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RESEARCH PAPER

A Ritz Formulation for Vibration Analysis of Axially Functionally Graded Timoshenko-Ehrenfest Beams

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Abstract

Dynamics of beams made of axially grading material has been analyzed in present work. Shear deformation and rotational inertia of the rectangular cross-sectional beam have been considered using Timoshenko-Ehrenfest beam model. Material properties of the beam have been assumed as a power-law function. Solution of the vibration problem of the axially functionally graded Timoshenko-Ehrenfest beam has been carried out with Ritz formulation. Present model has been validated with the previous literature works. Effects of power-law index parameter and grading material properties on the dynamics of axially functionally graded Timoshenko-Ehrenfest beam have been investigated. Transverse deflection and slope of the beam have been depicted in various cases. Present study can give useful results for designing of axially graded structural elements.

Keywords: Timoshenko-Ehrenfest beam, Axially graded material, Ritz Method, Weak form.

1. Introduction

Composite structures have made a great impact on material structure design, especially in aviation and space industry applications, at the end of 20th century. The trend is continuing with nano and functionally grading applications of composite materials. Axially grading materials are one of the popular functional materials which have variable material properties in the length direction of structure. Structural elements in a building, which are affected by various loads at different cross-sections, are frequently encountered problem for the engineers. Beam elements which are made of axially grading material can be appropriate solution for this kind of structural engineering problems.

Beam elements in structural engineering can be modeled with various continuum mechanics theories. Mostly used one is the unimodal Euler-Bernoulli Theory [1] which can give consistent results for smaller values of aspect (height/length) ratio. When the aspect ratio increases, rotation of the mid-axis of the beam have important shear deformation and rotational inertia effects on mechanics of the beam. Firstly, Bresse [2] interested with this problem considering the effects mentioned above. Then, Timoshenko [3,4] proposed the shear deformation and rotational inertia effects on beams in his so-called multimodal “Timoshenko beam theory”. Recently, Elishakoff [5–7] has proved that Ehrenfest had contributed to the Timoshenko beam theory and this theory must be named as “Timoshenko-Ehrenfest (T-E) beam theory”. In addition to the transverse displacement of beam, rotation of the mid-axis is defined as an independent displacement function in T-E beam theory which can give more accurate results than the Euler-Bernoulli model.

Vibration of Timoshenko-Ehrenfest beams had been considered firstly by Thomas and Abbas [8] with finite element modeling of the structure. Sarma and Varadan [9] used the Ritz Method for the nonlinear vibration analysis of T-E beams. Zhou and Cheung [10] investigated the vibration of tapered T-E beams with using static loading displacement function as an approximate functions in dynamic analysis. Ruta [11] used the Chebysev polynomials for

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the vibration analysis of nonprismatic T-E beam which rests in Pasternak foundation. Park and Hong [12] studied the energy flow model in T-E beams for the flexural waves.

Li [13] firstly modeled the beams which are functionally graded through the thickness, using Euler-Bernoulli, Rayleigh and Timoshenko-Ehrenfest beam theories. Free vibration and transverse wave propagation formulations were obtained analytically. Pradhan and Chakraverty [14] used the Ritz Method to investigate the vibration of transversely graded T-E beams. Gul, Aydogdu and Karacam [15] studied with the wave propagation and vibration of transversely graded T-E beams considering new frequency spectrums.

Nonlinear vibration of a T-E beam under the effect of moving load investigated in [16]. Attarnejad *et al*. [17] used basic displacement functions in vibration analysis of tapered T-E beams. Quintana and Grossi [18] studied the vibration of T-E beams with intermediate elastic supports by using Ritz Method. Ashgari *et al*. [19] used the modified couple stress theory for the transversely graded T-E beams. Yang and He [20] studied the vibration and buckling of size dependent axially graded T-E beams with modified couple stress theory.

Shahba *et al*. [21] investigated the axially graded tapered T-E beams with finite element modeling. Rajasekaran [22] studied the dynamics of centrifugally stiffened axially graded tapered T-E beams with differential transformation and differential quadrature element methods. Free vibration of axially graded T-E beams with non-uniform cross-section was modeled by Huang *et al*. [23]. They used auxiliary functions in solution for the deflection and rotation. Sarkar and Ganguli [24] obtained the closed-form solution for vibration of uniform cross-sectional simply-supported axially graded T-E beams. Tang et al. [25] presented closed form frequency equations of axially graded T-E beams in various boundary conditions. Free vibration of T-E beams in thermal environment [26], in stepped structures [27], embedded in two-parameter elastic foundation [28] and transient analysis with variable cross-sectional area [29] had been considered by researchers.

Dynamic analysis of bidirectional functionally graded T-E beams have been carried out by Hao and Wei [30]. Nguyen *et al*. [31] studied the forced vibration of bidirectional functionally graded T-E beams. Ghayesh [32–34] investigated the nonlinear forced vibration and of axially graded tapered T-E beams in his several papers. Cao and Gao [35] proposed the asymptotic development method in the dynamic analysis of axially graded T-E beams. Elishakoff and coworkers [36–39] proposed novel modified T-E beam models for various case studies including axially grading material structures. Nano-sized structures with functionally graded material assumption have been investigated by researchers [40–44] and special loading cases in specific nano structures have been studied in several works [45–52]. Also, dynamic analysis of axially functionally graded nano structures have been carried out in recent studies [53,54].

Present study considers the dynamic analysis of axially graded uniform cross-sectional T-E beams. Variation of material properties inside the beam are considered in power-law form. Energy formulation has been used in modeling of T-E beam. Approximate Ritz method has been used in solution of weak energy formulation. In addition to the literature, several boundary condition cases have been modeled and solved with Ritz formulation in the present work. Effects of power-law parameter and the grading material properties to the axially graded T-E beam vibration frequency have been investigated. Mode shapes for transverse deflection and rotation have been depicted.

2. Analysis

A rectangular cross-sectional beam is considered (Fig. 1). L, b and h are the length, width and height of the beam, respectively. x axis is the length direction and z axis is the transverse direction of the beam.

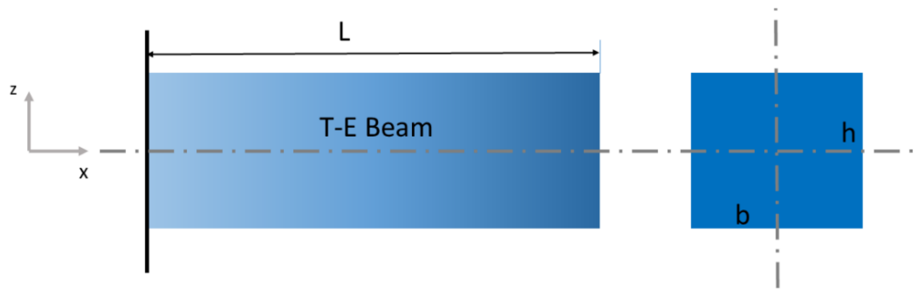

Figure 1. Continuum model for axially functionally graded timoshenko-ehrenfest beam.

According to Timoshenko-Ehrenfest beam theory, $w(x,t)$ is assumed as the transverse displacement and $\Phi(x,t)$ is the angle of rotation of the normal to the mid-surface of the beam, x is the position and t is the time. Strain-displacement relations can be interpreted as:

$$\varepsilon_{xx} = z\frac{\partial \Phi}{\partial x} \tag{1a}$$

$$\gamma_{xz} = \Phi + \frac{\partial w}{\partial x} \tag{1b}$$

where z is the coordinate measured from the mid-plane of the beam, ε_{xx} the normal strain, γ_{xz} the transverse shear strain. The strain energy for the beam can be written as:

$$U = \frac{1}{2}\int_0^L \int_A (\sigma_{xx}\varepsilon_{xx} + \sigma_{xz}\gamma_{xz})dA\,dx \tag{2}$$

where A is the cross-sectional area of the beam, σ_{xx} and σ_{xz} are the normal and transverse shear stresses which are defined as:

$$\sigma_{xx} = E(x)\varepsilon_{xx} = Ez\frac{\partial \Phi}{\partial x} \tag{3a}$$

$$\sigma_{xz} = G(x)\gamma_{xz} = G(x)\left(\Phi + \frac{\partial w}{\partial x}\right) \tag{3b}$$

where $E(x)$ is the Young Modulus and $G(x)$ is the Shear modulus of axially functionally graded beam. Relation between them can be expressed as:

$$G(x) = \frac{E(x)}{2(1+\nu)} \tag{4}$$

where ν is the Poison's ratio of the beam. By substituting equations (1a) and (1b) into equation (2), strain energy turns into:

$$U = \frac{1}{2}\int_0^L \left(M\left(\frac{\partial \Phi}{\partial x}\right)^2 + Q\left(\Phi + \frac{\partial w}{\partial x}\right)^2\right)dx \tag{5}$$

where M and Q are the bending moment and shear force, respectively.

$$M = \int_A z\sigma_{xx}\,dA = E(x)I\frac{\partial \Phi}{\partial x} \tag{6b}$$

$$Q = \int_A \sigma_{xz}\,dA = \kappa_s G(x)A\left(\Phi + \frac{\partial w}{\partial x}\right) \tag{6b}$$

where I is the second moment of area of beam and κ_s is the shear correction factor. The kinetic energy of the beam:

$$T = \frac{1}{2}\int_0^L \left(\rho(x)A\left(\frac{\partial w}{\partial t}\right)^2 + \rho(x)I\left(\frac{\partial \Phi}{\partial t}\right)^2\right)dx \tag{7}$$

where $\rho(x)$ is the mass density of the axially graded beam material. The Langrangian functional (F) of the vibration problem for the T-E beam can be written as:

$$F = \frac{1}{2}\int_0^L \left(E(x)I\left(\frac{\partial \Phi}{\partial x}\right)^2 + \kappa_s G(x)A\left(\Phi + \frac{\partial w}{\partial x}\right)^2\right)dx - \frac{1}{2}\int_0^L \left(\rho(x)A\left(\frac{\partial w}{\partial t}\right)^2 + \rho(x)I\left(\frac{\partial \Phi}{\partial t}\right)^2\right)dx \tag{8}$$

Transverse displacement and rotation of the beam functions may be defined as below with harmonic vibration assumption:

$$w(x,t) = W(x)\sin \omega t \tag{9a}$$

$$\Phi(x,t) = \theta(x)\sin \omega t \tag{9b}$$

Dimensionless form of the Eq. (8):

$$F = \frac{1}{2}\int_0^1 \left(\left(\frac{\partial \Phi}{\partial \bar{x}}\right)^2 + \alpha\left(\Phi + \frac{\partial \overline{W}}{\partial \bar{x}}\right)^2\right)d\bar{x} - \frac{1}{2}\int_0^L \left(\Omega^2(\overline{W})^2 + \frac{\Omega^2}{\beta}(\Phi)^2\right)d\bar{x} \tag{10}$$

where α is the shear deformation parameter, β is the slenderness ratio and Ω is the non-dimensional vibration frequency of the beam. $\bar{x}$ and $\overline{W}$ are the dimensionless beam length and transverse displacement, respectively. Mentioned parameters are defined below:

$$\alpha = \kappa_s \frac{G(x)}{E(x)}\beta \quad , \quad \beta = \frac{AL^2}{I} \quad , \quad \Omega^2 = \frac{\rho(x)A\omega^2 L^4}{E(x)I} \quad , \quad \bar{x} = \frac{x}{L} \quad , \quad \bar{W} = \frac{W}{L} \tag{11}$$

2.1. *Axially Graded Material*

The axially functionally graded (aFG) material considered as mixing of two materials in the present work. Variation of the material properties (elasticity modulus, shear modulus and density) are assumed in the following forms:

$$\begin{bmatrix} E(x) \\ G(x) \\ \rho(x) \end{bmatrix} = \begin{bmatrix} E_1 - E_0 \\ G_1 - G_0 \\ \rho_1 - \rho_0 \end{bmatrix} x^k + \begin{bmatrix} E_0 \\ G_0 \\ \rho_0 \end{bmatrix} \tag{12}$$

where k is the power-law index and E_0, G_0, ρ_0 and E_1, G_1, ρ_1 are the material properties at the left and right end of the aFG beam, respectively. s is the ratio of material property variations and is defined below. Material property variations can be seen in Fig. (2):

$$\frac{E_1}{E_0} = \frac{G_1}{G_0} = \frac{\rho_1}{\rho_0} = s \tag{13}$$

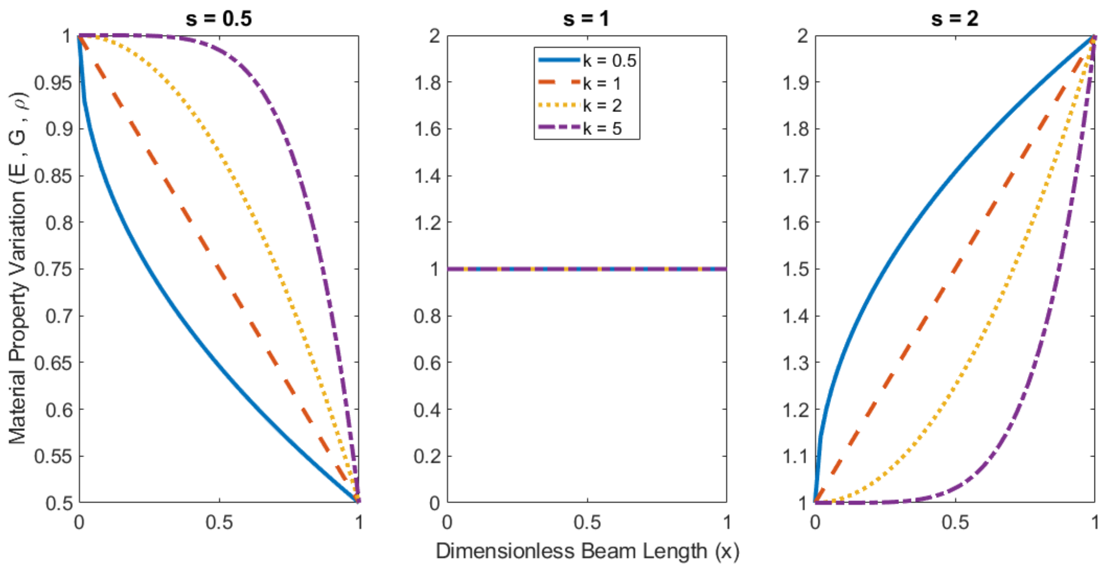

Figure 2. Variation of material properties in afg T-E beam.

2.2. *Ritz Method*

Analytical solution of the vibration problem of T-E beam is complicated for several boundary conditions [55]. In the present study, this complexity increases with axially grading material assumption. Therefore, an approximate variational method, Ritz method has been used in the solution of the present problem [53,54,56].

In the Ritz method, deflection and rotation functions can be defined in the following form [57]:

$$\bar{W}(\bar{x}) = \sum_{m=m_0}^{J} C_m \xi_m(\bar{x}) \tag{14a}$$

$$\Phi(\bar{x}) = \sum_{n=n_0}^{J} D_n \varphi_n(\bar{x}) \tag{14b}$$

where C_j and D_j are the unknown coefficients, $\xi_m(\bar{x})$ and $\varphi_n(\bar{x})$ are functions which fulfills at least geometric boundary conditions of the beam, respectively. Convergence of this function is satisfied if functions are mathematically complete set. To determine the vibration frequencies of beams following functional can be defined:

$$F = U_{max} - T_{max} \tag{15}$$

The functional should be minimized with respect to unknown coefficients given in Eq. (14):

$$\frac{\partial F}{\partial C_{\overline{m}}} = \int_0^1 \left(\alpha \frac{\partial \xi_m(\bar{x})}{\partial \bar{x}} \left(D_n \varphi_n(\bar{x}) + C_m \frac{\partial \xi_m(\bar{x})}{\partial \bar{x}}\right) - \Omega^2 \varphi_n(\bar{x})(D_n \varphi_n(\bar{x}))\right) d\bar{x} = 0 \quad , \quad \overline{m} = \overline{m}_0, \dots, J \tag{16a}$$

$$\frac{\partial F}{\partial D_{\overline{n}}} = \int_0^1 \left(\frac{\partial \varphi_n(\bar{x})}{\partial \bar{x}} \left(D_n \frac{\partial \varphi_n(\bar{x})}{\partial \bar{x}}\right) + \alpha \varphi_n(\bar{x}) \left(D_n \varphi_n(\bar{x}) + C_m \frac{\partial \xi_m(\bar{x})}{\partial \bar{x}}\right) - \frac{\Omega^2}{\beta} \varphi_n(\bar{x})(D_n \varphi_n(\bar{x}))\right) d\bar{x} = 0 \quad , \quad \overline{n} = \overline{n}_0, \dots, J \tag{16b}$$

These equations should be solved simultaneously in homogeneous system of linear equations which size is equal to sum of number of unknowns (C_m, D_n). Those equations can be described as an eigen-value problem:

$$\left(\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} - \Omega^2 \begin{bmatrix} P_{13} & 0 \\ 0 & P_{23} \end{bmatrix}\right) \begin{bmatrix} C_m \\ D_n \end{bmatrix} = 0 \tag{17}$$

where P_{11}, P_{12}, P_{21} and P_{22} are the stiffness matrix elements and P_{13} and P_{23} are the mass matrix elements. The mode shapes corresponding to any Ω is found by substituting that value into Eq. (15) and solving for the eigenvector components. Inserting these components into Eq. (15) gives mode shape of axially graded T-E beam.

$\xi_m(\bar{x})$ and $\varphi_n(\bar{x})$ functions should be assumed as in following form:

$$\xi_m(\bar{x}) = (\bar{x} - 0)^{c_1} (\bar{x} - 1)^{c_2} (\bar{x}^{m-1}) \tag{18a}$$

$$\beta_j(\bar{x}) = (\bar{x} - 0)^{d_1} (\bar{x} - 1)^{d_2} (\bar{x}^{m-1}) \tag{18b}$$

where c_1 and c_2 parameters define the transverse displacement function boundary conditions and d_1 and d_2 parameters define the rotation function boundary conditions. c_1 and c_2 parameters should be selected as 0, 1 and 1 for the free, simply supported and clamped boundary conditions, respectively. d_1 and d_2 parameters should be selected as 0, 0 and 1 for the same boundary conditions.

Assumptions about $\xi_m(\bar{x})$ and $\varphi_n(\bar{x})$ polynomials can be seen below in general form:

Table 1: Transverse Deflection and Slope Functions for Various Boundary Conditions.

Boundary Condition	$\xi_m(\bar{x})$	$\varphi_n(\bar{x})$
S – S	$(\bar{x}-0)^1 (\bar{x}-1)^1(\bar{x}^{m-1})$	$(\bar{x}-0)^0 (\bar{x}-1)^0(\bar{x}^{n-1})$
C – S	$(\bar{x}-0)^1 (\bar{x}-1)^1(\bar{x}^{m-1})$	$(\bar{x}-0)^1 (\bar{x}-1)^0(\bar{x}^{n-1})$
C – F	$(\bar{x}-0)^1 (\bar{x}-1)^0(\bar{x}^{m-1})$	$(\bar{x}-0)^1 (\bar{x}-1)^0(\bar{x}^{n-1})$
C – C	$(\bar{x}-0)^1 (\bar{x}-1)^1(\bar{x}^{m-1})$	$(\bar{x}-0)^1 (\bar{x}-1)^1(\bar{x}^{n-1})$
S – F	$(\bar{x}-0)^1 (\bar{x}-1)^0(\bar{x}^{m-1})$	$(\bar{x}-0)^0 (\bar{x}-1)^0(\bar{x}^{n-1})$

3. Numerical Results and Discussion

In this section, transverse vibration analysis of the axially graded T-E beam has been carried out for various power-law index parameters and end side material properties. Dimensionless material properties are assumed in the analysis except the Poisson ratio (ν=0.3) of beam. The shear correction factor for the rectangular cross-section is calculated according to equation in below [55]:

$$\kappa_s = \frac{10(1+\nu)}{12+11\nu} \tag{19}$$

Convergence of the Ritz method is satisfied with the comparison with Tang et al. [25]'s study for the first five frequencies of the homogenous T-E beam in Table 1. While increasing the J to 10, present model converges to literature results. Especially first three mode frequencies are found almost identical for most of boundary conditions. Highest percentage error is calculated as %0.295 for 5th mode frequency at C-C boundary condition. Therefore, all results presented in this study have been obtained using J=10.

Table 2: Comparison of the First Five Dimensionless Frequencies of T-E beam.

Boundary Condition		J	Mode Number 1	2	3	4	5
S – S	Present	8	9.1631	31.0854	58.2548	88.5601	120.2256
		9	9.1631	31.0851	58.2544	87.1330	119.2301
		10	9.1631	31.0850	58.2470	87.1272	116.4933
	Tang et al. [25]		9.1607	31.0643	58.1930	86.9867	116.1501
C – S	Present	8	11.0825	27.1144	44.8485	59.2041	63.3665
		9	11.0825	27.1144	44.8436	59.2036	63.3664
		10	11.0825	27.1144	44.8436	59.2030	63.3398
	Tang et al. [25]		11.0825	27.1144	44.8435	59.2030	63.3395
C – F	Present	8	3.2271	14.4689	31.5028	47.9189	62.5408
		9	3.2271	14.4689	31.5025	47.9112	62.3602
		10	3.2271	14.4689	31.5025	47.9091	62.3511
	Tang et al. [25]		3.2271	14.4689	31.5025	47.9090	62.3470
C – C	Present	8	13.8348	28.5179	45.6673	61.8775	68.3033
		9	13.8348	28.5179	45.6660	61.8675	68.2919
		10	13.8348	28.5179	45.6660	61.8622	68.2839
	Tang et al. [25]		13.8347	28.5179	45.6659	61.8620	68.2836

Effect of aspect ratio to the dimensionless frequency of aFG T-E beam is shown in Table 3. Length scale affects the rigidity of the aFG T-E beam and increasing h/L ratio decreases the frequency. Second or right end side material properties also enhances the rigidity of the beam and decreases the frequency on stiffening case (*s*=2).

Table 3: Dimensionless Fundamental Frequencies for aFG T-E beams (*k* =2).

		s = 0.5			s = 2		
h / L		0.01	0.1	0.5	0.01	0.1	0.5
Boundary Condition	S – S	9.8262	9.4614	5.7750	9.8080	9.4777	5.9727
	C – S	15.7130	14.4006	6.5763	14.8415	13.6042	6.2609
	C – F	4.2502	4.1693	3.0284	2.7941	2.7535	2.1210
	C – C	21.4955	18.6069	7.0160	22.9564	19.8744	7.5099
	S – F	16.5541	15.7704	9.9809	14.4881	13.8413	7.8387

Variation of dimensionless frequencies with power-law index and right end side material properties can be seen in Figs. (3-7) for various boundary conditions. (*s*=1) curve defines the homogenous material property behavior which can be interpreted from Fig. (2). Clamped and simply supported boundary conditions restrict the dynamic behavior of aFG T-E beam. Therefore, frequency decreases in (*s*=0.5) case and increases (*s*=2) case with the effect of average rigidity of aFG T-E beam. Free end condition has not restrictive characteristics and in fact, polynomials in Table-1 can not satisfy the free end condition exactly which can be interpreted from Table-1. Thus, frequency increases in (s=0.5) case and decreases in (s=2) case with help of free end condition.

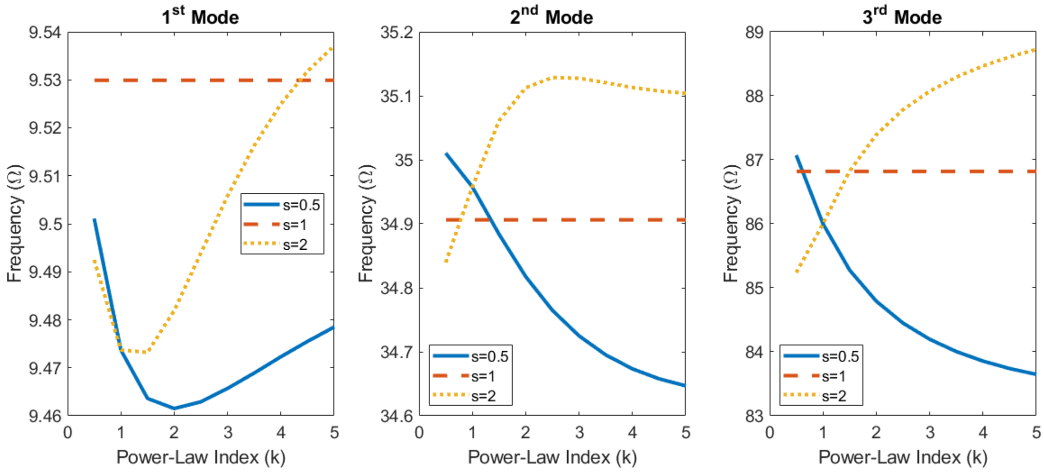

Figure 3. Variation of *Ω* with Power-Law Index (*k*) on S-S Boundary Condition.

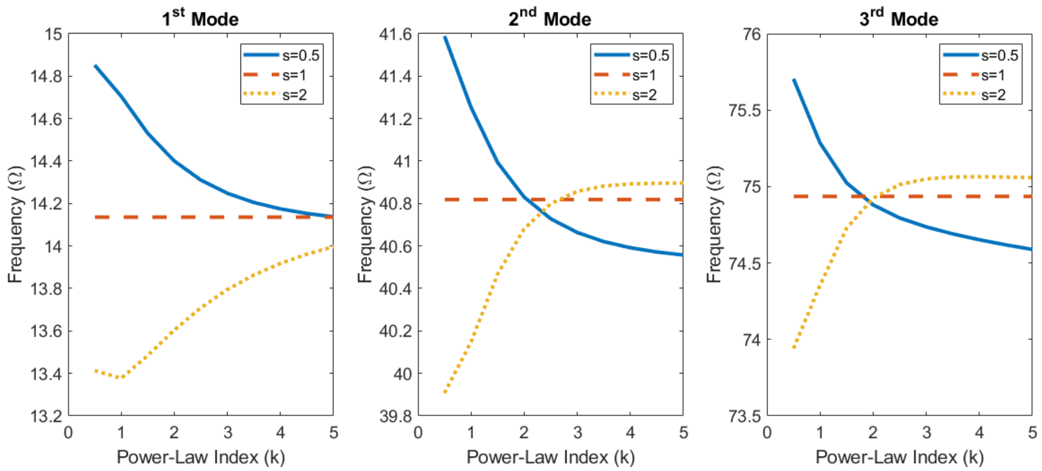

Figure 4. Variation of *Ω* with Power-Law Index (*k*) on C-S Boundary Condition.

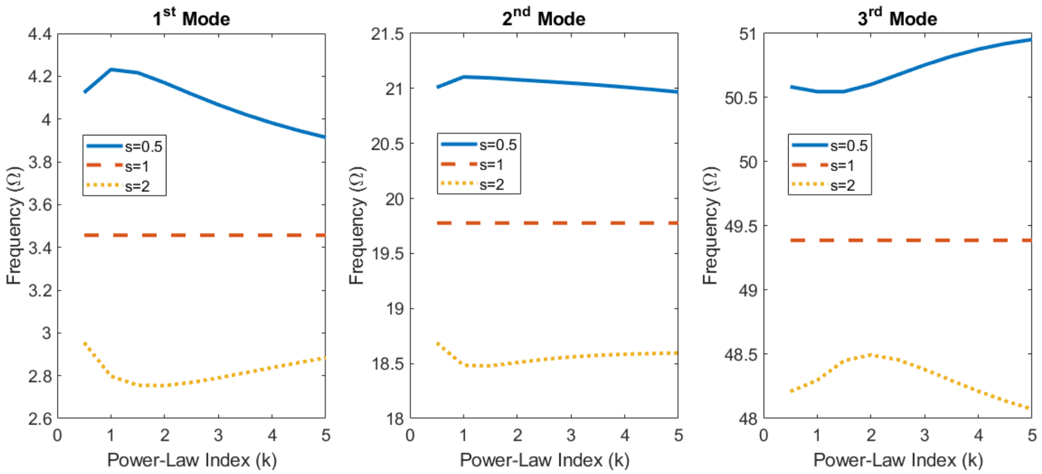

Figure 5. Variation of *Ω* with Power-Law Index (*k*) on C-F Boundary Condition.

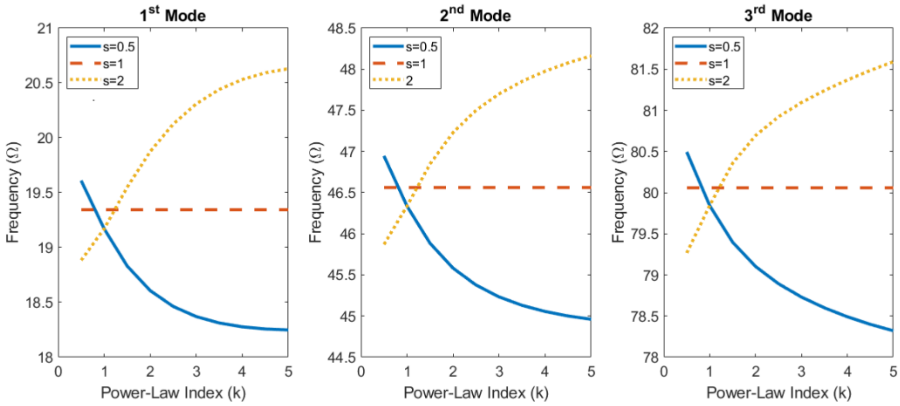

Figure 6. Variation of *Ω* with Power-Law Index (*k*) on C-C Boundary Condition.

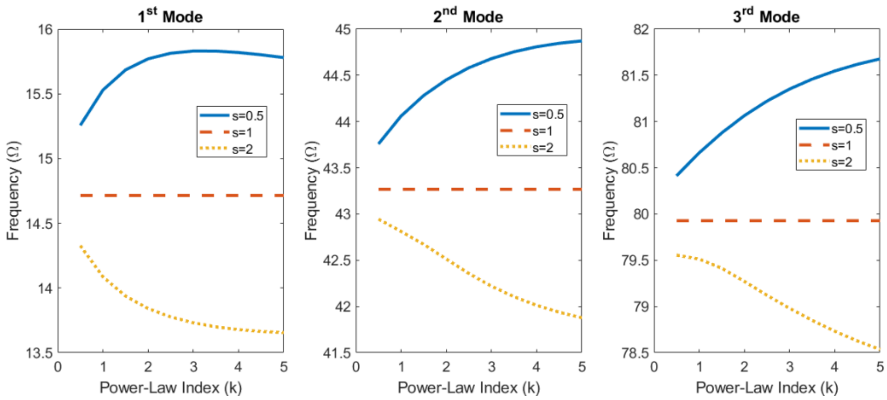

Figure 7. Variation of *Ω* with Power-Law Index (*k*) on S-F Boundary Condition.

First three transverse deflection and rotation mode shapes of aFG T-E beam are depicted in Figs. (8-12). As in the frequency variation, (s=1) curves define the mode shape of homogenous T-E beam. aFG T-E beam mode shapes are obtained for (k=3) power-law index parameters. Mode shapes almost same in clamped and simply supported boundary cases but maximum deflection at nodal points changes position with the effect of right end side material properties. This situation can be seen clearly on 3rd mode shapes. Effect on amplitudes can be seen obviously in free end condition cases. Related with the frequency behavior, increasing second material properties increase the amplitude of aFG T-E beam.

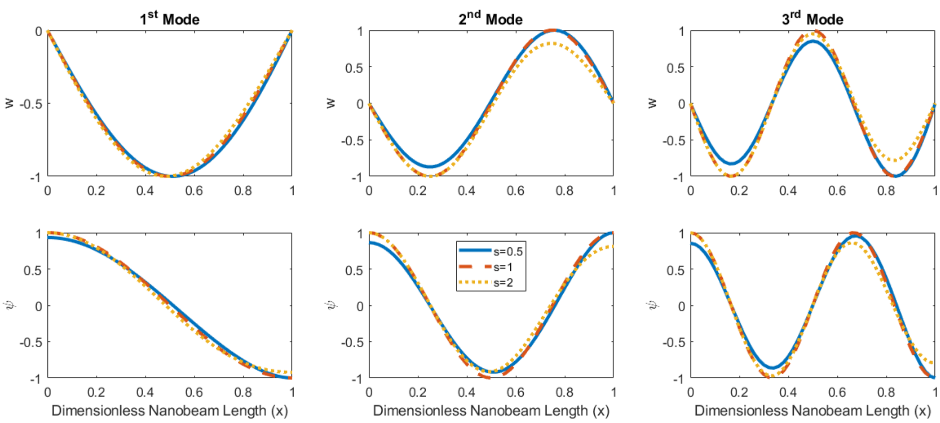

Figure 8. Right End Material Property Effects on Transverse Deflection and Slope of T-E Beam on S-S Case.

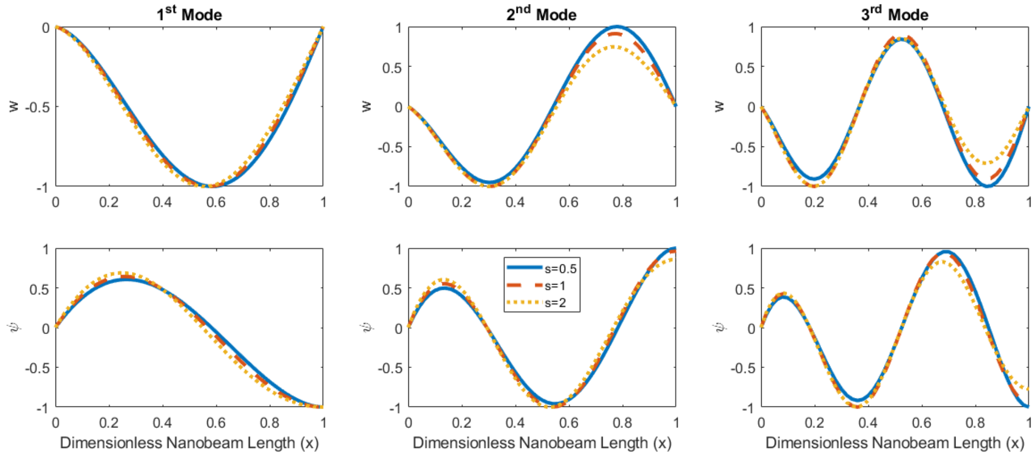

Figure 9. Right End Material Property Effects on Transverse Deflection and Slope of T-E Beam on C-S Case.

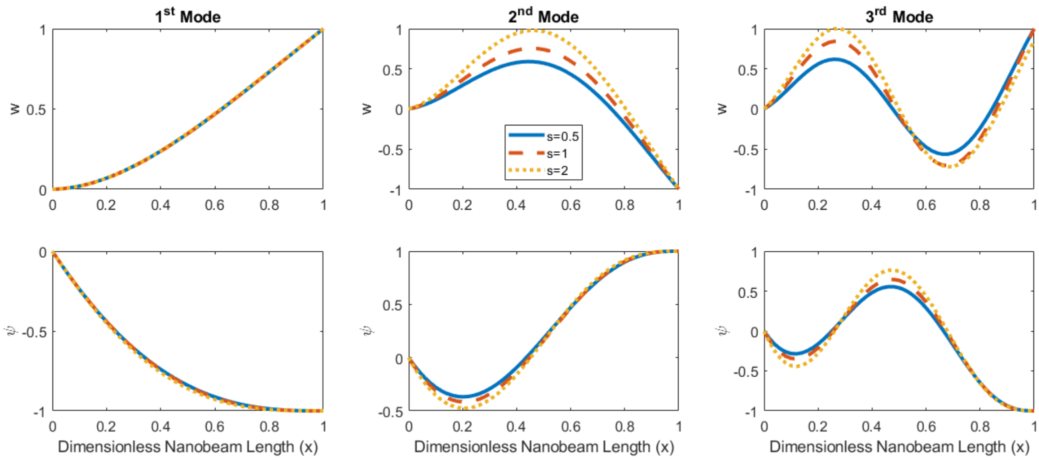

Figure 10. Right End Material Property Effects on Transverse Deflection and Slope of T-E Beam on C-F Case.

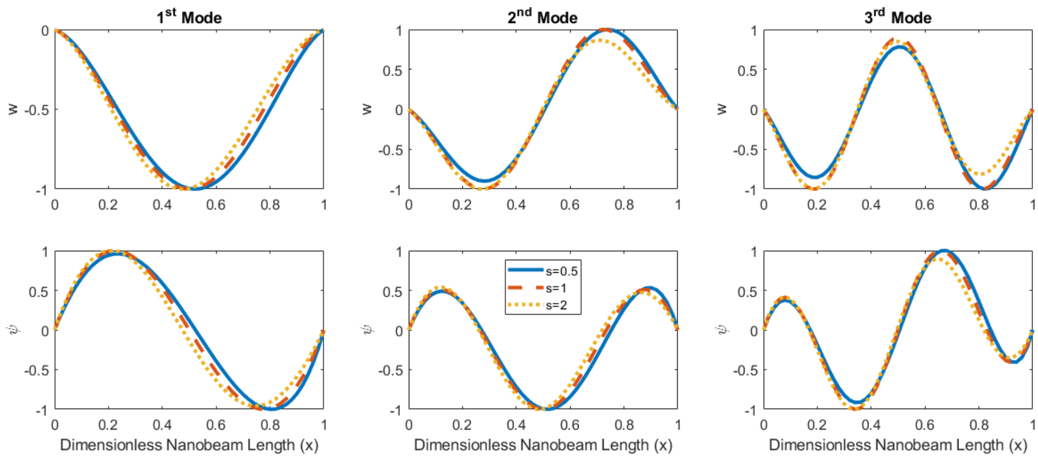

Figure 11. Right End Material Property Effects on Transverse Deflection and Slope of T-E Beam on C-C Case.

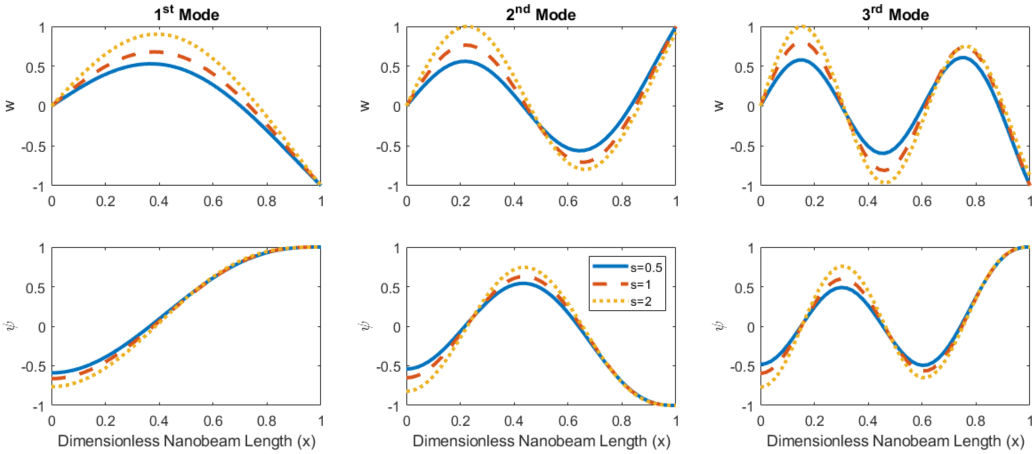

Figure 12. Right End Material Property Effects on Transverse Deflection and Slope of T-E Beam on S-F Case.

4. Conclusion

Vibration of axially functionally graded beams has been investigated. Shear deformation and rotational inertia of the rectangular cross-sectional beam has been considered with using Timoshenko-Ehrenfest beam theory. Axially grading material variation has been assumed in power-law formulation. Weak energy formulation has been used in the dynamic modeling of aFG T-E beam. Solution of the vibration problem has been utilized by using approximate Ritz Method. Variation of material properties and characteristics show important effects on dynamics of aFG T-E beam. Restrictive boundary conditions (clamped and simply supported) increases the rigidity of beam and vice versa non-restrictive (free) boundary condition decreases. Second material property affects the in which nodal point maximum deflection occurs.

Present study could be useful at designing of newly developed structural elements or shafts which are subjected various load in different cross-sections.

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