viscoelastic nanobeams. Based on the nonlocal strain gradient theory, Ebrahimi and Barati [60] offered a model for buckling behavior of higher order shear deformable curved nanobeams. Nanobeams were made of functionally graded material and mechanical properties varied according to power-law model. They investigate the effect of boundary conditions, material inhomogeneity parameter and length parameter. Xu et al. [61] investigated bending and buckling behavior of Euler-Bernoulli beams using of nonlocal strain gradient theory. Based on the nonlocal strain gradient theory, Li et al. [62] investigated bending, buckling and vibration behaviors through the length functionally graded Euler-Bernoulli size-dependent beams. They used Hamilton's principle to obtain the governing equation and associate boundary conditions. Then, generalized differential quadrature method was employed to solve these equations. Finally, they studied the effect of grading index and size-dependent parameter on the mechanical behaviors of beams. Li and Hu [63] derived torsional motion equation of nanotubes in the framework of nonlocal elasticity theory. Nanotubes made of functionally graded materials which the mechanical property was varying in the radius and length directions. Results show that torsional frequencies have reverse relation to nonlocal parameter.

Torsional vibration of nano-cone based on nonlocal strain gradient elasticity theory presented by Adeli et al. [20].

Functionally Graded Material (FGM) is one of the newest concepts in composite design. The properties of the FGM vary continuously from one point to another. Several articles dealing with different aspects of FGM have been published in recent years [12, 19, 22, 64-83]. It need to be cited that maximum of the above-noted analyses are associated with FGMs with material properties varying in a single direction only. However, there are practical occasions which require tailored grading of properties in two or even three directions. As reported by Steinberg

[84], Two-direction materials are needed in spacecraft design. Therefore, it is of great significance to develop novel FGMs with properties varying in two or three directions (2D or 3D FGMs) to withstand a more general temperature field.

In this article, using nonlocal strain gradient theory, buckling analysis of TDFGMs Euler-Bernoulli nano-beams is presented. The effects of changes of some important parameters are investigated.

2. Analysis

Consider a nano-beam of length L, width b, and thickness h made of three-directional functionally graded materials. Cartesian coordinates (x,y,z) are considered. The modulus of elasticity E are assumed to vary as arbitrary functions in axial, thickness and width directions, as indicated below [85]:

$$E(x,y,z)=X(x)Y(y)Z(z) \tag{1}$$

where $X(x)$, $Y(y)$ and $Z(z)$ are arbitrary functions.

Nonlocal elasticity theory, in contrast to classical elasticity, the stress tensor at an arbitrary point x in the domain of the material depends not only on the strain tensor at x but also on strain tensor at all other points in the domain. According to this theory, the structural stiffness of nanomaterials is smaller than that of the corresponding bulk material. Strain gradient theory in addition to the strain tensor, strain gradients are also considered in writing the strain energy density. Unlike to nonlocal elasticity theory, this theory predicts that the structural stiffness of nanomaterials is larger than that of the corresponding bulk material. Recently, by considering this problem, Lim et al. [11] demonstrate the nonlocal strain gradient theory that can predict both increase and decrease in the structural stiffness [85, 86].

In the nonlocal strain gradient theory, the total stress of Euler-Bernoulli beam are defined as

$$\sigma_{xx}^{t}=\sigma_{xx}-\frac{d\sigma_{xx}^{(1)}}{dx} \tag{2}$$

here σ_{xx} and $\sigma_{xx}^{(1)}$ are the classical and higher-order nonlocal stress tensors, respectively. The constitutive equation of the nonlocal strain gradient theory can be expressed as;

$$\left[1-\mu^{2}\nabla^{2}\right]\sigma_{xx}^{t}=E\left(1-l^{2}\nabla^{2}\right)\varepsilon_{xx} \tag{3}$$

where μ and l is the nonlocal and strain gradient material length scale parameter introduced to consider the significance of nonlocal strain gradient stress field.

Components of displacement vector (u_1, u_2, u_3) for Nano-beams based on Euler-Bernoulli beam theories can be expressed as

$$\begin{cases} u_1 = -z\left(\partial w/\partial x\right) \\ u_2 = 0 \\ u_3 = w\left(x,t\right) \end{cases} \tag{4}$$

Assuming the small deformations, the only nonzero strain of the Euler-Bernoulli beam theory is

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} \tag{5}$$

The governing equations of the FGM Euler-Bernoulli beam can be obtained, using the concept of minimum total potential energy principle. According to the principle of minimum total potential energy, the first variation in total potential energy must be zero.

$$\delta\Pi = \delta U + \delta V = 0 \tag{6}$$

in which δU and δV represent the variation of strain energy and that of virtual potential energy of axial load P, respectively. According to the nonlocal strain gradient theory developed by Lim et al. [48], variation of the strain energy density of an isotropic linear elastic material with volume Ω experiencing an infinitesimal displacement is defined as

$$\delta U = \int_\Omega \left(\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{xx}^{(1)}\nabla\delta\varepsilon_{xx}\right)dV = -\int_0^L M_{xx}\frac{\partial^2 \delta w}{\partial x^2}dx - M_{xx}^{(1)}\delta\frac{\partial^2 w}{\partial x^2}\bigg|_0^L$$

$$= -M_{xx}^{(1)}\delta\frac{\partial^2 w}{\partial x^2}\bigg|_0^L - M\delta\frac{\partial w}{\partial x}\bigg|_0^L + \frac{\partial M}{\partial x}\delta w\bigg|_0^L - \int_0^L \frac{\partial^2 M_{xx}}{\partial x^2}\delta w dx \tag{7}$$

where $\nabla\delta\varepsilon_{xx}$ is first-order strain gradient. M_{xx} and $M_{xx}^{(1)}$ is defined as follows:

$$M_{xx} = \int_A z\sigma_{xx}^t dA \tag{8}$$

$$M_{xx}^{(1)} = \int_A z\sigma_{xx}^{(1)} dA \tag{9}$$

The first variation of the work due to the axial compressive force is given by:

$$\delta V = -\int_0^L P\frac{dw}{dx}\frac{d\delta w}{dx}dx = \int_0^L P\frac{d^2 w}{dx^2}\delta w dx \tag{10}$$

By substituting Eqs. (7) and (10) into Eq. (6) and using integration by parts, the motion equation is expressed as

$$\frac{\partial^2 M}{\partial x^2} = P\frac{d^2 w}{dx^2} \tag{11}$$

The classical boundary conditions is as follows

$$\begin{cases} w = 0 \quad or \quad \dfrac{dM}{dx} = 0 \\ M = 0 \quad or \quad \dfrac{dw}{dx} = 0 \end{cases} \tag{12}$$

and the non-classical boundary conditions

$$\frac{d^2 w}{dx^2} = 0 \qquad \text{or} \qquad M^{(1)} = 0 \tag{13}$$

By combining constitutive equation of the nonlocal strain gradient theory (3) and the equilibrium equation (11), we obtain the size-dependent Navier equation of TDFG beams on the basis of the nonlocal strain gradient theory and the Euler–Bernoulli beam theory

$$P\frac{d^2w}{dx^2} - \mu^2 P\frac{d^4w}{dx^4} = -\frac{d^2X}{dx^2}I_2\frac{d^2w}{dx^2} - 2\frac{dX}{dx}I_2\frac{\partial^3 w}{\partial x^3} + \left(l^2\frac{d^2X}{dx^2} - X\right)I_2\frac{d^4w}{dx^4}$$
$$+2l^2 I_2\frac{dX}{dx}\frac{d^5w}{dx^5} + l^2 I_2 X\frac{d^6w}{dx^6} \tag{14}$$

For convenience, the following nondimensionalizations are used:

$$\bar{w} = \frac{w}{L}, \qquad \bar{x} = \frac{x}{L}, \qquad \bar{P} = \frac{PL^2}{I_2}, \qquad \bar{\mu} = \frac{\mu}{L}, \quad \bar{l} = \frac{l}{l} \tag{15}$$

The non-dimensional governing equation expression can be obtained as

$$\bar{P}\left(\frac{d^2\bar{w}}{d\bar{x}^2} - \bar{\mu}\frac{d^4\bar{w}}{d\bar{x}^4}\right) = -\frac{d^2X_1}{d\bar{x}^2}\frac{d^2\bar{W}}{d\bar{x}^2} - 2\frac{dX_1}{d\bar{x}}\frac{d^3\bar{W}}{d\bar{x}^3} + \left(\bar{l}^2\frac{d^2X_1}{d\bar{x}^2} - X_1\right)\frac{d^4\bar{W}}{d\bar{x}^4}$$
$$+2\bar{l}^2\frac{dX_1}{d\bar{x}}\frac{d^5\bar{W}}{d\bar{x}^5} + \bar{l}^2 X_1\frac{d^6\bar{W}}{d\bar{x}^6} \tag{16}$$

Solving the obtained governing equation gives the critical buckling load ($\bar{P}$) of the TDFGM Euler-Bernoulli nano-beams based on nonlocal strain gradient elasticity theory using generalized differential quadrature method (GDQM).

3. Results and discussion

It is proposed that the modulus of elasticity and density of the nano-beam vary in the x , y and z directions, as follows [87]

$$E(x,y,z) = e^{\frac{n_1}{L}x}\left[\left(\frac{2y+b}{2b}\right)^{n_2} + k\left(1 - \left(\frac{2y+b}{2b}\right)^{n_2}\right)\right]\left[E_c\left(\frac{2z+h}{2h}\right)^{n_3} + E_m\left(1 - \left(\frac{2z+h}{2h}\right)^{n_3}\right)\right] \tag{17}$$

$$\rho(x,y,z)=e^{\frac{n_4}{L}x}\left[\left(\frac{2y+b}{2b}\right)^{n_5}+k\left(1-\left(\frac{2y+b}{2b}\right)^{n_5}\right)\right]\left[E_c\left(\frac{2z+h}{2h}\right)^{n_6}+E_m\left(1-\left(\frac{2z+h}{2h}\right)^{n_6}\right)\right] \tag{18}$$

where, k, n_1, n_2, n_3, n_4, n_5 and n_6 are constant material parameters.

To verify the results, the results of this paper are compared with the results of previous work. To compare the results, strain gradient material length scale parameter and constant material parameters are considered zero. A comparison of the results is given in Table 1. Comparison of the results showed that the results of this study have acceptable accuracy.

Table 1. Comparison of the results of this work with other works.

$\bar{\mu}$	Ref.	
0	Present work	39.4783
	(Ghannadpour et al., 2013)	39.4784
	(Wang et al., 2006)	39.4786
	(Pradhan & Phadikar, 2009)	39.4784
	(Nejad et al., 2016)	39.4784
0.2	Present work	15.3068
	(Ghannadpour et al., 2013)	15.3068
	(Wang et al., 2006)	15.3068
	(Nejad et al., 2016)	15.3068
1	Present work	0.9753
	(Ghannadpour et al., 2013)	0.9753
	(Pradhan & Phadikar, 2009)	0.9753
	(Nejad et al., 2016)	0.9753

In Figure 1, the convergence of the buckling load calculated by the GDQ method is investigated. According to this figure, it can be concluded that with increasing nodes, the response converges. Considering 15 nodes ensures convergence.

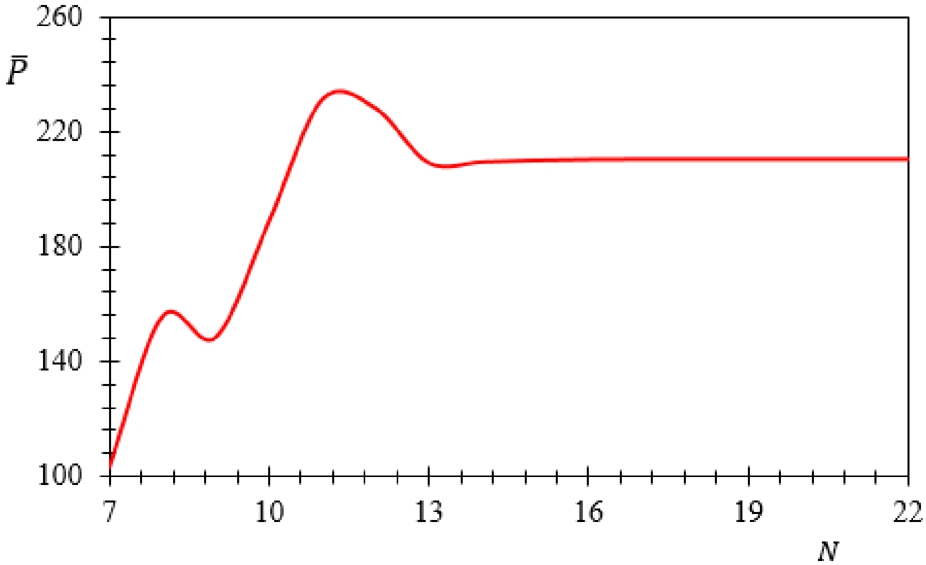

Fig. 1. The convergence of the buckling load calculated by the GDQ method for $\bar{\mu}=2, \bar{l}=2, n_1=2$

Figure 2 shows the changes in dimensionless buckling load in terms of nonlocal length scale parameter for different strain gradient length scale parameter. The results show that with increasing μ, the buckling load decreases, which shows that in the theory of nonlocal elasticity, the buckling load is lower than in the classical theory. In other words, the theory of non-local elasticity expresses the softening for a nanoscale material. In addition, with increasing strain gradient, the buckling load increases. This shows that by increasing the strain gradient length scale parameter, the material is more stable in nanoscale than in macro.

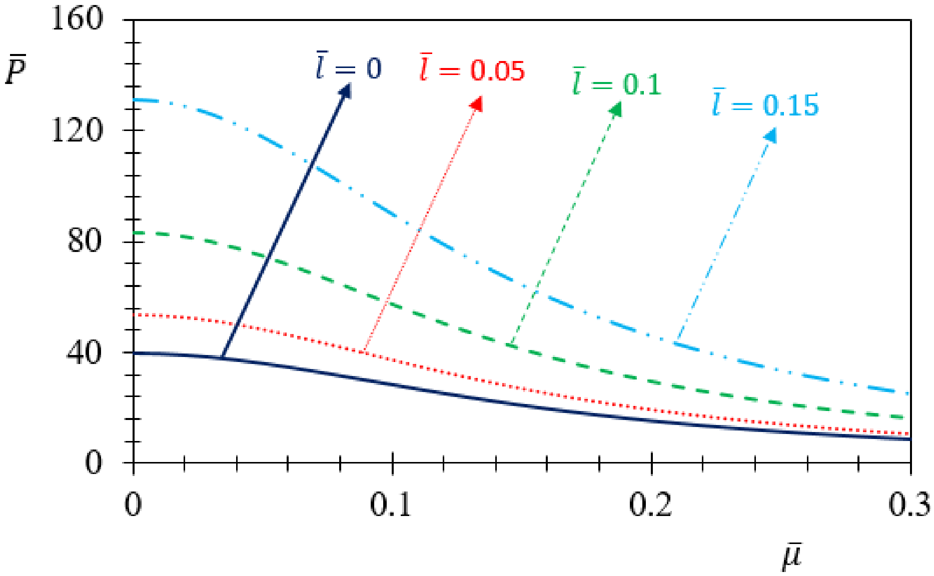

Fig. 2. changes in dimensionless buckling load in terms of nonlocal length scale parameter for different strain gradient length scale parameter and $n_1 = 0$

Dimensionless buckling load changes in terms of n_1 material parameter are shown in Figure 3. This figure shows that the buckling load increases with increasing n_1 . The reason for the increase in buckling load is the increase in modulus of elasticity with increasing n_1 .

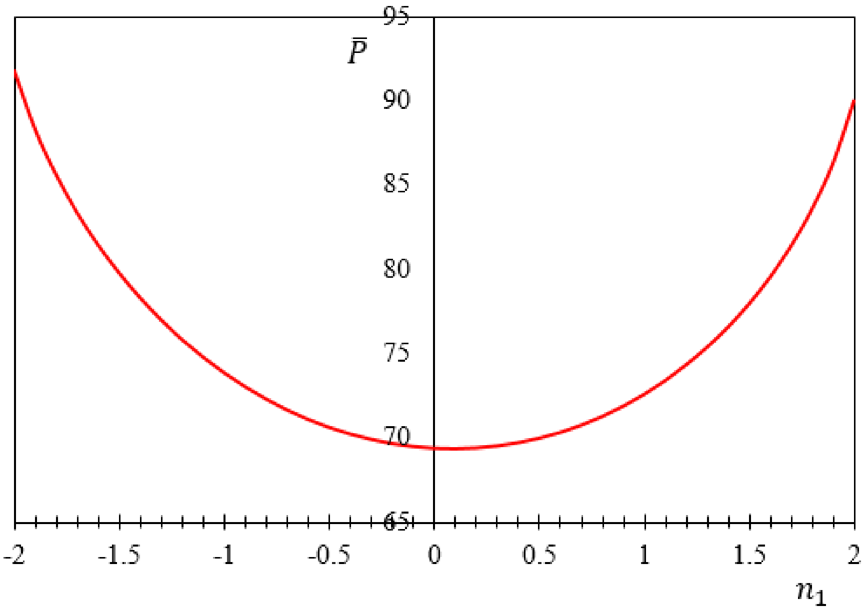

Fig. 3. Dimensionless buckling load changes in terms of n_1 **material parameter for** $\bar{\mu} = 0.2, \bar{l} = 0.2$

The buckling load changes in terms of n_2 material parameter for different bouckling modes are shown in Figure 4. This figure shows that the buckling load for all modes decreases with increasing n_2 size. The reason for the decrease in buckling load is the decrease in modulus of elasticity with increasing n_2 .

Dimensional buckling load changes according to the material parameter n_3 for 4 buckling modes are shown in Figure 5. This figure shows that with increasing n_3 size, the buckling load increases for 4 buckling modes. The reason for increasing the buckling load is the increase in modulus of elasticity with increasing n_3 .

Figure 6 shows the buckling in terms of beam length for 3 states $(\bar{u} = 0\ \bar{\iota} = 0)$, $(\bar{u} = 1\bar{\iota} = 2)$ and $(\bar{u} = 2\ \bar{\iota} = 1)$. The results show that for the case that $(\bar{u} > \bar{\iota})$ softening is predicted for the material but for the case that $(\bar{u} < \bar{\iota})$ the hardening is predicted.

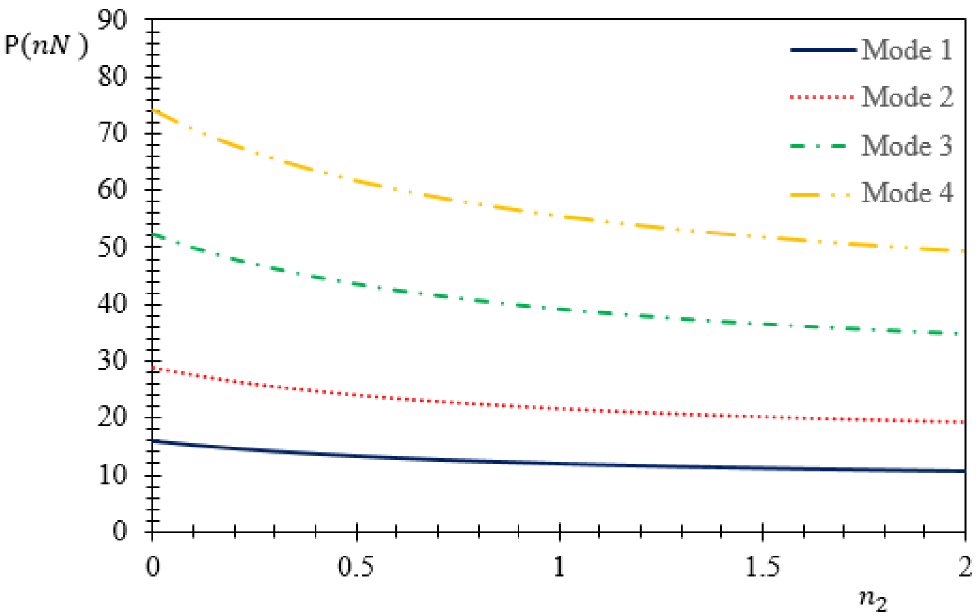

Fig. 4. The buckling load changes in terms of n_2 material parameter for different bouckling modes for $\bar{\mu} = 0.2, \bar{l} = 0.2, n_1 = 0, n_3 = 0$

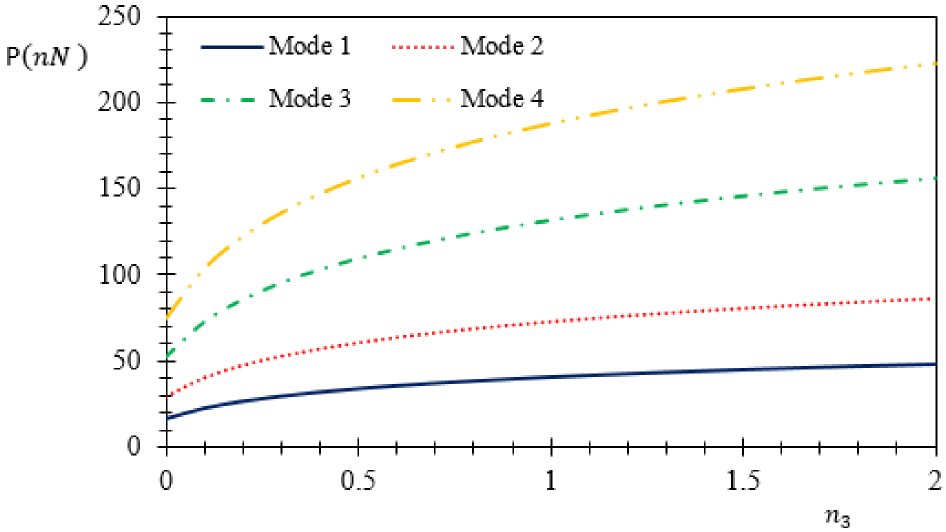

Fig 5. The buckling load changes in terms of n_3 material parameter for different buckling modes for

$$\bar{\mu}=0.2, \bar{l}=0.2, n_1=0, n_2=0$$

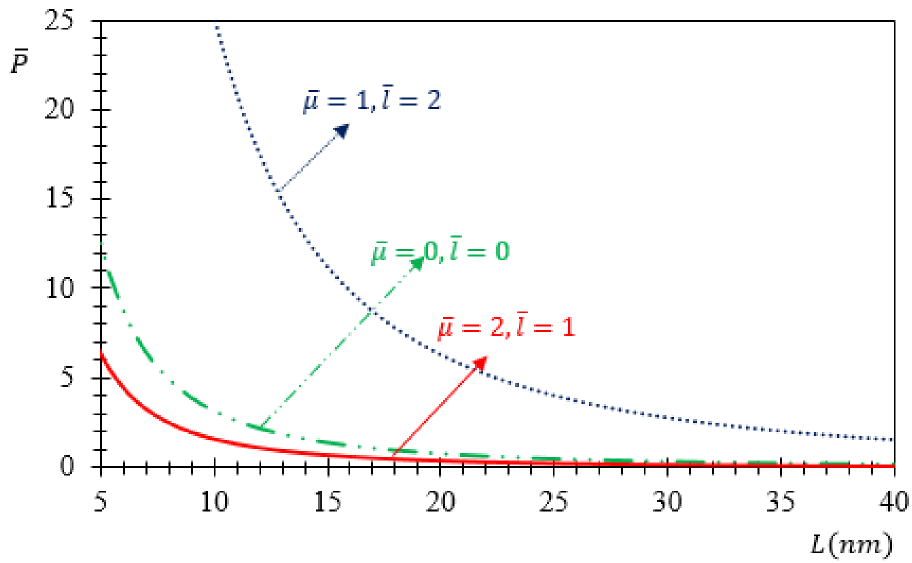

Fig. 6. The buckling load in terms of beam length for 3 states $(\bar{u}=0\ \bar{\imath}=0)$, $(\bar{u}=1\bar{\imath}=2)$ **and** $(\bar{u}=2\ \bar{\imath}=1)$

4. Conclusion

In this paper, the effect of size for Bernoulli Euler nano beam made of three-dimensional functionally graded material is investigated using a combination of strain gradient theory and non-local elasticity. The results of this study showed that the strain gradient theory predicts hardening of the material at the nanoscale but the theory of non-local elasticity predicts the softening at the nanoscale. The combination of these two theories is able to predict both the hardening and softening effects at the nanoscale. In addition, the three-dimensional functionally graded material in this paper has been investigated that the types of material changes in all directions of the beam have been considered with an arbitrary mathematical model. This model can be extended to a variety of nanostructures with different molecular structures.

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