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RESEARCH PAPER

Fourier series method for finding displacements and stress fields in hyperbolic shear deformable thick beams subjected to distributed transverse loads

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Abstract

This paper presents a systematic formulation of the hyperbolic shear deformation theory for bending problems of thick beams; and the Fourier series method for solving the resulting system of coupled differential equations and ultimately finding the displacements and stress fields. Hyperbolic sine and cosine functions are used in formulating the displacement field components such that transverse shear stress free conditions are achieved at the top and bottom surfaces of the beam, thus obviating the shear correction factors of the first order shear deformation theories. The vanishing of the first variation of the total potential energy functional is used to obtain the system of coupled differential equations for the domain and the boundary conditions. The domain equations are solved using Fourier series method for simply supported ends for linearly distributed and uniformly distributed loads. The solutions are found as infinite series with good convergence. Solutions obtained for the axial and transverse displacements, and normal and shear stresses at critical points on the beam agree remarkably well with previous solutions, and for normal stresses, the errors of the present method are less than 0.5% for aspect ratio of 4 and less than 1.9% for aspect ratio of 10.

Keywords: Hyperbolic shear deformation beam theory; Fourier series method; thick beams; total potential energy functional; first variation of total potential energy functional.

1. Introduction

Beams are structural members which carry transverse loads which may be applied at points on the beam or distributed over the entire span or parts of the span. The load may be static or dynamic. Beams may also be subjected to compressive loads which may be concentrically or eccentrically applied; in which case the behaviour in buckling becomes important. Beam problems in static flexure, dynamic and stability have been extensively studied by various researchers using different techniques. The ratio of the beam thickness to the span has been found to govern the classification of beams as thin, moderately thick and thick.

Euler and Bernoulli independently developed a theory of beams using the hypothesis that straight lines that are on the cross-section which are originally perpendicular to the neutral axis before the beam bending deformation remain straight and perpendicular to the neutral axis after deformation.

The hypothesis of orthogonality of straight lines on the cross-section before and after bending deformation effectively implies that transverse shear strains are ignored, and this limits the scope of the resulting formulation to thin beams only where transverse shear deformations do not have significant impacts on the flexural, vibration or stability behaviours of the beam [1 – 5].

The Euler-Bernoulli beam theory EBBT is satisfactory for thin beams, but unsatisfactory for moderately thick and thick beams [1 – 5].

Timoshenko [6] presented first order shear deformation theory (FSDT) which extends the classical EBBT to account for transverse shear deformation. For FSDT the orthogonality criterion is modified so that a line on the plane cross-section initially perpendicular to the neutral axis before deformation may not remain perpendicular to the neutral axis after deformation. Hence for Timoshenko beams, the foundational hypothesis is that plane cross-sections that are initially normal to the neutral axis of the beam before deformation would remain plane but would not necessarily be normal to the neutral axis after deformation. FSDT assumes constant transverse shear strain through the beam thickness, thus violating the transverse shear stress free conditions on the top and bottom beam surfaces. The theory thus has the major short coming of requiring problem dependent shear modification factors to appropriately represent the strain energy of deformation.

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Other first order shear deformation theories were developed by Reissner, Mindlin, Pakhare et al [7] and others. Senjanovic et al [8] derived a new FSDT with in-plane shear influence considered. Their new formulation which employed Hamilton's principle assumed the coupling of flexural and in-plane shear vibrations. They confirmed their formulation with illustrative problems to demonstrate its effectiveness and accuracy compared with the Timoshenko theory.

Ike [9] presented a variational formulation of the Timoshenko beam theory under static flexure and solved the resulting formulation in closed form.

Ike et al [10] used the modified single finite Fourier cosine integral transform method to satisfactorily obtain buckling loads and critical buckling loads of first order shear deformable beams with fixed ends.

Ike et al [11] used the Laplace transform method to solve the elastic buckling problems of moderately thick beams modelled using the single variable FSDT, and obtained exact solutions for the buckling loads for various boundary conditions considered.

Onah et al [12] solved the elastic buckling problems of moderately thick beams modelled using the single variable FSDT by the rigorous mathematical tool of the method of trial functions. They obtained exact solutions to the resulting eigenvalue-eigenvector problem for different considered boundary conditions of the beam.

Shimpi et al [13] used a displacement based formulation to derive a two-variable refined theory for shear deformable beams. They assumed linearly elastic homogeneous isotropic material and uniform rectangular cross-section. Their theory gave quadratic variation of transverse shear strain through the beam thickness and transverse shear stress free boundary conditions at the top and bottom surfaces of the beam. They obtained two fourth order partial differential equations which are uncoupled for static problems but inertially coupled for dynamic problems. They validated the theory by solved bending and vibration problems which gave results comparable with previous solutions in the literature.

Shimpi et al [14] developed a simple one-variable shear deformation theory for beam with prismatic rectangular cross-section and isotropic homogeneous material. They obtained a fourth order equation of equilibrium for the domain which is a close analogue of the equation of the Euler-Bernoulli theory for the cases of bending problems.

Levinson [15] developed a novel theory for rectangular beams which considers warping of the cross-section, and satisfies the transverse shear stress free boundary conditions at the beam surfaces. The theory does not need shear correction factors and the governing domain equations are a pair of coupled differential equations.

Gao and Wang [16] used elasticity theory to derive a refined theory of rectangular beams without adhoc assumptions. Shi and Voyiadjis [17] derived a new beam theory governed by sixth order differential equations for the analysis of shear deformable beams.

Ike and Oguaghamba [18] used the Fourier series method to solve the governing domain equations of equilibrium of bending problems of thick beams modelled within the framework of trigonometric shear deformation theory. They obtained exact solutions with the framework of the theory used. Ike [19] used the Ritz variational method to solve the flexural problems of third order shear deformable beams with simply supported ends; and obtained results that agreed with previously obtained solutions.

Ghugal and Nakhate [20], and Ghugal and Shimpi [21] presented trigonometric shear deformation theories for thick beams and used them to successfully solve flexural and vibration problems of isotropic thick beams by seeking closed form analytical solutions. Ghugal and Shimpi [22] presented a review of refined shear deformation theories for laminated beams of isotropic and anisotropic materials.

Ghugal and Dahake [23] used the refined shear deformation beam theory to solve bending problems of thick beams subjected to parabolic load. Pote Rohit et al [24] presented a refined beam theory for solving flexural problems of composite beams.

Pakhare et al [25] presented the analysis of stability problems of thick isotropic shear deformable beams. Sayyad and Ghugal [26] derived single variable refined beam theories for the flexural stability and eigenfrequency analysis of beams made of homogeneous materials.

Ghugal [27] presented a new refined bending theory for thick beams that include transverse shear and transverse normal strain effects. Ghugal [28] presented a single variable parabolic shear deformation theory for solving problems of the static bending and vibration of thick isotropic beams.

Hyperbolic shear deformation theory for flexure of thick beams have been derived using virtual work principles by Darak and Bajad [29]. Ghugal and Sharma [30] presented the hyperbolic shear deformation theory for the flexure and flexural vibration of thick isotropic beams. Sayyad and Ghugal [31] derived a new hyperbolic shear deformation theory for the flexural analysis of thick beams and used it to accurately solve for stresses and displacements in thick beams.

Timoshenko and Goodier [32], and Ghugal [33] used the theory of elasticity formulations to solve the thick beam bending problems. They thus obtained exact elasticity solutions for the thick beam flexure problems based on two-dimensional elasticity theory.

Krisha Murthy [34] presented higher order deformation beam theories. Ghugal [35] studied the bending and vibration behaviour of deep beams using trigonometric shear deformation theory. Reddy [36 – 38] applied energy and variational methods to the analysis of shear deformable beams and plates. Naik et al [39] presented refined beam theory and applied it to bending analysis of deep beams under different load conditions.

Canales and Mantari [40] presented closed form solutions to the static bending problems of deep rectangular beams for different restraint conditions. They used Carrera's Unified Formulation (CUF) in order to account for shear deformation theories of arbitrary order. They applied a boundary discontinuous Fourier technique to account for clamped boundaries in the analytical solution obtained.

Their formulation has some advantages over Navier – type methods which apply only to simply supported boundaries. They used the virtual work principle to obtain the domain equations. They obtained numerical solutions for beams submitted to bending and torsion, and validated their work by favourable comparison with finite element solutions.

A generalization of the Fourier series method called the boundary discontinuous Fourier method has been developed. It extends the application of the Fourier series method beyond simply supported ends and allows applications to clamped ends.

More complex problems of elasticity theory involving beams, curved pipes and nanobeams have been studied by Barati et al [41]. Nejad and Hadi [42, 43], Ghumare and Sayyad [44] and Zidi et al [45]. Fouseca et al [46] have presented numerical study of curved pipes submitted to in-plane loading condition.

Fonseca et al [47] have employed trigonometric functions in the formulation of a multi-nodal finite tabular element. Karamanli [48] derived solutions for buckling of functionally graded beams modelled using Reddy's third order shear deformation beam theory. Sayyad and Ghugal [49] have also presented buckling solutions for functionally graded sandwich beams modelled with unified beam thoery.

Fonseca et al [50] also employed Fourier series method in their numerical analysis studies of piping elbows under in-plane bending and internal pressure. Their formulation used thin shell displacement theory where the displacement is assumed in the form of higher order polynomials or trigonometric functions for rigid beam displacement.

Fonseca et al [51] have also presented a semi-analytical derivation using Fourier trigonometric series method to solve the bending problem in curved pipes. They used a displacement finite element formulation and Fourier series basis functions to derive the governing equations for the pipe element, assumed as part of a toroidal shell. They solved the resulting system of differential equations using computational software tools.

Recently Hadi et al [52 – 55] have investigated the elasticity, vibration and buckling behaviours of nanobeams, beams and plates made of functionally graded materials (FGM). Hosseini et al [56, 57] have presented elasticity analysis of FGM structures with variable thickness. Nejad et al [58 – 62] have presented studies on elasticity, buckling and vibrations of FGM beams using nonlocal elasticity theory and consistent couple stress theory. Shishesaz et al [63] and Shishesaz and Hosseini [64] have used the strain gradient theory to study FGM cylinders for thermoelastic loading and for radial pressure conditions.

Other seminal works on the subject of this paper which make important contributions to the literature include Mohammadi et al [65], Gorgani et al [66], Khoram et al [67, 68], Daneshmehr [69], Mazarei et al [70], Gharibi et al [71], Noroozi et al [72] and Barati et al [73].

This study undertakes a systematic first principles presentation of the Hyperbolic Shear Deformation Theory for thick isotropic beams and used the Fourier series method in a systematic way to solve the resulting system of domain equations.

Case Study

The case study considered is a thick beam with rectangular cross-sections which is modelled using Hyperbolic Shear deformation theory. The theory is presented.

2. Formulation of Hyperbolic Shear Deformation Beam Theory (HSDBT)

2.1. Thick Beam Considered

The considered beam which is subjected to transverse distributed load of intensity $q(x)$ is assumed to be made of homogeneous, isotropic, linearly elastic material. The domain is defined with reference to the three dimensional (x, y, z) Cartesian coordinates as $0 \le x \le l,\ -b/2 \le y \le b/2,\ -t/2 \le z \le t/2$ where l is the length in the x-direction, b is the width in the y-direction, t is the depth of the beam. x, y, z are the Cartesian coordinate axes. Figure 1 shows the beam under consideration.

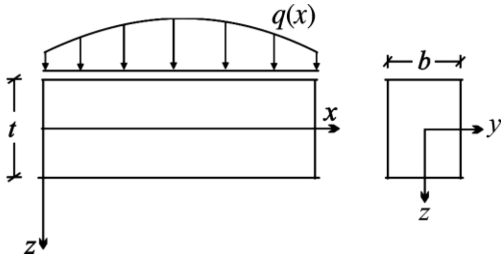

Figure 1: Longitudinal and cross-sectional views of the thick beam under bending in the *xz* coordinate plane due to transverse load *q*(*x*).

Assumptions

The assumptions are:

(i) The displacement in the longitudinal direction is made up of components due to bending deformation and shear deformation.
(ii) The transverse component of the displacement field depends only on the longitudinal coordinate, x.
(iii) The material constitutive equations are one dimensional.
(iv) Body forces are neglected but can be considered by including them in the applied loads.
(v) Beam material is homogeneous, isotropic and linearly elastic.

The displacement field

The displacement field components are:

$$u(x,z) = -z\frac{dw}{dx} + g(z)\,\varphi(x) \quad (1)$$

where $g(z)$ is a shape function which determines the shearing stress variation across the thickness of the beam. $g(z)$ which satisfies shear stress free conditions at the beam boundaries $z = \pm\frac{t}{2}$ is given by:

$$g(z) = \left(z\cosh\frac{1}{2} - t\sinh\frac{z}{t} \right) \quad (2)$$

$\varphi(x)$ is the warping function which measures the rotation of the cross-section of the beam at its neutral axis. $\varphi(x)$ is an unknown function which is to be determined.

$$v(x,z) = 0 \quad (3)$$

$$w(x,z) = w(x) \quad (4)$$

u is the displacement in the x direction, v is the displacement in the y direction, w is the transverse displacement in the z direction.

Strain fields

The strain fields are found using the strain-displacement equations of linear elastic theory as:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{d^2 w}{dx^2} + g(z)\frac{d\varphi(x)}{dx} \tag{5}$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{dw}{dx} = g'(z)\,\varphi(x) = \left[\cosh\left(\frac{1}{2}\right) - \cosh\left(\frac{z}{t}\right)\right]\varphi(x) \tag{6}$$

ε_{xx} is the axial (normal) strain, and γ_{xz} is the transverse shear strain

From Equation (6), $\gamma_{xz}\left(x,\, z = \pm t/2\right) = 0$

Stress fields

The stress fields are obtained using the stress-strain law as:

$$\sigma_{xx} = E\varepsilon_{xx} = -zE\frac{d^2 w}{dx^2} + E\left(z\cosh\frac{1}{2} - t\sinh\frac{z}{t}\right)\frac{d\varphi}{dx} \tag{7}$$

$$\tau_{xz} = G\gamma_{xz} = G\left(\cosh\frac{1}{2} - \cosh\frac{z}{t}\right)\varphi(x) \tag{8}$$

where E is the Young's modulus, G is the shear modulus of the beam material.

From Equation (8), $\tau_{xz}\left(x,\, z = \pm t/2\right) = 0$

2.6 Total potential energy functional Π

The total potential energy functional Π is:

$$\Pi = \frac{1}{2}\int_{-b/2}^{b/2}\int_{0}^{l}\int_{-t/2}^{t/2} (\sigma_{xx}\varepsilon_{xx} + \tau_{xz}\gamma_{xz})dydxdz - \int_0^l q(x)w(x)dx \tag{9}$$

For equilibrium,

$$\delta\Pi = 0 \tag{10}$$

where δ is the variational operator.

Thus,

$$b\int_0^l\int_{-t/2}^{t/2} (\sigma_{xx}\delta\varepsilon_{xx} + \tau_{xz}\delta\gamma_{xz})dxdz - \int_0^l q(x)\delta w(x)dx = 0 \tag{11}$$

$$b\int_0^l\int_{-t/2}^{t/2}\left\{E\left(-z\frac{d^2w}{dx^2} + \left(z\cosh\frac{1}{2} - t\sinh\frac{z}{t}\right)\frac{d\varphi}{dx}\right)\times\ \delta\left(-z\frac{d^2w}{dx^2} + \left(z\cosh\frac{1}{2} - t\sinh\frac{z}{t}\right)\frac{d\varphi}{dx}\right) + G\left(\cosh\frac{1}{2} - \cosh\frac{z}{t}\right)\varphi(x)\delta\left(\cosh\frac{1}{2} - \cosh\frac{z}{t}\right)\varphi(x)\right\}dxdz - \int_0^l q(x)\,\delta w(x)dx = 0 \tag{12}$$

Integration by parts and simplification by bringing like terms together after using Green's theorem in Equation (12) successively, gives the following:

$$\int_0^l\left(EI\frac{d^4w}{dx^4} - c_0EI\frac{d^3\varphi}{dx^3} - q(x)\right)\delta w dx + \int_0^l\left(c_0EI\frac{d^3w}{dx^3} - c_1EI\frac{d^2\varphi}{dx^2} - c_2GA\varphi(x)\right)\delta\varphi(x)dx + \left(EI\frac{d^2w}{dx^2} - c_0EI\frac{d\varphi}{dx}\right)\frac{d\delta w}{dx}\Bigg]_0^l + \left(-EI\frac{d^3w}{dx^3} + c_0EI\frac{d^2\varphi}{dx^2}\right)\delta w\Bigg]_0^l + \left(-c_0EI\frac{d^2w}{dx^2} + c_1EI\frac{d\varphi}{dx}\right)\delta\varphi\Bigg]_0^l = 0 \tag{13}$$

c_0, c_1, c_2 are the stiffness coefficients.

I is the moment of inertia.

The domain equations of equilibrium are obtained from the conditions for the vanishing of the integrals in Equation (13). Using $\delta w = 0$ and $\delta\varphi = 0$ the following coupled Euler-Lagrange equations which are the equations of equilibrium of the beam flexure problem are obtained as:

$$EI\frac{d^4w}{dx^4}-c_0EI\frac{d^3\varphi}{dx^3}-q(x)=0 \tag{14}$$

$$c_0EI\frac{d^3w}{dx^3}-c_1EI\frac{d^2\varphi}{dx^2}-c_2GA\varphi=0 \tag{15}$$

$$A=bt \tag{16}$$

where A is the cross-sectional area of the beam.

The stiffness coefficients are:

$$c_0=\cosh\left(\frac{1}{2}\right)-12\left(\cosh\left(\frac{1}{2}\right)-2\sinh\left(\frac{1}{2}\right)\right)= 0.102401712 \tag{17}$$

$$c_1=\cosh^2\left(\frac{1}{2}\right)+6(\sinh(1)-1)- 24\cosh\left(\frac{1}{2}\right)\left(\cosh\left(\frac{1}{2}\right)-2\sinh\left(\frac{1}{2}\right)\right)= 0.010608508 \tag{18}$$

$$c_2=\cosh^2\left(\frac{1}{2}\right)+\frac{1}{2}(\sinh(1)+1)-4\cosh\left(\frac{1}{2}\right)\sinh\left(\frac{1}{2}\right)= 0.0087385269 \tag{19}$$

The boundary conditions are:

Either $V_x=EI\frac{d^3w}{dx^3}-c_0EI\frac{d^2\varphi}{dx^2}=0$ (20)

or w is known

Either $M_x=EI\frac{d^2w}{dx^2}-c_0EI\frac{d\varphi}{dx}=0$ (21)

or $\frac{dw}{dx}$ is known

Either $M_x=-c_0EI\frac{d^2w}{dx^2}+c_1EI\frac{d\varphi}{dx}=0$ (22)

or φ is given.

V_x and M_x are the shear force and bending moment resultants.

3. Methodology

The problem considered is a thick beam under two cases of distributed loads; namely:

(a) uniformly distributed load, and

(b) linearly distributed load as shown in Figure 2

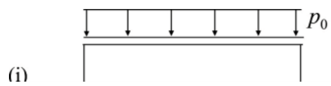

Figure 2: (i) thick beam subjected to uniform distributed load over the span, (ii) thick beam subjected to linearly distributed load over the entire span.

The boundary conditions for $w(x)$ and $\varphi(x)$ at the simple supports $x = 0$, and $x = l$ are:

$$w(x=0)=w(x=l)=0$$
$$w''(x=0)=w''(x=l)=0 \quad (23)$$
$$\varphi'(x=0)=\varphi'(x=l)=0$$

Thus, suitable functions that satisfy the conditions are constructed using Fourier series as:

$$w(x)=\sum_{n=1}^{\infty} w_n \sin\frac{n\pi x}{l} \quad (24)$$

$$\varphi(x)=\sum_{n=1}^{\infty} \varphi_n \cos\frac{n\pi x}{l} \quad (25)$$

where w_n and φ_n are the generalized displacement parameters for $w(x)$ and $\varphi(x)$. w_n and φ_n are the Fourier coefficients of $w(x)$ and $\varphi(x)$ respectively.

Let $q(x)=\sum_{n=1}^{\infty} p_n \sin\frac{n\pi x}{l}$ (26)

where p_n is the Fourier coefficient of $q(x)$ given by Fourier series theory as:

$$p_n=\frac{2}{l}\int_0^l q(x)\sin\frac{n\pi x}{l}\,dx \quad (27)$$

The reasons for the choice of Fourier series are as follows: (a) the Fourier series chosen in Equations (24) and (25) for $w(x)$ and $\varphi(x)$ satisfies all the boundary conditions stated in Equation (23). Also the Fourier series theory shows the representation of any distribution of loading function $q(x)$ as the Fourier series given by Equation (26).

Then, the governing Equations (14) and (15) become:

$$EI\sum_{n=1}^{\infty}\left(\frac{n\pi}{l}\right)^4 w_n \sin\frac{n\pi x}{l} - c_0 EI\sum_{n=1}^{\infty}\varphi_n\left(\frac{n\pi}{l}\right)^3 \sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} p_n \sin\frac{n\pi x}{l} \quad (28)$$

$$c_0 EI\sum_{n=1}^{\infty} -w_n\left(\frac{n\pi}{l}\right)^3 \cos\frac{n\pi x}{l} + c_1 EI\sum_{n=1}^{\infty}\varphi_n\left(\frac{n\pi}{l}\right)^2 \cos\frac{n\pi x}{l} + c_2 GA\sum_{n=1}^{\infty}\varphi_n \cos\frac{n\pi x}{l} = 0$$

Hence,

$$-c_0 EI\sum_{n=1}^{\infty} w_n\left(\frac{n\pi}{l}\right)^3 \cos\frac{n\pi x}{l} + c_1 EI\sum_{n=1}^{\infty}\varphi_n\left(\frac{n\pi}{l}\right)^2 \cos\frac{n\pi x}{l} + c_2 GA\sum_{n=1}^{\infty}\varphi_n \cos\frac{n\pi x}{l} = 0 \quad (29)$$

Orthogonalizing, the equations give after simplification;

$$\begin{pmatrix} EI\left(\frac{n\pi}{l}\right)^4 & -c_0 EI\left(\frac{n\pi}{l}\right)^3 \\ -c_0 EI\left(\frac{n\pi}{l}\right)^3 & \left(c_1\left(\frac{n\pi}{l}\right)^2 EI + c_2 GA\right) \end{pmatrix}\begin{pmatrix} w_n \\ \varphi_n \end{pmatrix} = \begin{pmatrix} p_n \\ 0 \end{pmatrix} \quad (30)$$

Solving, using Cramer's rule, w_n is found as:

$$w_n = \frac{\begin{vmatrix} p_n & -c_0 EI\left(\frac{n\pi}{l}\right)^3 \\ 0 & \left(c_1\left(\frac{n\pi}{l}\right)^2 EI + c_2 GA\right) \end{vmatrix}}{\begin{vmatrix} EI\left(\frac{n\pi}{l}\right)^4 & -c_0 EI\left(\frac{n\pi}{l}\right)^3 \\ -c_0 EI\left(\frac{n\pi}{l}\right)^3 & \left(c_1 EI\left(\frac{n\pi}{l}\right)^2 + c_2 GA\right) \end{vmatrix}} \quad (31)$$

Simplifying Equation (31) gives:

$$w_n = \frac{p_n\left(c_1EI\left(\frac{n\pi}{l}\right)^2 + c_2GA\right)}{EI\left(\frac{n\pi}{l}\right)^4\left(c_1EI\left(\frac{n\pi}{l}\right)^2 + c_2GA\right) - \left(c_0EI\left(\frac{n\pi}{l}\right)^3\right)^2} \quad (32)$$

$$\varphi_n = \frac{\begin{vmatrix} EI\left(\frac{n\pi}{l}\right)^4 & p_n \\ -c_0EI\left(\frac{n\pi}{l}\right)^3 & 0 \end{vmatrix}}{\begin{vmatrix} EI\left(\frac{n\pi}{l}\right)^4 & -c_0EI\left(\frac{n\pi}{l}\right)^3 \\ -c_0EI\left(\frac{n\pi}{l}\right)^3 & \left(c_1EI\left(\frac{n\pi}{l}\right)^2 + c_2GA\right) \end{vmatrix}} \quad (33)$$

Simplifying Equation (33) gives:

$$\varphi_n = \frac{p_n c_0 EI\left(\frac{n\pi}{l}\right)^3}{EI\left(\frac{n\pi}{l}\right)^4\left(c_1EI\left(\frac{n\pi}{l}\right)^2 + c_2GA\right) - \left(c_0EI\left(\frac{n\pi}{l}\right)^3\right)^2} \quad (34)$$

Then from Equation (24),

$$w(x) = \sum_{n=1}^{\infty}\frac{p_n\left(c_1EI\left(\frac{n\pi}{l}\right)^2 + c_2GA\right)\sin\left(\frac{n\pi x}{l}\right)}{EI\left(\frac{n\pi}{l}\right)^4\left(c_1EI\left(\frac{n\pi}{l}\right)^2 + c_2GA\right) - \left(c_oEI\left(\frac{n\pi}{l}\right)^3\right)^2} \quad (35)$$

$$\varphi(x) = \sum_{n=1}^{\infty}\frac{p_n c_0 EI\left(\frac{n\pi}{l}\right)\cos\left(\frac{n\pi x}{l}\right)}{EI\left(\frac{n\pi}{l}\right)^4\left(c_1EI\left(\frac{n\pi}{l}\right)^2 + c_2GA\right) - \left(c_0EI\left(\frac{n\pi}{l}\right)^3\right)^2} \quad (36)$$

4. Results

The results for axial displacement u is found by using the expression for $w(x)$ and $\varphi(x)$ in Equation (1). Hence,

$$u = \sum_{n=1}^{\infty}\left(-z\left(\frac{n\pi}{l}\right)w_n + g(z)\,\varphi_n\right)\cos\frac{n\pi x}{l} \quad (37)$$

The axial displacement at $x = l/2$, $z = \pm t/2$ is given by

$$u\left(x = l/2, z = \pm t/2\right) = \sum_{n=1}^{\infty}\left(\mp\left(\frac{t}{2}\right)\left(\frac{n\pi}{l}\right)w_n + g\left(\pm t/2\right)\varphi_n\right)\cos n\pi \quad (38)$$

The normal stress σ_{xx} is found using Equation (7) as:

$$\sigma_{xx} = E\sum_{n=1}^{\infty}\left(-z\left(\frac{n\pi}{l}\right)^2 w_n - g(z)\left(\frac{n\pi}{l}\right)\varphi_n\right)\sin\frac{n\pi x}{l} \quad (39)$$

The normal stress at $x = \frac{l}{2}$, $z = \pm\frac{t}{2}$ is obtained by substitution as:

$$\sigma_{xx}\left(x=\frac{l}{2}, z=\pm\frac{t}{2}\right)= E\sum_{n=1}^{\infty}\left(\left(\pm\frac{t}{2}\right)\left(\frac{n\pi}{l}\right)^2 w_n - g\left(\pm\frac{t}{2}\right)\left(\frac{n\pi}{l}\right)\varphi_n\right)\sin\frac{n\pi}{2} \quad (40)$$

Similarly w at $x=\frac{l}{2}$ is found as:

$$w\left(x=\frac{l}{2}\right)=\sum_{n=1}^{\infty} w_n \sin\frac{n\pi x}{l}\bigg|_{x=l/2} = \sum_{n=1}^{\infty} w_n \sin\frac{n\pi}{2} \quad (41)$$

The shear stress distribution is found from Equation (8) as:

$$\tau_{xz}=\sum_{n=1}^{\infty} Gg'(z)\varphi_n \cos\frac{n\pi x}{l} \quad (42)$$

The shear stress at x = 0, z =0 is found as:

$$\tau_{xz}(x=0, z=0)=\sum_{n=1}^{\infty} Gg'(0)\varphi_n = \sum_{n=1}^{\infty} G\left(\cosh\frac{1}{2}-1\right)\varphi_n = 0.127626G\sum_{n=1}^{\infty}\varphi_n \quad (43)$$

The results are tabulated using dimensionless parameters for displacements and stresses defined to conform to literature as follows:

$$\bar{u}=\frac{bE}{p_0 t}u \quad (44)$$

$$\bar{w}=\frac{100Ebt^3}{p_0 l^4}w \quad (45)$$

$$\bar{\sigma}_{xx}=\frac{b}{p_0}\sigma_{xx} \quad (46)$$

$$\bar{\tau}_{xz}=\frac{b}{p_0}\tau_{xz} \quad (47)$$

5. Numerical results

The beam parameters considered are E = 210GPa, $\mu = 0.30$, where μ is the Poisson's ratio of the beam material. For uniformly distributed load, of intensity, p_0, the Fourier coefficient p_n is:

$$p_n=\frac{2}{l}\int_0^l p_0 \sin\frac{n\pi x}{l}dx = \frac{4p_0}{n\pi}, \; n = 1, 3, 5, 7, \ldots \quad (48)$$

For linearly varying load, $p(x)=\frac{p_1 x}{l}$, p_n is found from Fourier series theory as:

$$p_n=\frac{2}{l}\int_0^l \frac{p_1 x}{l}\sin\frac{n\pi x}{l}dx = \frac{2p_1}{l^2}\int_0^l x\sin\frac{n\pi x}{l}dx = \frac{2p_1}{n\pi} \quad (49)$$

6. Discussion

The Fourier series method has been successfully applied in this work to solve the flexural problems of thick beams modelled using Hyperbolic Shear Deformation Theory (HPSDT). The equations that were solved are a system of two coupled equations in terms of transverse deflection and warping function. The problems were solved for linearly distributed and uniformly distributed loads. The results obtained for displacements and stresses for the present study and previous studies for different aspect ratios for simply supported thick beams were shown in Table 1 for the case of linearly distributed loading over the domain. Table 1 shows that that present results agree remarkably well with previous results presented by Darak and Bajad who used a Modified Hyperbolic Shear

Deformation Beam Theory (MHPSDBT) and KrishnaMurthy who used a Higher Order Shear Beam Deformation Beam Theory (HOSDBT). The results for displacements and stresses in simply supported thick beam subjected to uniformly distributed load for various aspect ratios are shown in Table 2 for displacements and Table 3 for stresses. Tables 2 and 3 show that the present results agree remarkably well with the previous results obtained by Ike [19], Naik et al [39], Sayyad and Ghugal [31] and Reddy [36 – 38]. The present results are also in close agreement with the exact results from the elasticity theory with errors in σ_{xx} being generally less than 0.5% for l/t = 4, and less than 1.9% for l/t = 2.

Table 1: Comparison of dimensionless displacement and stress parameters for simply supported thick beams subjected to linearly distributed load $p(x) = p_1x/l$ for various aspect ratios

$$\bar{u}\left(x=\frac{3l}{4}, z=\frac{t}{2}\right),\quad \bar{w}\left(x=\frac{3l}{4}, z=0\right),\qquad \sigma_{xx}\left(x=\frac{3l}{4}, z=\frac{t}{2}\right),\quad \tau_{xz}(x=0, z=0)$$

$$\bar{u}=\frac{bE}{p_1 t}u,\quad \bar{w}=\frac{100Ebt^3}{p_1 l^4},\quad \bar{\sigma}_{xx}=\frac{b}{p_1}\sigma_{xx},\quad \bar{\tau}_{xz}=\frac{b}{p_1}\tau_{xz}$$

l/t	Reference / Theory	$\bar{u}$	% Diff with EBBT	$\bar{w}$	% Diff with EBBT	$\bar{\sigma}_{xx}$	% Diff with EBBT	$\bar{\tau}_{xz}$	% Diff with FSDBT
4	Present (HPSDBT)	5.5902	2.18	0.6874	18.29	5.4449	3.71	0.9989	24.86
	Krishna Murthy [34] (HOSDBT)	5.5902	2.18	0.6874	18.29	5.4450	3.71	0.9988	24.85
	Ghugal [35] (TSDBT)	5.1100	–6.60	0.6872	18.26	7.6927	40.53	1.2007	50.09
	Darak and Bajad [29] (MHPSDBT)	5.5903	2.18	0.6874	18.29	5.4451	3.72	0.9959	24.49
	Timoshenko [6] (FSDBT)	5.9375	6.35	0.6877	18.34	5.2500	0	0.80	0
	Darak and Bajad [29] (EBBT)	5.4708	0	0.5811	0	5.2500	0	–	–
10	Present (HPSDBT)	85.7799	0.35	0.5981	0	33.0075	0.59	2.4997	24.99
	Krishna Murthy [34] (HOSDBT)	85.7798	0.35	0.5981	0	33.0075	0.59	2.4995	24.98
	Ghugal [35] (TSDBT)	84.05	–1.67	0.5981	0	34.0105	3.65	2.5801	72.51
	Darak and Bajad [29] (MHPSDBT)	85.7800	0.35	0.5981	0	33.0076	0.59	2.5005	25.025
	Timoshenko [6] (FSDBT)	92.7734	8.53	2.5981	0	32.8125	0	2	0
	Darak and Bajad [29] (EBBT)	85.4818	0	0.5981	0	32.8125	0	–	–

HPSDBT: Hyperbolic shear deformation beam theory.
HOSDBT: Higher order shear deformation beam theory.

% Difference for $\bar{u}$, $\bar{w}$ and $\bar{\sigma}_{xx}$ are calculated with respect to the EBBT presented by Darak and Bajad [29].

% Difference for $\bar{\tau}_{xz}$ is calculated with respect to the FSDBT of Timoshenko [6].

Table 2: Comparison of non-dimensional displacement parameters for simply supported thick beams subjected to uniformly distributed loads for various aspect ratios (*l*/*t*)

$$\bar{u}\left(x = l/2, z = t/2\right), \quad \bar{w}\left(x = l/2, z = 0\right)$$

l/*t*	Reference or Source	Model / Theory	$\bar{u}$	% Difference	$\bar{w}$	% Difference
2	Present study	HPSDBT	2.240	1.818	2.530	3.139
	Ike [19]	TODBT	–	–	2.532	3.465
	Timoshenko [6]	FSDBT	2.000	–9.091	2.532	3.465
	Naik et al [39]	EBBT	2.000	–9.091	1.563	–36.282
	Reddy theory [36–38], Naik et al [39]	HSDBT	2.245	2.045	2.532	3.221
	Naik et al [39]	RBBT	2.259	2.682	2.529	3.098
	Timoshenko and Goodier [32]	Exact Elasticity	2.200	0	2.453	0
4	Present study		16.420	3.924	1.803	1.008
	Ike [19]		–	–	1.806	1.176
	Sayyad and Ghugal [31]	TSDBT	16.487	4.348	1.804	1.064
	Timoshenko [6]	FSDBT	16.000	1.265	1.806	1.176
	Naik et al [39]	EBBT	16.000	1.265	1.563	–12.437
	Reddy [36–38], Naik et al [39]		16.504	4.456	1.806	1.176
	Naik et al [39]		16.535	4.652	1.805	1.120
	Timoshenko and Goodier [32]	Exact Elasticity	15.800	0	1.785	0
10	Present study		251.10	0.641	1.602	0.250
	Ike [19]		–	–	1.602	0.250
	Sayyad and Ghugal [31]	TSDBT	251.23	0.693	1.601	0.188
	Timoshenko [6]		250.00	0.20	1.602	0.250
	Naik et al [39]		250.00	0.20	1.563	–2.19
	Reddy [36 – 38], Naik et al [39]		251.27	0.709	1.602	0.25
	Naik et al [39]		251.35	0.745	1.601	0.188
	Timoshenko and Goodier [32]	Exact Elasticity	249.50	0	1.598	0

TSDBT: Trigonometric shear deformation beam theory
MHPSDBT: Modified hyperbolic shear deformation beam theory
FSDBT: First order shear deformation beam theory
EBBT: Euler-Bernoulli beam theory
RBBT: Refined thick beam bending theory

Table 3: Comparison of non-dimensional stress parameters for simply supported thick beams subjected to uniformly distributed loads for various aspect ratios (*l*/*t*)

$$\bar{\sigma}_{xx}\left(x=\frac{l}{2}, z=0\right), \quad \tau_{xz}(x=0, z=0)$$

l/t	Reference / Model	$\bar{\sigma}_{xx}$	% Difference	$\bar{\tau}_{xz}$	% Difference
2	Present (HPSDBT)	3.260	1.875	1.4110	–5.933
	Ike [19]	3.261	1.90625	1.4115	–5.667
	Timoshenko [6] (FSDBT)	3.465	3.00	0.984	–34.4
	Naik et al [39] (EBBT)	3.00	–6.25	–	–
	Reddy (HSDBT) [36 – 38], Naik et al [39]	3.261	1.960	1.415	–5.667
	Naik et al (RBBT) [39]	3.278	2.438	1.451	–3.267
	Timoshenko & Goodier [32] (Exact)	3.20	0	1.50	0
4	Present (HPSDBT)	12.260	0.4918	2.90	–3.333
	Ike [19]	12.623	0.5164	2.908	–3.067
	Sayyad and Ghugal [31]	12.254	0.443	2.882	–3.933
	Timoshenko [6] (FSDBT)	12.000	–1.6393	1.969	–34.367
	Naik et al (EBBT) [39]	12.00	–1.6393	–	–
	Reddy (HSDBT) [36 – 38], Naik et al [39]	12.263	0.516	2.908	–3.067
	Naik et al [39] (RBBT)	12.280	0.656	2.993	–0.233
	Timoshenko & Goodier [32] (Exact)	12.20	0	3.00	0
10	Present (HPSDBT)	75.260	0.0798	7.350	–2.000
	Ike [19]	75.268	0.0904	7.361	–1.853
	Sayyad and Ghugal [31]	75.259	0.0785	7.312	–2.507
	Timoshenko [6] (FSDBT)	75.000	–0.266	4.922	–34.373
	Naik et al [39] (EBBT)	75.000	–0.266	–	–
	Reddy (HSDBT) [36 – 38], Naik et al [39]	75.268	0.090	7.361	–1.853
	Naik et al [39] (RBBT)	75.284	0.112	7.591	1.2133
	Timoshenko & Goodier [32] (Exact)	75.20	0	7.50	0

7. Conclusion

In this paper a systematic presentation and formulation of the HPSDT for thick, isotropic, homogeneous beams has been done. The resulting equations are variationally consistent. The formulation ensured that transverse shear stress free boundary conditions are achieved at the top and bottom surfaces of the beam, and no shear correction factor is required. The vanishing of the first variation of the total potential energy functional is used to obtain the governing equations of equilibrium and the boundary conditions.

Fourier series method is used to obtain the solutions for the unknown transverse displacement $w(x)$ and warping function $\varphi(x)$ as infinite series. The displacements and stresses are found as single series of infinite terms with good convergence. The results are validated by good agreement with previous results obtained for the two load cases considered.

Nomenclature

x, y, z	three dimensional coordinates or Cartesian coordinates
l	length of beam in the x-direction
b	width of beam in the y-direction
t	depth or thickness of beam
$q(x)$	intensity of transverse distributed load
$u(x, z)$	displacement in the x-direction
v	displacement in the y-direction
$w(x)$	transverse displacement in the z-direction
$g(z)$	shape function which determines the shearing stress variation across the beam thickness
$\varphi(x)$	warping function which measures the rotation of the beam cross-section at its neutral axis

ε_{xx}	normal strain
γ_{xz}	transverse shear strain
$g'(z)$	derivative of $g(z)$ with respect to z
σ_{xx}	normal stress
τ_{xz}	transverse shear stress
E	Young's modulus of elasticity
G	shear modulus of the beam material
Π	total potential energy functional
δ	variational operator
$\int$	integral operator
$\iint$	double integral operator
$\iiint$	triple integral operator
c_0, c_1, c_2	stiffness coefficients
A	cross-sectional area
I	moment of inertia
V_x	shear force
M_x	bending moment resultant
$\sum$	summation
w_n	generalized displacement parameter for transverse displacement
φ_n	generalized displacement parameter for the warping function
p_n	Fourier series coefficient of the transverse loading function
n	integer
∞	infinity
$\bar{u}$	dimensionless form of u
$\bar{w}$	dimensionless form of w
$\bar{\sigma}_{xx}$	dimensionless form of σ_{xx}
$\bar{\tau}_{xz}$	dimensionless form of τ_{xz}
EBBT	Euler-Bernoulli beam theory
FSDT	First order shear deformation theory
FSDBT	First order shear deformation beam theory
HSDBT	Hyperbolic shear deformation beam theory
HOSDBT	Higher order shear deformation beam theory
TSDBT	Trigonometric shear deformation beam theory
MHPSDBT	Modified hyperbolic shear deformation beam theory
RBBT	Refined thick beam bending theory
FGM	Functionally Graded Materials
HPSDBT	Hyperbolic shear deformation beam theory

References

[1] C.C. Ike, Fourier sine transform method for the free vibration of Euler-Bernoulli beam resting on Winkler foundation, International Journal of Darshan Institute on Engineering Research and Emerging Technologies, Vol. **7** No.1, pp 1 – 6, 2018. DOI: 10.32692/IJDI-ERET/7.1.2018.1801.

[2] C.C. Ike, Point collocation method for the analysis of Euler-Bernoulli beam on Winkler foundation, International Journal of Darshan Institute on Engineering Research and Emerging Technologies, Vol. **7,** No.2, pp 1 – 7, 2018.

[3] C.C. Ike, E.U. Ikwueze, Ritz method for the analysis of statically indeterminate Euler-Bernoulli beams, Saudi Journal of Engineering and Technology, Vol.3, No 3, pp 133 – 140, 2018. DOI:10.21276/sjeat.2018.3.3.3

[4] C.C. Ike, E.U. Ikwueze, Fifth degree Hermittian polynomial shape functions for the finite element analysis of clamped simply supported Euler-Bernoulli beam, American Journal of Engineering Research, Vol.**7** No.4, pp 99 – 105, 2018.

[5] B.O. Mama, O.A. Oguaghamba, C.C. Ike, Quintic polynomial shape functions for the finite element analysis of elastic buckling loads of Euler-Bernoulli beam resting on Winkler foundation, Proceedings, 2nd NIEEE Nsukka Chapter Conference on Sustainable Infrastructure Development in Developing Nations, pp 122-128, 2020.

[6] S.P. Timoshenko, On the correction for shear of the differential equation for transverse vibrations of prismatic bars. Philosophical Magazine and Journal of Science, Vol. 41, No 6, pp 744 – 746, 1921. https://doi.org/10.1080/14786442108636264.
[7] K.S. Pakhare, M. Mitra, R.P. Shimpi, Development of single variable new first order shear deformation theories for plates and beams, Master of Technology Dissertation, Department of Aerospace Engineering, Indian Institute of Technology, Bombay, 2016.
[8] I. Senjanovic, N. Vladimir, N. Hadzik, M. Tomic, New first order shear deformation beam theory with in-plane shear influence" Engineering Structures, Vol. 110, No. 1, pp. 169 – 183, 2016. https://doi.org/10.1016/j.engstructures.2015.11.032
[9] C.C. Ike, Timoshenko beam theory for the flexural analysis of moderately thick beams – variational formulation and closed form solutions. Tecnica Italiana – Italian Journal of Engineering Science, Vol.63, No.1, pp. 34–45, 2019. https://doi.org/10.18280/ti-ijes.630105
[10] C.C. Ike, C.U. Nwoji, H.N. Onah, B.O. Mama, M.E. Onyia, Modified single finite Fourier cosine integral transform method for finding the critical buckling loads of first order shear deformable beams with fixed ends. Revue des Composites et des Materiaux Avances, Vol. 29, No. 6, pp. 357–362, 2019. https://doi.org/10.18280/rcma.290603
[11] C.C. Ike, C.U. Nwoji, B.O. Mama, H.N. Onah, M.E. Onyia, Laplace transform method for the elastic buckling analysis of moderately thick beams, International Journal of Engineering Research and Technology, Vol. 12, No.10, pp. 1626 – 1638, 2019.
[12] H.N. Onah, C.U. Nwoji, M.E. Onyia, B.O. Mama, C.C. Ike, Exact solutions for the elastic buckling problems of moderately thick beams, Revue des Composites et des Materiaux Avances, Vol.30, No.2, pp. 83 – 93, 2020. https://doi.org/10.18280/rcma.300205
[13] R.P. Shimpi, P.J. Guruprasad, K.S. Pakhare, Simple two variable refined theory for shear deformable isotropic rectangular beams, Journal of Applied and Computational Mechanics, Vol. 6, No.3, pp 394 – 415, 2020. DOI:10.22055/JACM.2019.29555.1615.
[14] R.P. Shimpi, R.A. Shetty, A. Guha, A simple single variable shear deformation theory for a rectangular beam. Proceedings of the Institution of Mechanical Engineers Part C. Journal of Mechanical Engineering Science, Vol. 231, No.24, pp. 4576 – 4591, 2017. https://doi.org/10.1177/0954406216670682
[15] M. Levinson, A new rectangular beam theory, Journal of Sound and Vibration, Vol. 74, No.1, pp. 81 – 87, 1981. https://doi.org/10.1016/0022-460X(81)90493-4
[16] Y. Gao, M. Wang, The refined theory of deep rectangular beams based on general solutions of elasticity, Science in China: Series G Pysics, Mechanics and Astronomy, Vol. 49, No 3, pp 291 – 303, 2006. https://doi.org/10.1007/s11433-006-0291-0
[17] G. Shi, G. Z. Voyiadjis, A sixth order theory of shear deformable beams with variational consistent boundary conditions, Journal of Applied Mechanics, Vol. 78, No.2, pp 021019, (11pgs), 2011. https://doi.org/10.1115/1.4002594
[18] C.C. Ike, O.A. Oguaghamba, Trigonometric shear deformation theory for bending analysis of thick beams: Fourier series method, Proceedings, Conference on Engineering Research, Technology Innovation and Research, 2020.
[19] C.C. Ike, Ritz variational method for the flexural analysis of third order shear deformable beams, Proceedings, Conference on Engineering Research, Technology Innovation and Practice, pp. 64, 2020.
[20] Y. Ghugal, V. Nakhate, Flexure of thick beams using trigonometric shear deformation theory, The Bridge and Structural Engineer – The Quarterly Journal of the Indian National Group of International Association for Bridge and Structural Engineering, Vol. 39, No.4, pp 1 – 17, 2010.
[21] Y. M. Ghugal, R.P. Shimpi, A trigonometric shear deformation theory for flexure and free vibration of isotropic thick beams, Proceeding of Structural Engineering Convention (SEC 2000), IIT Bombay Mumbai India, pp 255 – 263, 2020.
[22] Y. Ghugal, R. Shimpi, A review of refined shear deformation theories for isotropic and anisotropic laminated beams, Journal of Reinforced Plastics and Composites, Vol. 20, No 3, pp 255 – 272, 2001.
[23] Y. Ghugal, A. Dahake, Flexural analysis of deep beam subjected to parabolic load using refined shear deformation theory, Applied and Computational Mechanics, Vol 6, pp 163 – 172, 2012.
[24] A. Pote Rohit, U.S. Ansari, A.S. Sayyad, Refined beam theory for flexural analysis of composite beam, International Journal of Recent Scientific Research, Vol **7**, No 7, pp 12382 – 12385, 2016.
[25] K.S. Pakhare, R.P. Shimpi, P.J. Guruprasad, Buckling analysis of thick isotropic shear deformable beams, Proceedings of ICTACEM 2017 International Conference on Theoretical, Applied, Computational and Experimental Mechanics, IIT Kharagpur, India, pp 1 – 7, 2017.
[26] A.S. Sayyad, Y.M. Ghugal, Single variable refined beam theories for the bending, buckling and free vibration of homogeneous beams. Applied and Computational Mechanics, Vol 10, No 2, pp 123 – 138, 2016.

[27] Y.M. Ghugal, A new refined bending theory for thick beam including transverse shear and transverse normal strain effects, Departmental Report Applied Mechanics Department, Government College of Engineering, Aurangabad India, pp 1 – 96, 2006.

[28] Y.M. Ghugal, A single variable parabolic shear deformation theory for flexure and flexural vibration of thick isotropic beams. Proceedings of Third International Conference on Structural Engineering Mechanics and Computation (SEMC-2007), Cape Town, South Africa, pp 77 – 78. 10 – 12 September, 2007.

[29] P.G. Darak, M.N. Bajad, Static flexural analysis of thick beam by hyperbolic shear deformation theory, International Journal of Science, Engineering and Technology Research, Vol 6, No 6, pp 977 – 983, 2017.

[30] Y. Ghugal, R. Sharma, Hyperbolic shear deformation theory for flexure and vibration of thick isotropic beams, International Journal of Computational Methods, Vol 6, No 4 pp 585 – 604, 2009.

[31] A. Sayyad, Y. Ghugal, Flexure of thick beams using new hyperbolic shear deformation theory, International Journal of Mechanics, Vol 5, No3, pp 113 – 122, 2011.

[32] S.P. Timoshenko, J.N. Goodier, Theory of Elasticity, 3rd International Edition, McGraw Hill, Singapore, 1970.

[33] Y. Ghugal, A two-dimensional exact elasticity solution of thick beam, Departmental Report-1 Department of Applied Mechanics Government Engineering College, Aurangabad India, pp 1 – 96, 2006.

[34] A.V. Krishna Murthy, Towards a consistent beam theory. AIAA Journal, Vol 22: 811 – 816, 1984.

[35] Y.M. Ghugal, Flexure and vibration of thick beams using trigonometric shear deformation theory, Journal of Experimental and Applied Mechanics, Vol 1, No 1, pp 1 – 27, 2010.

[36] J.N. Reddy, Energy and Variational Methods in Applied Mechanics, Wiley, New York, 1984.

[37] J. N. Reddy, Simple higher order theory for laminated composite plates, ASME Journal of Applied Mechanics, Vol 51, No 4, pp 745 – 752.1984

[38] J. N. Reddy, Canonical relationships between bending solutions of classical and shear deformation beam and plate theories, Annals of Solid and Structural Mechanics, Vol 1, No 1, pp 9 – 27, 2010.

[39] U.P. Naik, A.S. Sayyad, P. N. Shinde, Refined beam theory for bending of thick beams subjected to various loading. Elixir Applied Mathematics,Vol 43, pp 7004 – 7015, 2012.

[40] F.G. Canales and J.L. Mantari Boundary discontinuous Fourier analysis of thick beams with clamped and simply supported edges via CUF. Chinese Journal of Aeronautics Vol 30, No 5, pp 1708 – 1718. http://dx.doi.org/10.1016/j.cja.

[41] A Barati, A. Hadi, M.Z. Nejad, R.Noroozi, On the vibration of bi-directional functionally graded nanobeams under magnetic field, Mechanics Based Design of Structures and Machines, Vol 50, No 2 pp 468 - 485 2022. Doi: 10.1080/15397734.2020.1719507.

[42] M.Z. Nejad, A. Hadi, Eringen's non-local elasticity theory for bending analysis of bi-directional functionally graded Euler-Bernoulli nanobeams, International Journal of Engineering Science Vol 106, pp 1 – 9, 2016.

[43] M.Z. Nejad, A. Hadi, Non-local analysis of free vibration of bi-directional functionally graded Euler-Bernoulli beams, International Journal of Engineering Science Vol 105, pp 1 – 11, 2016.

[44] S.M. Ghumare, A.S. Sayyad, A new fifth order shear and normal deformation theory for static bending and elastic buckling of P-FGM beams, Latin American Journal of Solids and Structures, Vol 14, pp 1893 – 1911, 2017.

[45] M. Zidi, M.S.A. Houari, A. Tounsi, A. Bessaim, S.R. Mahmoud, A novel simple two-unknown hyperbolic shear deformation theory for functionally graded beams, Structural Engineering and Mechanics, Vol 64, No 2, pp 145 – 153, 2017.

[46] E.M.M. Fonseca, F.J.M.Q. de Melo, Numerical solution of curved pipes submitted to in-plane loading conditions, Thin Walled Structures Vol 48, No 2, pp 103 – 109, doi:10.1016/j.tws.2009.09.004.

[47] E.M.M. Fonseca, F.J.M.Q. de Melo, C.A.M. Oliveira, Trigonometric function used to formulate a multi-nodal finite tubular element Journal Mechanics Research Communications Vol 34, No 1, pp 54 – 62, 2007. Doi:10.1016/j.mechrescom.2006.06.008.

[48] A. Karamanli, Analytical solutions for buckling behaviour of two directional functionally graded beams using a third order shear deformable beam theory, Academic Platform Journal of Engineering and Science, Vol 6, No 2, pp 164 – 178, 2018.

[49] A.S. Sayyad, Y.M. Ghugal, "On the buckling analysis of functionally graded sandwich beams using a unified beam theory," Journal of Computational Applied Mechanics Vol 51, No 2, December 2020, pp 443 – 453. Doi: 10.22059/jcamech.2020.310180.557.

[50] E.M.M. Fonseca, F.J. Melo, C.A.M. Oliveira, "Numerical analysis of piping elbows for in-plane bending and internal pressure" Thin-Walled Structures Vol 44, No 4, pp 393 – 398, 2006. Doi: 10.1016/j.tws.2006.04.005.

[51] E.M.M. Fonseca, F.J. Melo, C.A.M. Oliveira, Determination of flexibility factors in curved pipes with end restraints using a semi-analytic formulation, International Journal of Pressure Vessels and Piping Vol 19, No 12, pp 829 – 840, December 2002. Doi: 10.1016/S0308-0161(02)00102-3.

[52] A. Hadi, M.Z. Nejad, M. Hosseini Vibrations of thick-dimensionally graded nanobeams. International Journal of Engineering Science Vol 128, pp 12 – 23, 2018.
[53] A. Hadi, M.Z. Nejad, A. Rastgoo, M. Hosseini Buckling analysis of FGM Euler – Bernoulli nano-beams with 3D – varying properties based on consistent couple – stress theory. Steel and Composite Structures, An International Journal Vol 26 No 6, pp 663 – 672, 2018.
[54] A. Hadi, A. Rastgoo, A.R. Daneshmehr, F. Ehsani Stress and strain analysis of functionally graded rectangular plate with exponentially varying properties. Indian Journal of Material Science, Vol 2013. 206239, http://dx.doi.org/10.1155/2013/206239.
[55] A. Hadi, A.R. Daneshmehr, S.M.N Mehrian, M. Hosseini, F. Ehsani Elastic analysis of functionally graded Timoshenko beam subjected to transverse loading. Technical Journal of Engineering and Applied Sciences Vol 3 No 13, pp 1246 - 1254, 2013.
[56] H. Hosseini, M. Shishesaz, K.N. Tahan, A. Hadi Stress Analysis of rotating nano-disks of variable thickness made of functionally graded materials. International Journal of Engineering Materials, Vol 109, pp 29 – 53, 2016.
[57] M. Hosseini, M. Shishesaz, A. Hadi Thermoelastic analysis of rotating functionally graded micro/nanodisks of variable thickness. Thin Walled Structures Vol 134, pp 508 – 523, 2019.
[58] M.Z. Nejad, A. Hadi, A. Rastgoo Buckling analysis of arbitrary two-dimensional functionally graded Euler-Bernoulli nano-beams based on nonlocal elastic theory. International Journal of Engineering Science Vol 103, pp 1 – 10, 2016.
[59] M.Z. Nejad, A. Rastgoo, A. Hadi Exact elasto-plastic analysis of rotating disks made of functionally graded materials. International Journal of Engineering Science, Vol 85, pp 47 – 57, 2014.
[60] M.Z. Nejad, A. Hadi, A. Farajpour Consistent couple-stress theory for free vibration analysis of Euler-Bernoulli nano-beams made of arbitrary bi-directional functionally graded materials. Structural Engineering and Mechanics, An International Journal Vol 63 No 2, pp 161 – 169, 2017.
[61] M.Z. Nejad, N. Alamzadeh, A. Hadi Thermoelastic plastic analysis of FGM rotating thick cylindrical pressure vessels in linear elastic fully plastic condition. Composites Part B: Engineering Vol 154 pp 410 – 422, 2018.
[62] M.Z. Nejad, A. Hadi, A. Omidvari, A. Rastgoo Bending analysis of bi-directional functionally graded Euler-Bernoulli nanobeams using integral form of Eringen's non-local elasticity theory. Structural Engineering and Mechanics, An International Journal Vol 67, No 4 pp 417 – 425, 2018.
[63] M. Shishesaz, M. Hosseini, K.N. Tahan, A. Hadi Analysis of functionally graded nanodisks under thermoelastic loading based on the strain gradient theory. Acta Mechanica Vol 228 No 12, pp 4141 – 4168, 2017.
[64] M. Shishesaz, M. Hosseini Mechanical behavior of functionally graded cylinders under radial pressure based on strain gradient theory. Journal of Mechanics, Vol 35, No 4, pp 441 – 454, 2019.
[65] M. Mohammadi, M. Shishesaz, A. Hadi Primary and secondary resonance analysis of porous functionally graded nanobeam resting on a nonlinear foundation subjected to mechanical and electrical loads. European Journal of Mechanics – A/Solids Vol 77, pp 103793, 2019.
[66] H.H. Gorgani, M.M. Adeli, M. Hosseini Pull-in behavior of functionally graded micro/nanobeams for MEMs and MEMs switches. Microsystem Technologies, Vol 25, No 8, pp 3165 – 3173, 2019.
[67] M.M. Khoram, M. Hosseini, A. Hadi, M. Shishesaz Bending analysis of bidirectional FGM Timoshenko nanobeam subjected to mechanical and magnetic forces and resting on Winkler – Pasternak foundation. International Journal of Applied Mechanics, Vol 12 No 8 pp 2050093, 2020.
[68] M. Mousavi Khoram, M. Hosseini, M. Shishesaz A concise review of nanoplates. Journal of Computational Applied Mechanics Vol 50 No 2, pp 420 – 429, 2019.
[69] A. Daneshmehr, A. Rajabpoor, A. Hadi Size dependent free vibration analysis of nanoplates made of functionally graded materials based on nonlocal elasticity theory with higher order theories. International Journal of Engineering Science, Vol 95, pp 23 – 35, 2015.
[70] Z. Mazarei, M.Z. Nejad, A. Hadi Thermo-elastic-plastic analysis of thick-walled spherical pressure vessels made of functionally graded materials. International Journal of Applied Mechanics Vol 8 No 04 pp 1650054, 2016.
[71] M. Gharibi, M. Z. Nejad, Elastic analysis of functionally graded rotating thick cylindrical pressure vessels with exponentially-varying properties using power series method of Frobenius. Journal of Computational Applied Mechanics Vol 48, No 1, pp 89 – 98, 2017.
[72] R. Noroozi, A. Barati, A. Kazemi, S. Norouzi, A. Hadi Torsional vibration analysis of bi-directional FG nano-cone with arbitrary cross-section based on nonlocal strain gradient elasticity. Advances in nano research Vol 8 No 1, pp 13 – 24, 2020.
[73] A. Barati, M.M. Adeli, A. Hadi Static torsion of bi-directional functionally graded microtube based on the couple stress theory under magnetic field. International Journal of Applied Mechanics Vol 12 No 2, pp 2050021, 2020.