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A Survey on Tenacity Parameter Part II

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ABSTRACT In this paper we study the edge tenacity of graphs. We will be primarily interested in edge-tenacious graphs, which can be considered very stable and are somewhat analogous in edge tenacity to honest graphs in edge-integrity. We show several results about edge-tenacious graphs as well as find numerous classes of edge-tenacious graphs.
The Cartesian Products of graphs like hypercube, grids, and tori are widely used to design interconnection networks in multiprocessor computing systems. These considerations motivated us to study tenacity of Cartesian products of graphs. We find the tenacity of Cartesian product of complete graphs (thus setting a conjecture stated in Cozzens and al.) and grids.
The Middle Graph, M(G) of a graph G is the graph obtained from G by inserting a new vertex into every edge of G and by joining by edges those pairs of these new vertices which lie on adjacent edges of G

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1 Abstract continued

We give the edge-tenacity of the middle graph of specific families of graphs and its relationships with other parameters. We discuss about tenacity and its properties in stability calculation. We indicate relationships between tenacity and connectivity, tenacity and binding number, tenacity and toughness. We also give good lower and upper bounds for tenacity.
Also In this paper, the maximum graphical structure is obtained when the number of vertices p of a connected graph G and tenacity $T(G) = T$ are given. Finally the method of constructing the sort of graphs are also presented.

I Edge-tenacious graphs

1.1 PRELIMINARIES:

The stability of a (communication or transportation) network composed of (processing) nodes and (communication or transportation) links is of prime importance to network designers. As the network begins losing links or nodes, eventually, there is a loss in its effectiveness. Thus, it is desirable that networks be constructed to be as stable as possible, not only with respect to the initial disruption, but also with respect to the possible reconfiguration of the network after disruption. Many graph theoretical parameters have been used in the past to describe the stability of communication networks. Most notably, the vertex-connectivity and edge-connectivity have been frequently used. The difficulty with these parameters is that they do not take into account what remains after the graph is disconnected. Consequently, a number of other parameters have recently been introduced that attempt to cope with this difficulty, including toughness and edge-toughness in [10], integrity and edge-integrity in [3,4], and tenacity in [15]. Several of these deal with two fundamental questions:
(1) How many vertices can still communicate after the loss of links or nodes?
(2) How difficult is it to reconnect the network?
Let S be a set of edges or vertices of G. Question (1) is sometimes analyzed by considering $\tau(G-S)$, the order (number of vertices) of a largest component of $G-S$. Question (2) is sometimes analyzed by considering $\omega(G-S)$, the number of components of $G-S$. The integrity (edge-integrity) minimizes $\mid S \mid + \tau(G-S)$ over all vertex (edge) sets S; the toughness (edge-toughness) minimizes $\mid S \mid /[\omega(G-S)]$ over all vertex (edge) sets S. The tenacity (edge-tenacity) minimizes $[\mid S \mid + \tau(G-S)]/[\omega(G-S)]$ over all vertex (edge) sets S. In toughness, the "cost" to an "attacker" of destroying S is the size of S and the "reward" is measured by the number of components left after destroying S (since creating more components makes it harder to reconnect a network). In tenacity, the "cost" also takes into account the size of the largest remaining component, since a larger remaining component means the "attack" was not quite as successfull. Integrity disregards "reward." An "attacker" wishes to make the ratio of cost to reward as small

as possible, whereas a "network designer wishes to make the smallest such ratio as large as possible.
In this paper, we study the edge-tenacity of graphs. Barry L. Piazza, Fred S. Robert, Sam K. Stueckl studied " Edge-Tenacious Networks" [40].
Given a set S of edges of G, the score of S is defined as $sc(S) = [|S|+\tau(G-S)]/[\omega(G-S)]$. Formally, the edge-tenacity of a graph G is defined as $T'(G) = min\{sc(S)\}$, where the minimum is taken over all edge-sets S of G. A subset S of $E(G)$ is said to be a $T' - set$ of G if $T'(G) = sc(S)$. Note that if G is disconnected, then the set S may be empty. Throughout this paper, we use ω and τ to represent $\omega(G-S)$ and $\tau(G-S)$, respectively, when G and S are clear from the context. We also use p and q to represent the number of vertices (order) and the number of edges (size), respectively, of a graph. The edge-connectivity of G will be denoted $\lambda = \lambda(G)$. Definitions and notation not otherwise defined here can be found in [6].
A graph is called edge-tenacious if $T'(G) = sc(E(G))$. Edge-tenacious graphs are somewhat analogous to honest graphs, as introduced in [2]. They can be considered very stable, because to minimize the ratio of cost to reward, an attacker needs to destroy all of the edges in the network. Thus, attacks tend to be "expensive" and so the networks are relatively invulnerable. The main results of this paper will show that many network topologies used to design highly reliable computer, communication, and transportation networks are edge-tenacious.
In the remainder of this section, we give some elementary bounds on the edge-tenacity of a graph. In Section II, we show that many graphs are edge-tenacious, and, in particular, the r-regular, r-edge-connected graphs are. In Section III, we consider the consequences of relaxing either of the conditions of r-regularity or r-edge-connectivity.
We now give several elementary bounds on the edge-tenacity of a graph.

Theorem 1.1. If G is connected and $S \subseteq E$, then $sc(S) \geq 1$ with equality if and only if G is a tree and $S = E$.

Proof. Let $S \subseteq E$. Since G is connected, $|S| \geq \omega - 1$. Thus, if $\tau \geq 2$, $sc(S) \geq (\omega+1)/\omega > 1$. If $\tau = 1$, then $|S| = q$, $\omega = p$, and $sc(S) = (q+1)/p \geq 1$ with equality if and only if $q = p - 1$. $\square$

Corollary 1.1. If G is connected, then $T'(G) \geq 1$ with equality if and only if G is a tree.

It follows from Corollary 1.1 that every tree is edge-tenacious, i.e., every connected graph with $q = p - 1$ is edge-tenacious. We shall show below that every connected graph with $q \leq p + 1$ is edge-tenacious, and this result is the best possible.

Theorem 1.2. [40], If G is a spanning subgraph of H, then $T'(G) \leq T'(H)$.

Proof. Let S be a subset of $E(H)$ and let $S' = S \cap E(G)$.Then, $|S'| \leq |S|$, $\tau(G - S') \leq \tau(H - S)$, and $\omega(G - S') \geq \omega(H - S)$. The result follows from the definition. $\square$

Theorem 1.3. For any graph G, $T'(G) \geq (\lambda/2) + (1/p)$.

Proof. Let S be a $T' - set$ that leaves ω components. Since G has edge-connectivity λ, each of these components must have at least λ edges of S with one end in the component. Thus, $|S| \geq (\lambda\omega)/2$ and $sc(S) \geq (\lambda/2) + (\tau/\omega)$. Since $\tau \geq 1$ and $\omega \leq p$, it follows that $T'(G) \geq (\lambda/2) + (1/P)$. □

(The reader might wish to compare the result of [11] that the edge-toughness is exactly $\lambda/2$.)

Theorem 1.4. For all G, $T'(G) \leq (q + 1)/p$.

Proof. The removal of E will leave p isolated vertices. □

Recall that a graph is edge-tenacious if $T'(G) = sc(E(G)) = \frac{q+1}{p}$. We will call G strictly edge-tenacious if E is the unique set whose score equals $T'(G)$. We now state one result relating edge-tenacious and strictly edge-tenacious graphs and then prove several results giving insight into the structure of $G - S$, where S is a T'-set.

Theorem 1.5. If G is edge-tenacious and $gcd(p, q + 1) = 1$, then G is strictly edge-tenacious.

Proof. If G is edge-tenacious, then for any $T' - set$ S of G, $sc(S) = [|S| + \tau(G - S)]/[\omega(G - S)] = (q + 1)/p$. But since $\omega(G - S) \leq p$ and $gcd(p, q + 1) = 1$, we have $w(G - S) = p$ and so $S = E$. □

The following theorem gives the possible relationships between scores for two arbitrary subsets of E. This result gives us a very usefhl tool for deciding whether a set is a $T' - set$.

Theorem 1.6. Let S and S' be subsets of E such that $|S'| - |S| = a$, $\omega(G - S') - \omega(G - S) = b$ and $\tau(G - S') - \tau(G - S) = -c$. Then,
(i) $sc(S') < sc(S)$ if and only if $[(a - c)/b] < sc(S)$,
(ii) $sc(S') = sc(S)$ if and only if $[(a - c)/b] = sc(S)$,
and
(iii) $sc(S') > sc(S)$ if and only if $[(a - c)/b] > sc(S)$.

Proof. Since $sc(S') = [(|S| + a) + (\tau(G - S) - c)]/[\omega(G - S) + b]$, the results follow from basic algebraic manipulations. □

Note that if S is a subset of S' we are considering the case where the deletion of the a additional edges in $S' - S$ creates b additional components and reduces the order of a largest remaining component by c. Throughout the remainder of this paper, $a, b, andc$

will be as defined in the above theorem. This result yields several interesting and useful corollaries about the components of $G - S$ for a $T' - set\ S$.

Corollary 1.2. If S is a $T' - set$ and C is a nontrivial component of $G - S$, then $\lambda(C) \geq T'(G)$.

Proof. Otherwise, choose $S' = S \cup T$, where T is a minimum edge-cut-set of C. Then, $a = \lambda(C)$, $b = 1$, and $c \geq 0$, so $[(a - c)/b] \leq \lambda(C) < T'(G)$. Thus, $sc(S') < sc(S)$, a contradiction to the fact that S is a $T' - set$. □

Corollary 1.3. If S is a $T' - set$ of a connected graph G, then $G - S$ contains no bridge.

Proof. Suppose e is a bridge of $G - S$. Then, by Theorem 1.1, $sc(S) > 1$. Let $S' = S \cup \{e\}$. Then, $a = 1$, $b = 1$, and $c \geq 0$, so $[(a - c)/b] \leq 1$, a contradiction. □

Corollary 1.4. If S is a $T' - set$ of a connected, nontrivial graph G and $G - S$ has a unique component C of maximum order, then C is 3-edge-connected.

Proof. Since $G - S$ has a unique component of maximum order, $S \neq E$ and $sc(S) > 1$. Suppose $\lambda(C) \leq 2$. Let $S' = S \cup T$, where T is a minimum edge-cut-set of C. Then, $a = \lambda(C) \leq 2$, $b = 1$, and $c \geq 1$, so $[(a - c)/b] \leq 1$, a contradiction. □

Corollary 1.5. If there exists a $T' - set\ S$ of a connected graph G with $\tau(G - S) = 3$, then $G - S$ contains at least three components that are copies of K_3.

Proof. Since, by Corollary 1.3, $G - S$ contains no bridge, maximum order components must be copies of K_3. By Corollary 1.4, $G - S$ contains at least two copies of K_3. Thus, suppose that $G - S$ contains exactly two copies of K_3. Let $S' = S \cup E(2K_3)$. Then, $a = 6$, $b = 4$, and $c = 2$, since S has no bridges, so $[(a - c)/b] = 1 < T'(G)$, a contradiction. □

Corollary 1.6. If there exists a $T' - setS$ of a connected graph G with $\tau(G - S) = 4$, then $G - S$ contains at least two nontrivial components.

Proof: Suppose $G - S$ has a unique nontrivial component C. By Corollary 1.4, $C \cong K_4$. Let $S' = S \cup E(K_4)$. Then, $a = 6$, $b = 3$, and $c = 3$, so that $[(a - c)/b] = 1 < T'(G)$, a contradiction. □

The next theorem gives an improved lower bound on the edge-tenacity of a graph. This bound is especially useful for non-edge-tenacious graphs.

Theorem 1.7. If S is a $T' - set$ of a connected graph G with $p \geq 9$ and $\omega(G - S) \neq p$, then $T'(G) \geq (\lambda/2) + [3/(p - 6)]$.

Proof. Let S be a $T' - set$ leaving $\omega \neq p$ components. As in the proof of Theorem 1.3, $sc(S) \geq (\lambda/2) + (\tau/\omega)$. From Corollaries 1.3, 1.5,and1. 6, we have $(\tau/\omega) \geq min\{3/(p-6), 4/(p-5), k/(p-k+1)|5 \leq k \leq p\}$. It is easy to show that for $p \geq 9$ this minimum is $3/(p-6)$. □

We now define a class of graphs that will be useful as examples or counterexamples throughout this paper. For any graph H, define the class of graphs $G(H,m)$ as the class of graphs G having $V(G) = V(H) \cup \{v_1, v_2, ..., v_m\}$ and where $E(G)$ is $E(H)$ along with any $\omega(H) + m - 1$ additional edges that result in a connected graph. More generally, for $t \geq 2$, define the class of graphs $G_t(H,m)$ as the set of graphs G having $V(G) = V(H) \cup \{v_1, v_2, ..., v_m\}$ and where $E(G)$ is $E(H)$ along with any $(t/2)[\omega(H)+m]$ additional edges that result in a t-edgeconnected graph.

Remark 1.1. For certain graphs G and certain values of m, the class $G_t(H,m)$ may be empty.

Remark 1.2. If there exists a $t - regular$, $t - edge - connected$ graph of order $\omega(H)+m$, then for appropriate graphs H, $G_t(H,m)$ will be nonempty. For example, if H is K_{t+1}, then $G_t(H,m)$ is nonempty for appropriate values of m. In particular, it is nonempty if t is odd and m is odd and empty if t is odd and m is even.

Remark 1.3. None of the additional $(t/2)[\omega(H)+m]$ edges will be added to any component of H.

We note that the bound from Theorem 7 is the best possible since equality is obtained for all graphs in the class $G_2(3K_3, m)$ for $m \geq 15$. This is shown by observing that if S consists of the edges added to $E(H)$, then if $m \geq 15$, $sc(E(G)) \geq sc(S)$ and $sc(S) = (\lambda/2) + [3/(p-6)]$.

1.2 EDGE-TENACIOUS NETWORKS:

Unless otherwise stated, all graphs henceforth will be connected. The following two theorems give lower bounds on the order and size of non-edge-tenacious graphs. These results are shown to be the best possible.

Theorem 1.8. If the order of a connected graph G is at most 10, then G is edge-tenacious. Furthermore, if the order is at most 9, then G is strictly edge-tenacious.

Proof. If there exists a graph G that is not edge-tenacious, then there exists a non-edge-tenacious graph G' with T'-set S [so $sc(S) < sc(E)$] having the following properties:
(i) The components of $G' - S$ are complete, and
(ii) $|S| = \omega(G' - S) - 1$.

This is because adding edges to components of $G-S$ does not increase $sc(S)$ and arranging these components in a connected graph using as few edges between components as possible (i.e., arranging them in a tree structure) minimizes the score.
The set of orders of the components of $G'-S$ forms a partition P of p, with maximum element max P, which satisfies the following:

(i) P has no 2's, by Corollary 1.3;
(ii) if max $P=3$, then there are at least three 3's, by Corollary 1.5; and
(iii) if max $P=4$, then there is at least one other element of P larger than 1, by Corollary 1.6.

Hence, $sc(S)=(|P|-1+maxP)/|P|$. It is easy to check that all such partitions of $p\leq 10$ have $sc(S)\geq sc(E)$, a contradiction. Of these, the only partitions with max $P>1$ that have $sc(S)=sc(E)$ are (6,1,1,1, 1) and (7,1,1,1) and have $p=10$. □

Theorem 1.8 is the best possible result, since the graphs in the classes $G(K_6,5)$, $G(K_7,4)$, $G(K_7-e,4)$, $G(K_8,3)$, and $G(K_8-e,3)$ are non-edge-tenacious graphs with order 11. (In fact, these graphs are the only non-edgetenacious graphs of order 11.) Note that Theorem 1.8 also improves Theorem 1.7, since if $p\leq 9$, then G is strictly edge-tenacious and the premise is vacuously true.

Theorem 1.9. If the size of a connected graph G is at most 17, then G is edge-tenacious.

Proof. Suppose there exists a non-edge-tenacious graph G with $q\leq 17$. Then, there exists a non-edge-tenacious graph G' with T'-set S [so $sc(S)<sc(E)$] having the property that $|S|=\omega(G'-S)-1$.
The set of orders of the components of $G'-S$ forms a partition P of p, with maximum element max P, which satisfies the following:

(i) P has no 2's, by Corollary 1.3;
(ii) if max $P=3$, then there are at least three 3's, by Corollary 1.5; and
(iii) if max $P=4$, then there is at least one other element of P larger than 1, by Corollary 1.6.

Now, by Theorem 1.8, $p\geq 11$, and since G is connected, $p\leq 18$. But if $p=18$, then G is a tree and every edge is a bridge, so, by Corollary 1.3, G must be edge-tenacious. Thus, $p\leq 17$. Clearly, since $q\leq 17$, $(18/p)>sc(S)=[(|P|-1+maxP)/|P|]$, which holds if and only if $|P|>[(\max P-1)p/(18-p)]$.lt is easy to see that the only possible partitions satisfying this are as follows:
(i) (3,3,3,1,1), $p=11$, and $q=13$.
(ii) (4,4,1,1,1), $p=11$, and $12\leq q\leq 16$.
(iii) (4,3,1,1,1,1), $p=11$, and $q=14$; since the 4 corresponds to a unique largest component, this component is necessarily a K_4 since, by Corollary 1.4, it must be 3-edge-

connected. The values for q in cases (iv), (vi), and (viii) are found using similar arguments.

(iv) (5,1,1,1,1,1,1), $p = 11$, and $14 \leq q \leq 16$.
(v) (3,3,3,1,1,1), $p = 12$, and $q = 14$.
(vi) (4,3,1,1,1,1,1), $p = 12$, and $q = 15$.
(vii) (3,3,3,1,1,1,1), $p = 13$, and $q = 15$.
(viii) (4,3,1,1,1,1,1,1), $p = 13$, and $q = 16$.
(ix) (3,3,3,1,1,1,1,1), $p = 14$, and $q = 16$.

It is easy to check that the graphs corresponding to these partitions are edge-tenacious and so we have a contradiction. □

This result is the best possible since the graphs in the classes $G(2K_4, 5)$ and $G(K_5, 8)$ are non-edge-tenacious graphs with $q = 18$. (In fact, these graphs are the only non-edge-tenacious graphs of size 18.)

The following two lemmas and theorem show that if a graph is sparse enough then it is edge-tenacious. Again, the theorem will be shown to be best possible.

Lemma 1.1. If G is a connected graph with $q = p+1$ and $S = \{e|e \text{ is a bridge in } G\}$, then the subgraph induced by the nontrivial components of $G-S$ has one of the following forms:

(i) The union of two disjoint cycles.
(ii) Two cycles whose intersection is a vertex.
(iii) Two cycles whose intersection is a path.

Proof. Let T be a spanning tree of G and let u_1v_1 and u_2v_2 be the edges of $G - E(T)$. There is a unique $u_1 - v_1$ path P_1 in T and a unique $u_2 - v_2$ path P_2 in T. Let $S = \{e|e$ is a bridge in $G\}$. Then, the subgraph H induced by the nontrivial components of $G - S$ will be the graph induced by the vertices on the paths P_1 and P_2. If P_1 and P_2 do not intersect, H will be the union of two cycles. If P_1 and P_2 have a single vertex in common, H will be two cycles whose intersection is this vertex. If P_1 and P_2 have more than one vertex in common, their intersection must be a path; otherwise, T would contain a cycle. Thus, in this case, H will be two cycles whose intersection is this path. □

Lemma 1.2. If G is connected and $q \leq p+1$, then no subgraph of G is 3-edge-connected.

Proof. Suppose that H is a 3-edge-connected subgraph of G. Then, $q(H) \geq [3p(H)]/2 \geq p(H) + 2$, since $p(H) \geq 4$. Since G is connected, there are at least $p(G - H)$ edges in G that are not in H. Hence, $q(G) \geq q(H) + p(G - H) \geq p(H) + 2 + p(G - H) = p(G) + 2$, a contradiction. □

Theorem 1.10. If G is a connected graph with $q \leq p+1$, then G is strictly edge-tenacious.

Proof. Since G is connected, $q \geq p-1$. The proof is done in three cases:

First, if $q = p-1$, then G is a tree and so every edge is a bridge and, hence, by Corollary 1.3, G is strictly edge-tenacious.
If $q = p$, then G is unicyclic and so every edge not on the cycle is a bridge. Thus, every T'-set for G will contain all noncycle edges. Also, if any cycle edge is deleted, then all of the remaining cycle edges will then be bridges, and so will also be deleted. Hence, the only possible T'-sets for G are E and $S = \{uv|uv$ is not on the cycle in $G\}$. Consider the set S. The graph $G - S$ has a unique largest component that by Corollary 1.4 must be 3-edge-connected, a contradiction to Lemma 1.2. Hence, the only possible T'- set is E and so G is strictly edge-tenacious.
Finally, consider $q = p+1$. Suppose that $S \neq E$ is a T'-set of G. As in the above cases, S must contain all bridges in G. By Lemma 1.2, no subgraph of G is 3-edgeconnected and so, by Corollary 1.4, $G - S$ cannot have a unique largest component. Since none of the components can contain any bridges, $G - S$ must consist of 2 disjoint cycles of the same order, say x, along with isolated vertices. Considering S' from Theorem 1.6 as E, we have $a = 2x$, $b = 2x - 2$, and $c = x - 1$. Hence, $(a - c)/b = (x + 1)/(2x - 2) \leq 1 < sc(S)$, by Theorem 1.1. This implies that $sc(E) < sc(S)$, by Theorem 1.6, a contradiction. Thus, G is strictly edge-tenacious. □

The above result is the best possible, since the graphs in the classes $G(3K_3, m)$, for $m \geq 10$, have $q = p+2$ and are non-edge-tenacious. Note that $G(3K_3, 9)$ is edgetenacious but not strictly edge-tenacious.
We now prove that all r-regular, r-edge-connected graphs are edge-tenacious.

Theorem 1.11. If G is r-regular and r-edge-connected, then G is strictly edge-tenacious.

Proof. If $p < 9$, the result follows by Theorem 1.8. Suppose that $p \geq 9$. From Theorems 1.3 and 1.4, $(r/2) + (1/p) \leq T'(G) \leq (q + 1l)/p = (r/2) + (1/p)$. Thus, $T'(G) = sc(E(G))$ and G is edge-tenacious. Hence, by Theorem 1.7, G is strictly edge-tenacious. This is because $T'(G) = (r/2) + (1/p) < (\lambda/2) + [3/(p - 6)]$. □

It follows from Theorem 1.11 that many of the topologies widely used to design highly reliable computer, communication, and transportation networks are edge-tenacious. We close this section with a number of results illustrating this point. A further such result is contained in Corollary 1.12 below.

Corollary 1.7. The complete graph K_p and the complete n-partite graph $K_{m,m,\ldots,m}$, where $p = nm$, are strictly edge-tenacious.

In the next four results, we use concepts of power and product as defined in [6] and inflation as defined in [11].

Corollary 1.8. The nth power of the p-cycle, C_p^n, is strictly edge-tenacious for $1 \leq n \leq \lfloor p/2 \rfloor$.

Corollary 1.9. If G_i is r_i-regular and r_i-edge-connected for $i = 1, 2, ..., n$, then $G_1 \times G_2 \times ... \times G_n$, is strictly edge-tenacious.

Corollary 1.10. The n -cubes are strictly edge-tenacious.

Corollary 1.11. If G is an inflation of an r-regular r-edgeconnected graph, then G is strictly edge-tenacious.

As an aside, we show that all of the complete bipartite graphs, $K_{m,n}$ are strictly edge-tenacious. Note that if $n = m$, then this result follows from Corollary 1.7.

Theorem 1.12. For $m \leq n$, $K_{m,n}$ is strictly edge-tenacious.

Proof. Suppose that $S \neq E$ is a T'-set. Note that any nontrivial component of $G - S$ must be of the form $K_{x,y}$. By Theorem 1.3 and Corollary 1.2, each of these nontrivial components must have edge-connectivity at least $(m/2) + [1/(m+n)]$. It follows that $G - S$ has at most one nontrivial component. So, $sc(S) = (mn - xy + x + y)/(m + n - x - y + 1)$. Using derivatives, it is easy to see that the minimum value for this function occurs at one of the four points (x, y) obtained when $x \in \{1, m\}$ and $y \in \{1, n\}$. It is now easy to check that $sc(E) < sc(S)$, a contradiction. Hence, E is the only T'-set of $K_{m,n}$. $\square$

We conjecture the following:

Conjecture 1. The graph $K_{m_1,m_2,...m_n}$ is, strictly edge-tenacious.

1.3 RELAXATION OF THEOREM 1.11:

In this section, we explore what happens when we weaken either the r-edge-connected hypothesis or the r-regular hypothesis in Theorem 1.11.
We first define another class of graphs that will be useful as examples or counterexamples. For $r \geq 2$, $1 \leq n \leq (r/2)$, $m \geq 2$, define the graph $F(r, m, n)$ as follows:
$V(F(r, m, n)) = \bigcup_{k=1}^{m} V_k$ and

$$E(F(r, m, n)) = \bigcup_{k=1}^{m} (B_k - M_k) \cup \bigcup_{k=1}^{m} C_k;$$

where $V_k = \{v_{i,k} | 0 \leq i \leq r\}$, $B_k = \{v_{i,k}v_{j,k} | 0 \leq i < j \leq r\}$, $M_k = \{v_{i,k}v_{r-i,k} | 0 \leq i \leq n-1\}$, $C_k = \{v_{i,k}v_{r-i,k+1} | 0 \leq i \leq n-1\}$ and all arithmetic on k is done modulo m. Let

$G_k =< V_k >$. The graph $F(r, m, n)$ is constructed from m copies of K_{r+1} by deleting a matching of size n from each complete graph, giving $\bigcup_{k=1}^{m} G_k$, and then adding n edges connecting "corresponding" reduced degree vertices in G_k and G_{k+1}. The resulting graph is r-regular and has $\lambda = 2n$.
Note that if S is $\bigcup_{k=1}^{m} C_k$, then $T'(F(r, m, n)) \leq sc(S) = n + (r + 1)/m$. This bound is sufficient to prove Theorem 1.14. However, for further results, it is helpful to get an exact formula for $T'(F(r, m, n))$.

Theorem 1.13. [40], For $r \geq 2$, $1 \leq n \leq (r/2)$,and $m \geq 2$,

$T'(F(r, m, n)) =$

$$\begin{cases} n+\frac{r+1}{m}, & n<\frac{r}{2},\ m>\frac{2r(r+2)}{(r+1)(r-2n)} \\ \frac{r}{2}+\frac{1}{m(r+1)}, & otherwise \end{cases}$$

Theorem 1.14. If r is odd, there are r-regular graphs that are $(r-1)$-edge-connected and not edge-tenacious. If r is even, there are r-regular graphs that are $(r-2)$-edgeconnected and not edge-tenacious.

As an aside, we may use the graphs $F(r, m, n)$ to examine the possible values of the ratio $R(G) = [sc(E)]/[sc(S)]$, when G is a non-edge-tenacious graph with T'-set S. If $r \geq 3$, $m > [2r(r + 2)]/[(r + 1l)(r - 2n)]$,and $n < (r/2)$, then $G = F(r, m, n)$ is non-edge-tenacious with $T'(G) = n + (r + 1)/m$, by Theorem 1.13. Hence, $R(G) = [mr(r + 1) + 2]/[2(r + 1)(mn + r + 1)]$ and so $R(G)$ tends to $r/(2n) = r/\lambda$ as m goes to ∞. In the case where $n = 1$, so $\lambda = 2$, this limit is $r/2$, while if $n = \lfloor (r - 1)/2 \rfloor$ (so λ is either $r - 1$ or $r - 2$) this limit is either $r/(r - 1)$ or $r/(r - 2)$. Hence, in the first case, the limit tends to ∞ as r goes to ∞ , and in the second case, the limit tends to 1 as r approaches ∞ . Therefore, we can see that $R(G)$ can get arbitrarily large or arbitrarily close to 1 , the two possible extremes.
We may also use the graphs $F(r, m, n)$ to prove the following theorem, suggested and proved independently by Fetterman .

Theorem 1.15. For any positive rational number a/b , $a \geq b$, there exists a graph G such that $T'(G) = a/b$.

Proof. If $a = b$, then the result follows from Corollary 1. So suppose $a > b$. Then, consider the graph $F(6a - 6b - 1, 6b, 1)$. Simple algebra shows that the first set of conditions from Theorem 1.13 are satisfied. Hence, $T'(F(6a - 6b - 1, 6b, 1)) = (6a)/(6b) = a/b$. □

Next, we consider relaxing the condition in Theorem 1.11 that G is r-regular, $r \geq 2$, but maintaining the condition that $\lambda(G) = r$. Note that under this condition each vertex must have degree at least r. Our relaxation of the regularity condition is in the form of allowing the sum of the degrees to increase above pr. Given a fixed graph with $\lambda = r \geq 2$, consider $\sum_{i=1}^{p} d_i = pr + \epsilon$, where d_i, is the degree of vertex v_i. Then, define ϵ_i, to be the largest ϵ such that all graphs with $\lambda = r$ and $\sum_{i=1}^{p} d_i \leq pr + \epsilon$ are edge-tenacious. Note that we may define a similar concept for $\lambda = 1$. In this case, we will consider ,$\sum_{i=1}^{p} d_i = 2(p-1) + \epsilon$ for a fixed graph with $\lambda = 1$. Define ϵ_i to be the largest ϵ such that all graphs with $\lambda = 1$ and ,$\sum_{i=1}^{p} d_i \leq 2(p-1) + \epsilon$ are edge-tenacious. We now prove several results giving bounds on ϵ and ϵ_r.

Theorem 1.16. $\epsilon_l = 4$.
Proof. First note that ϵ_l must be even. By Theorem 1.10, $\epsilon_l \geq 4$. Also, for $m \geq 10$, the graphs in $G(3K_3, m)$ are non-edge-tenacious, implying that $\epsilon_l < 6$ and, hence, the result. □

Theorem 1.17. Let G be a connected graph with $\lambda = r$ and $\sum_{i=1}^{p} d_i = pr + \epsilon$. If $c < 4[(p+3)/(p-6)]$, then G is strictly edge-tenacious.

Proof. First, if $p \leq 9$, then G is strictly edge-tenacious, by Theorem 1.8. Thus, assume $p > 9$. Now,

$$sc(E) = \frac{\frac{pr+\epsilon}{2} + 1}{p} < \frac{r}{2} + \frac{3}{p-6}$$

if and only if $\epsilon < 4[(p+3)/(p-6)]$. It follows then from Theorem 1.7 that for any T'-set S, $\omega(G-S) = p$ and so G is strictly edge-tenacious. □

The next result concerns the Harary graphs $H(p,k)$ defined in [24] and gives another example of a well-known network topology that is strictly edge-tenacious.

Corollary 1.12. The Harary graphs $H(p,k)$ are strictly edge-tenacious.

Theorem 1.18. For $r \geq 2$, $\epsilon_r \geq max\{r, 4\}$.

Proof. Let G be a non-edge-tenacious graph with $\lambda = r$, $\sum_{i=1}^{p} d_i = pr + \epsilon$, and T'-set S. We know from the proof of Theorem 1.3 that $sc(S) \geq (r/2) + (\tau/\omega)$ and so $[(q+1)/p] = (r/2) + [(\epsilon+2)/(2p)] > sc(S) \geq (r/2) + (\tau/\omega)$. Thus $\epsilon > (2p\tau - 2\omega)/\omega$.
First, since S is a T'-set we have $\tau \geq 3$, by Corollary 1.3. Hence, $\epsilon > [(6p)/\omega] - 2 > 4$. Hence, $\epsilon_r \geq 4$. Finally, suppose C is a nontrivial component of $G - S$. Then, from Corollary 1.2 and Theorem 1.3, $\lambda(C) \geq T'(G) > (r/2)$ and so $\tau \geq p(C) > (r/2) + 1$. This implies that $\omega \leq p - \tau + 1 < p - (r/2)$. Thus,

$$\epsilon > \frac{2p(r+2)}{2p-r} - 2 = \frac{r(2p+2)}{2p-r} > r$$

Hence, $\epsilon_r \geq r$. □

Theorem 1.19. For $r \geq 2$, $\epsilon_r \leq 2r$.

Proof. For $r \geq 2$, consider any graph G in the class $G_r(3K_{r+1} - M, k)$, where M is a matching of size m. Thus, $p = 3(r+1) + k$, $pr = 3r(r+1) + kr$, $\sum_{i=1}^{p} d_i = 3r(r+1) + r(k+3) - 2m$ and hence $\epsilon = 3r - 2m$. Let S be the set of $(r/2)(k+3)$ additional edges that were added to attain edge-connectivity r in the construction of G. Now, it is easy to show that

$$sc(S) = \frac{\frac{r(k+3)}{2} + r + 1}{k+3} < \frac{\frac{3r(r+1)+r(k+3)-2m}{2} + 1}{3(r+1) + k} = sc(E)$$

if and only if $k(r - 2m) > 6r^2 + 3r + 6m$. Assuming this inequality holds, since $6r^2 + 3r + 6m > 0$, we have $r - 2m > 0$ and so $m \leq \lfloor (r-1)/2 \rfloor$. Since we are interested in minimizing ϵ, we choose $m = \lfloor (r-1)/2 \rfloor$ and so

$$(\epsilon = \begin{cases} 2r+1 & if\ \ r\ is\ odd \\ 2r+2 & if\ \ r\ is\ even \end{cases}$$

If r is odd, we must have k odd and $k > 6r^2 + 6r - 3$, whereas if r is even, we must have $k > 3r^2 + 3r - 3$. If k and m are chosen to satisfy these conditions, G is nonedge-tenacious with

$$\epsilon = \begin{cases} 2r+1 & if\ \ r\ is\ odd \\ 2r+2 & if\ \ r\ is\ even \end{cases}$$

thus;

$$\epsilon_r \begin{cases} 2r & if\ \ r\ is\ odd \\ 2r+1 & if\ \ r\ is\ even \end{cases}$$

Finally, if r is even, then pr is even and so ϵ_r is even and, hence $\epsilon_r \leq 2r$. □

Corollary 1.13. $\epsilon_2 = 4$.
From the last two theorems, for $r \geq 3$ we have $r \leq \epsilon_r \leq 2r$. We conjecture the following:

Conjecture 1.2. For $r \geq 3$, $\epsilon_r = 2r$.

Besides the two conjectures given above, there are a number of open problems. These include problems involving finding, characterizing, or recognizing non-edge-tenacious graphs that are maximal or minimal with respect to this property. For example, it can be shown that $G(K_5, 8)$ is both a maximal and a minimal non-edge-tenacious graph since either deleting or adding an edge will make this graph edge-tenacious. Other problems are in finding the edge-tenacity of non-edge-tenacious graphs and in relating edge-tenacity to other parameters, such as the diameter and edge-integrity.

II Tenacity of Complete Graph Products and Grids

One way of measuring the stability of a network (computer, communication, or transportation) is through the ease (or the cost) with which one can disrupt the network. The connectivity gives the minimum cost to disrupt the network, but it does not take into account what remains after disruption. One can say that the disruption is more successful if the disconnected network contains more components and much more successful if, in addition, the components are small. As nicely explained in [15] and [40], one can associate the cost with the number of vertices destroyed to get small components and associate the reward with the number of components remaining after destruction. The tenacity measure is a compromise between the cost and the reward by minimizing the cost:reward ratio. Thus, a network with a large tenacity performs better under external attack. In this sense, the following parameters are successively better for the measurement of stability; see [15] for a comparison. Before we formally define these parameters, we recall some standard notation and terminology from [5] and [15].
Edge-analogs of these concepts are defined similarly; see [3, 4, 5, 40].
Let G_1,G_2, . . . ,G_r be graphs. The Cartesian product $G_1 \times G_2 \times ... \times G_r$ has vertex set $V(G_1) \times V(G_2) \times ... \times V(G_r)$ with two vertices $u = (g_1, g_2, ..., g_r)$ and $v = (h_1, h_2, ..., h_r)$ adjacent iff for exactly one i, $g_i \neq h_i$ and (g_i, h_i) is an edge in G_i.
As usual, let P_n, C_n, and K_n, respectively, denote the path, cycle, and complete graph on n vertices. It is well known that Cartesian products like hypercubes $(K_2 \times ... \times K_2)$, grids $(P_{n_1} \times ... \times P_{n_k})$, and tori $(C_{n_1} \times ... \times C_{n_k})$ are highly recommended for the design of interconnection networks in multiprocessor computing systems. Hence, there is a large literature containing the study of the stability of these graphs. In [15], among other results, the following theorem and conjecture are stated:
Theorem A [15]. If $m \leq n$, then
$\frac{m^2+mn-2m+2}{2m} \leq T(K_m \times K_n) \leq \frac{mn-n+\lceil \frac{n}{m} \rceil}{m}$.
Conjecture [15]. If $2 \leq m \leq n$, then $T(K_m \times K_n) = (mn - n + \lceil \frac{n}{m} \rceil)/m$.
S. A. Choudum, N. Priya proved this conjecture, [9]. In this paper, we show the proof of this conjecture and find the tenacity of grid graphs $p_{n_1} \times p_{n_2} \times \cdots \times p_{n_k}$.

2 Tenacity of $K_m \times K_n$:

Theorem 2.1. If $1 \leq m \leq n$, then $T(K_m \times K_n) = \frac{mn-n+\lceil \frac{n}{m} \rceil}{m}$.
Proof. Let $G = K_m \times K_n$. For any $S \subseteq V(G)$, the components of $G - S$ have the following property:
(1) By the definition of $K_m \times K_n$, the neighborhood of the vertex (i, j) is

$$\{(i, 1), (i, 2), ..., (i, n)\} \cup \{(1, j), (2, j), ..., (m, j)\} - \{(i, j)\}.$$

So, if $S \subseteq V(G)$ and (i, j) is a vertex of a component C in $G - S$, then for every other component D of $G - S$, $V(D) \cap \{(i, 1), (i, 2), ..., (i, n)\} = \phi$ and $V(D) \cap \{(1, j), (2, j), ..., (m, j)\} = \phi$. Hence, S contains every vertex of $\{(i, 1), (i, 2), ..., (i, n)\}$ not in $V(C)$ and every vertex of $\{(1, j), (2, j), ..., (m, j)\}$ not in $V(C)$.

Let F be the family of all T-sets A in G with the maximum number of components in $G-A$. Let S be an element of F with minimum order. We shall show that the components of $G-S$ satisfy the following properties (2)–(6) and complete the proof.
2) Every component C of $G-S$ is of the form $K_s \times K_t$. Let $C_1, C_2, ..., C_\omega$ be the components of $G-S$, with $|V(C_i)| = m_i \times n_i$, $i = 1, 2, ..., \omega$, where $\omega = \omega(G-S)$.
(3) Given any $i \in \{1, 2, ..., m\}$ (or $j \in \{1, 2, ..., n\}$), there exists a component containing some(i, p) [respectively, (p, j)], where $p \in \{1, 2, ..., n\}$ (respectively, $p \in \{1, 2, ..., m\}$).
(4) Clearly, by (1), (2), and (3), $\sum_{i=1}^{\omega} m_i = m$ and $\sum_{i=1}^{\omega} n_i = n$.
(5) For every component C_i, either $m_i = 1$ or $n_i = 1$.
(6) There is no component with $m_i > 1$ and $n_i = 1$.
Thus, all the components are of the form $K_1 \times K_{n_i}$, $|V(Ci)| = n_i$, $\omega(G-S) = m$, $|S| = mn - \sum_{i=1}^{m} n_i = mn - n$, and $\tau(G-S) \geq \lceil n/m \rceil$. We thus get the required lower bound:

$$T(G) = \frac{|S|+\tau(G-S)}{\omega(G-S)} \geq \frac{mn-n+\lceil \frac{n}{m} \rceil}{m}.$$

Proof of (2). It is enough if we show that whenever $(i, r), (i, s), (j, r) \in V(C)$ then $(j, s) \in V(C)$. On the contrary, suppose that $(j, s) \notin V(C)$. Define $S' = S - (j, s)$. By (1), $(j, s) \in S$, (j, s) is adjacent with $(i, s), (j, r)$ in $G-S'$ and $[C \cup \{(j, s)\}]$ is a component of $G-S'$, so $|S'| = |S| - 1$, $\omega(G-S') = \omega(G-S), \tau(G-S') \leq \tau(G-S) + 1$. Hence,

$$sc(S') = \frac{|S'|+\tau(G-S')}{\omega(G-S')} \leq \frac{|S|-1+\tau(G-S)+1}{\omega(G-S)} = sc(S)$$

and so S' is a T-set. But this contradicts the fact that $|S|$ is of minimum order.
Proof of (3). Assume the contrary; so, $S \supseteq \{(i, j) : 1 \leq j \leq n\}$ for some i, $1 \leq i \leq m$. Let C be a component of $G-S$ and (k, r) be an element of C. Define $S' = S - (i, r)$. Then, $[C \cup \{(i, r)\}]$ is a component in $G-S'$, $|S'| = |S| - 1$, $\tau(G-S') \leq \tau(G-S) + 1$, and $\omega(G-S') = \omega(G-S)$. Hence, S' is a T-set as above, contradicting the fact that S is of minimum order.
Proof of (5). Assume that there is a component $C_i = K_{m_i} \times K_{n_i}$ with $m_i > 1$ and $n_i > 1$. For notational convenience, let $m_i = s$, $n_i = t$. Without loss of generality, assume that $C_1 = K_s \times K_t$, where $V(K_s) = \{1, 2, ..., s\}$ and $V(K_t) = \{1, 2, ..., t\}$. Let $S' = S \cup \{(s, 1), (s, 2), ..., (s, t-1)\} \cup \{(1, t), (2, t), ..., (s-1, t)\}$. Then, $|S'| = |S| + s + t - 2$, and the components of $G-S'$ are $C_2, ..., C_\omega$ [where $\omega = \omega(G-S)$], a singleton component containing the vertex (s, t) and a component $[(V(K_s) - \{s\}) \times (V(K_t) - \{t\})] \subset C_1$. So, $\tau(G-S') \leq \tau(G-S)$ and $\omega(G-S') = \omega(G-S) + 1$. We now estimate the size of S. Let $[x, y]$ denote the set of all integers z, such that $x \leq z \leq y$. Since C_1 is a component in $G-S$, $S \supseteq [1, s] \times [t+1, n] \cup [s+1, m] \times [1, t]$. So, $|S| \geq s(n-t) + (m-s)t = s(n_2 + n + 3 + ... + n_\omega) + (m_2 + m_3 + ... + m_\omega)t$
since $\sum_{i=1}^{\omega} n_i = t + \sum_{i=2}^{\omega} n_i = n$ and $\sum_{i=1}^{\omega} m_i = s + \sum_{i=2}^{\omega} m_i = m$,
$\geq s(\omega - 1) + (\omega - 1)t = (\omega - 1)(s + t)$
Next, $\tau(G-S) \geq |C_1| = st \geq s + t - 2 \geq s + t - 2\omega$.
Adding the two inequalities, we get $|S| + \tau(G-S) \geq \omega(G-S)(s + t - 2)$. So,

$$\begin{aligned} sc(S') &= \tfrac{|S'|+\tau(G-S')}{\omega(G-S')} \\ &\leq \tfrac{|S|+\tau(G-S)+(s+t-2)}{\omega(G-S)+1} \\ &\leq \tfrac{|S|+\tau(G-S)}{\omega(G-S)} \end{aligned}$$
$$(since(s+t-2)\omega(G-S) \leq |S|+\tau(G-S)) = sc(S).$$

Hence, S' is a T-set with $\omega(G-S') > \omega(G-S)$, which is a contradiction to the choice of S.
Proof of (6). We first observe that if there is a component $C_i = K_{m_i} \times K_1$ with $m_i > 1$ then there is a component $C_j = K_1 \times K_{n_j}$ with $n_j > 1$; otherwise, $n_j = 1$ for every j, and we have the contradiction: $\omega + 1 \leq \sum_{i=1}^{\omega} m_i = m \leq n = \sum_{j=1}^{\omega} n_j = \omega$.
To prove (6), assume on the contrary that there is a component $C = K_s \times K_1$ with $s > 1$. By the above observation, there is a component $D = K_1 \times K_t$ with $t > 1$. Without loss of generality, assume that $V(C) = \{(1,a),(2,a),...,(s,a)\}$ and $V(D) = \{(b,1),(b,2),...,(b,t)\}$, where $b \notin \{1,2,...,s\}$ and $a \notin \{1,2,...,t\}$. Define $S' = S \cup \{(s,a)\} \cup \{(b,t)\} - \{(s,t)\}$.
Then, $|S'|+1$, $\omega(G-S') = \omega(G-S)+1$ with the new extra component being the singleton $\{(s,t)\}$, and $\tau(G-S') \leq \tau(G-S)$. We next estimate the size of S. Since $C = K_s \times K_1 (= C_k(say))$ is a component of $G-S$, $S \supset \{(s+1,a),(s+2,a),...,(m,a)\}$. So, $|S| \geq m-s = \sum_{j=1, j\neq k}^{\omega} m_j \geq \omega(G-S)-1$. Since $\tau(G-S) > 1$, we get $|S|+\tau(G-S) > \omega(G-S)$. But, then,

$$\begin{aligned} sc(S') &= \tfrac{|S'|+\tau(G-S')}{\omega(G-S')} \\ &\leq \tfrac{|S|+1+\tau(G-S)}{\omega(G-S)+1} \\ &\leq \tfrac{|S|+\tau(G-S)}{\omega(G-S)} \end{aligned}$$

$(since|S|+\tau(G-S) > \omega(G-S)) = sc(S)$.
We thus have a contradiction to the minimality of $sc(S)$.

2.1 Tenacity of grids $p_{n_1} \times p_{n_2} \times \cdots \times p_{n_k}$

To find the tenacity of grids, we require the tenacity of complete bipartite graph $K_{m,n}$ and paths.
Theorem B [15]. If $1 \leq m \leq n$, then $T(K_{m,n}) = (m+1)/n$.
Theorem 2.1. For every integer $n \geq 2$,

$$T(p_n) = \begin{cases} 1 & if\ n\, is\, odd \\ \frac{n+2}{n} & if\ n\, is\, even \end{cases}$$

Proof. At the outset, we observe that $\omega(P_n - S) \leq |S|+1$, for every$S \subseteq V(P_n)$. Let $1,2,...,n$ be the vertices and $(i,i+1)$ be the edges of P_n. Clearly, if H is a spanning subgraph of G, then $T(H) \leq T(G)$. Since,

$$p_n \subseteq \begin{cases} K_{\frac{n-1}{2},\frac{n+1}{2}} & if\ n\, is\, odd \\ K_{\frac{n}{2},\frac{n}{2}} & if\ n\, is\, even \end{cases}$$

it follows by Theorem B, that

$$T(p_n) \leq \begin{cases} 1 & if\ n\ is\ odd \\ \frac{n+2}{n} & if\ n\ is\ even \end{cases}$$

To establish the lower bound, we first claim that if S is a $T-set$ of P_n, then (i) every component of $P_n - S$ is K_1 or K_2, and (ii) there is at most one K_2 component. We assume the contrary and arrive at a contradiction. If $P_n - S$ contains a component $P_k = \{i+1, i+2, ..., i+k\}$, where $k \geq 3$, then defining $S' = S \cup \{i+2\}$, we have $|S'| = |S| + 1$, $\tau(P_n - S') \leq \tau(P_n - S)$, $\omega(P_n - S') = \omega(P_n - S) + 1$, and

$$sc(S') = \frac{|S'|+\tau(P_n-S')}{\omega(P_n-S')} \leq \frac{|S|+1+\tau(P_n-S)}{\omega(P_n-S)+1} < \frac{|S|+\tau(P_n-S)}{\omega(P_n-S)}$$

(since$|S| + \tau(P_n - S) \geq \omega(P_n - S) + 2) = sc(S)$, a contradiction to the minimality of $sc(S)$.

Next, if $P_n - S$ contains two K_2 components, say $(i, i+1)$ and $(j, j+1)$, where $j \geq i+3$, assume, without loss of generality, that $P_n - S$ has no edges $(r, r+1)$, where $i+3 \leq r \leq j-3$. Clearly, $i+2, j-1 \in S$. Let $S' = S \cup \{i+1, i+3, ..., j-2\} \cup \{j\} - \{i+2, i+4, ..., j-1\}$. Then, $|S'| \leq |S| + 1$, $\tau(P_n - S') \leq \tau(P_n - S), \omega(P_n - S') \geq \omega(P_n - S) + 1$,and

$$sc(S') = \frac{|S'|+\tau(P_n-S')}{\omega(P_n-S')} \leq \frac{|S|+1+\tau(P_n-S)}{\omega(P_n-S)+1} < \frac{|S|+\tau(P_n-S)}{\omega(P_n-S)}$$

(since$|S| + \tau(P_n - S) > \omega(P_n - S) + 2) = sc(S)$, a contradiction to the minimality of $sc(S)$.

We next complete the proof by taking a $T-set$ S of P_n and distinguishing two cases:

CASE 1.$\tau(P_n - S) = 1$.

Clearly, $|S| \geq \lfloor \frac{n}{2} \rfloor$, and $\omega(P_n - S) \leq \lceil \frac{n}{2} \rceil$. Hence, $T(P_n) = sc(S) = \frac{|S|+\tau(P_n-S)}{\omega(P_n-S)} \geq \frac{\lfloor \frac{n}{2} \rfloor + 1}{\lceil \frac{n}{2} \rceil} =$

$$\begin{cases} 1 & if\ n\ is\ odd \\ \frac{n+2}{n} & if\ n\ is\ even \end{cases}$$

CASE 2.$\tau(P_n - S) = 2$.

Clearly, $|S| \geq \lfloor \frac{(n-2)}{2} \rfloor$, and $\omega(P_n - S) \leq \lceil \frac{n}{2} \rceil$. Hence, $T(P_n) = \frac{|S|+\tau(P_n-S)}{\omega(P_n-S)} \geq \frac{\lfloor \frac{n-2}{2} \rfloor + 2}{\lceil \frac{n}{2} \rceil} =$

$$\begin{cases} 1 & if\ n\ is\ odd \\ \frac{n+2}{n} & if\ n\ is\ even \end{cases}$$

□

Theorem 2.1 was also proved by D.E. Mann, "The tenacity of trees", Ph.D. Thesis, Northeastern University, 1993.

Before proving our next theorem, we make a simple observation: If a graph G contains a Hamilton path, then so does $G \times P_n$.

Theorem 2.2. For all positive integers $n_1, n_2, ..., n_k$,

$$T(P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k}) = \begin{cases} 1 & if\ all\ n_i\ are\ odd \\ \frac{n_1 n_2 \cdots n_k + 2}{n_1 n_2 \cdots n_k} & if\ some\ n_i\ is\ even \end{cases}$$

Proof. By the above observation, it follows that$P_{n_1} \times P_{n_2} \times ... \times P_{n_k}$ contains a Hamilton path $P_{n_1 n_2 \ldots n_k}$. So, by Theorem A_2,

$$T(P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k}) \geq P_{n_1 n_2 \ldots n_k} = \begin{cases} 1 & if\ all\ n_i\ are\ odd \\ \frac{n_1 n_2 \cdots n_k + 2}{n_1 n_2 \cdots n_k} & if\ some\ n_i\ is\ even \end{cases}$$

If G is a bipartite graph with bipartition $[A, B]$ and H is a bipartite graph with bipartition $[C, D]$, then it is well known that $G \times H$ is a bipartite graph with bipartition $[(A \times C) \cup (B \times D), (A \times D) \cup (B \times C)]$. Hence, it follows that

$$P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k} \subseteq \begin{cases} K_{\frac{n_1 n_2 \ldots n_k - 1}{2}, \frac{n_1 n_2 \ldots n_k + 1}{2}} & if\ all\ n_i\ are\ odd \\ K_{\frac{n_1 n_2 \ldots n_k}{2}, \frac{n_1 n_2 \ldots n_k}{2}} & if\ some\ n_i\ is\ even \end{cases}$$

So, by Theorem B, we get

$$T(P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k}) = \begin{cases} 1 & if\ all\ n_i\ are\ odd \\ \frac{n_1 n_2 \cdots n_k + 2}{n_1 n_2 \cdots n_k} & if\ some\ n_i\ is\ even \end{cases}$$

□

Since a hypercube Q_n is the Cartesian product $P_2 \times P_2 \times \cdots \times P_2$ (n times), we have the following corollary,a result also proved by Stuart (see [15]).
Corollary 2.1 For every positive integer n. $T(Q_n) = \frac{(2^n+2)}{2^n}$. □

III On the edge-tenacity of the middle graph of a Networks

A communication network is modelled as an undirected and unweighted graph in which vertices represent the processing elements and edges represent the communication links. The stability of a network is of prime importance to network designers.As the network begins losing links or nodes, eventually it loses effectiveness. Communication networks are designed such that they are not easily disrupted under external attack and, moreover, such that they can easily be reconstructed if they are disrupted. These desirable properties of networks can be measured by various parameters such as connectivity and edge-connectivity. However, these parameters do not take into account what remains after the graph is disconnected. Consequently, a number of other parameters have recently been introduced in an attempt to cope with this. These include toughness and edge-toughness , integrity and edge-integrity, and tenacity.
We can say that the disruption is more successful if the disconnected network contains more components, and is much more successful if, in addition, the components are small. We can associate the cost with the number of edges destroyed to obtain small components and associate the benefit with the number of components remaining after destruction. The edge-tenacity is a compromise between the cost and the benefit obtained by minimizing the cost–benefit ratio. Thus a network with a large edge-tenacity performs better under

external attacks. In this sense, the following parameters are successively better for the measurement of stability.

- edge-connectivity $\lambda(G) = min\{|S| : S \subseteq E(G)$ is an edge set of $G\}$
- edge-integrity $I'(G) = min\{|S| + \tau(G - S) : S \subseteq E(G)\}$
- edge-toughness $t'(G) = \{min\frac{|S|}{\omega(G-S)} : S \subseteq E(G)$ is an edge set of $G\}$
- edge-tenacity $sc(S) = \frac{|S|+\tau(G-S)}{\omega(G-S)}$.

The edge-tenacity of a graph G is defined as $T'(G) = min\{sc(S)\}$, where the minimum is taken over all edge-sets S of G. A subset S of $E(G)$ is said to be a $T' - set$ of G if $T'(G) = sc(S)$. Aysun Aytaç, [1], in his paper gave the edge-tenacity of the middle graph of specific families of graphs and its relationships with other parameters.

DEFINITION 3.1. The middle graph $M(G)$ of a graph G is the graph obtained from G by inserting a new vertex into every edge of G and by joining by edges those pairs of these new vertices which lie on adjacent edges of G.

The definition of the endline graph of a graph is as follows. Let G be a graph and $V(G) = \{v_1, v_2, ..., v_n\}$. We add to G_n new vertices and n edges $\{u_i, v_i\}(i = 1, 2, ..., n)$, where u_i are different from any vertex of G and from each other. Then we obtain a new graph G' with $2n$ vertices, called the endline graph of G. Let us denote the line graph of a graph G by $L_(G)$. Then, from the definition of the endline graph and the middle graph of a graph, we have $L(G') \cong M(G)$. We saw the following results in section I.

THEOREM 3.1. If G is connected and $S \subseteq E$, then $sc(S) \geq 1$ with equality if and only if G is a tree and $S = E$.

COROLLARY 3.1. If G is connected, then $T'(G) \geq 1$ with equality if and only if G is a tree.

THEOREM 3.2. If G spans the subgraph of H, then $T'(G) \leq T'(H)$.

THEOREM 3.3. For any graph G, $T'(G) \geq \lambda/2 + 1/n$.

THEOREM 3.4. For all G, $T'(G) \leq (q + 1)/n$.

THEOREM 3.5. If G is r-regular and r-edge-connected, then $T'(G) = r/2 + 1/n$.

3.1 Relationships between the edge-tenacity and the edge-toughness and edge-integrity:

In this section, we consider relationships between the edge-tenacity and some other stability parameters, namely the edge-toughness and the edge-integrity.

Theorem 3.6. Let G be a connected graph such that $t'(G) = t'$, $I'(G) = I'$ and $T'(G) = T'$. Then $T' \geq t'I'/q$.

Proof. From the definition of $t'(G)$ and $I'(G)$

$t'I' = min\{\frac{|S|}{\omega(G-S)}[|S| + \tau(G - S)]\} = min\{\frac{|S|+\tau(G-S)}{\omega(G-S)}\}min\{|S|\} = T'min\{|S|\}$.

We know that $|S| < q$. It is easy to see that $T' \geq t'I'/q$. □

Theorem 3.7. Let G be a graph with n vertices and q edges such that $t'(G) = t'$, $I'(G) = I'$ and $T'(G) = T'$. Then $T' \leq t' - (1 - n)/2$.

Proof. From the definition of $t'(G)$ and $I'(G)$

$t'+I' = min\{\frac{|S|+\omega(G-S)+\omega(G-S)\tau(G-S)}{\omega(G-S)} = min\{\frac{|S|+\tau(G-S)}{\omega(G-S)}\}+min\{\frac{\omega(G-S)|S|+[\omega(G-S)-1]\tau(G-S)}{\omega(G-S)}\} = T' + min\{|S| + \tau(G - S)\} + min\{-\frac{\tau(G-S)}{\omega(G-S)}$.

We know that $2 \leq \omega(G-S) \leq n$ and $1 \leq \tau(G-S) \leq n-1$, and it is easy to see that

$$min\{-\frac{\tau(G-S)}{\omega(G-S)} \geq \frac{1-n}{2}$$

Therefore $t' \geq T' + (1-n)/2$. This implies$T' \leq t' - (1-n)/2$ □

Theorem 3.8. G is a connected graph with n vertices $I'(G) = I'$ and $T'(G) = T'$. Then $T' \geq I'/n$.

Proof. From the definition of I', we have

$$I' \leq |S| + \tau(G-S) \Rightarrow \frac{I'}{\omega(G-S)} \leq \frac{|S| + \tau(G-S)}{\omega(G-S)} \Rightarrow T' \geq \frac{I'}{\omega(G-S)}$$

We know that $\omega(G-S) \leq n$. Therefore $T' \geq I'/n$ □

3.2 The edge-tenacity of the middle graph of a graph:

In this section, we present some results for the edge-tenacity of the middle graph of specific graphs, i.e. part, cycle, complete, and star graphs with n vertices and q edges.

Theorem 3.9. Let $M(P_n)$ be the middle graph of P_n. Then

$$T'[M(P_n)] = 1 + \frac{(n-2)}{(2n-1)}$$

Proof. The number of vertices and the number of edges of graph $M(P_n)$ are $V[M(P_n)] = 2n-1$ and $E[M(P_n)] = 3n-4$, respectively. Initially, we observe that

$$\omega[M(P_n) - S] \leq |S| - [\lfloor \frac{|S|}{3} \rfloor - 1]$$

for every $S \subseteq E[M(P_n)]$. We claim that if S is a $T'-set$ of $M(P_n)$, then every component of $M(P_n) - S$ is K_1. We assume the contrary and arrive at a contradiction. If $M(P_n) - S$ contains a component K_2 then, defining $S' = S \cup \{(i, i+1) = e_i\}$, we have $S' = |S| + 1$,$\tau[M(P_n) - S'] \leq \tau[M(P_n) - S], \omega[M(P_n) - S'] = \omega[M(P_n) - S] + 1$. Then $sc(S') = \frac{|S'| + \tau[M(P_n) - S']}{\omega[M(P_n) - S']} \leq \frac{|S| + 1 + \tau[M(P_n) - S]}{\omega[M(P_n) - S] + 1} < \frac{|S| + \tau[M(P_n) - S]}{\omega[M(P_n) - S]} = sc(S) \Rightarrow sc(S') < sc(S)$. This contradicts the minimality of $sc(S)$.

Therefore $\tau[M(P_n) - S] = 1$. If $\tau[M(P_n) - S] = 1$, $|S| = E[M(P_n)] = 3n-4$ and $\omega[M(P_n) - S] = V[M(P_n)] = 2n-1$. Hence

$$T'[M(P_n)] = 1 + \frac{(n-2)}{(2n-1)}$$

□

Corollary 3.2. The relationship between the edge-tenacity of $M(P_n)$ and the edge-tenacity of P_n is $T'[M(P_n)] \leq T'(P_n) + T'(P_{n-1})$, i.e. $T'[M(P_n)] \leq 2$.

Proof We observe that the middle graph $M(P_n)$ consists of two parts, P_n and P_{n-1}. This is easy to see by using theorem 3.4.

Theorem 3.10. Let $M(C_n)$ be the middle graph of C_n. Then

$$T'[M(C_n)] = \frac{3n+1}{2n}$$

Proof. The $M(C_n)$ have $V[M(C_n)] = 2n$ vertices and $E[M(C_n)] = 3n$ edges. The proof of theorem 3.10 is similar to the proof of theorem 3.9. □

Corollary 3.3. The relationship between the edge-tenacity of $M(C_n)$ and the edge-tenacity of C_n is $T'[M(C_n)] < 2T'(C_n)$.

Theorem 3.11. Let $M(K_{1,n})$ be the middle graph of C_n. Then

$$T'[M(K_{1,n})] = \frac{2n+1}{n+1}, (n \geq 5)$$

Proof. The $M(K_1, n)$ graph has $V[M(K_1, n)] = 2n + 1$ vertices and $E[M(K_1, n)] = 2n + \left(\begin{array}{c} n \\ 2 \end{array} \right)$ edges. For $n \geq 5$, $\tau[M(K_1, n) - S] = |K_n| + 1$. The $+1$ comes from star graph because the graph K_n consists of n vertices and one vertex which is of highest degree in the star graph. All vertices of K_n are adjacent to this vertex. If $\tau[M(K_1, n) - S] = n+1$, then we remove $e_1, e_2, ..., e_n$ edges from $M(K_1, n)$. Therefore $|S| = n$ and $\omega[M(K_1, n) - S] = n + 1$, and it is easy to see that

$$T'[M(K_{1,n})] = \frac{2n+1}{n+1}$$

We assume the contrary and arrive at a contradiction as in the proof of theorem 3.9. If $\tau[M(K_1, n) - S'] = 1$, then

$$|S'| = 2n + \left(\begin{array}{c} n \\ 2 \end{array} \right)$$

and we have $V[M(K_1, n)] = 2n + 1$ components. Therefore

$$T'[M(K_{1,n})] = \frac{2n + \left(\begin{array}{c} n \\ 2 \end{array} \right) + 1}{2n+1}$$

We now consider the inequality

$$\frac{2n + \left(\begin{array}{c} n \\ 2 \end{array} \right) + 1}{2n+1} > \frac{2n+1}{n+1}$$

If we can demonstrate this inequality, we will arrive at a contradiction to the minimality of $sc(S)$. Suppose that above inequality is not true:

$$\frac{2n + \left(\begin{array}{c} n \\ 2 \end{array} \right) + 1}{2n+1} \leq \frac{2n+1}{n+1}$$

$$\frac{(n+2)(n+1)}{2(2n+1)} \leq \frac{2n+1}{n+1}$$

$$(n+1)^2 + (n+2) \leq 2(2n+1)^2$$

$$n^3 - 4n^2 - n - 2 \leq 0$$

$$(n+2)(n-1)^2 \leq 0$$

Since $n \geq 5$, the above inequality implies a contradiction. Therefore

$$\frac{2n + \binom{n}{2} + 1}{2n+1} > \frac{2n+1}{n+1}$$

This also contradicts the minimality of $sc(S)$. □

Theorem 3.12. If G is an $(n-1)$-regular graph with n vertices,

$$T'[M(G)] = \frac{n2 - 2n + 2}{n}$$

Corollary 3.4. If G is an $(n-1) - regular$ graph with n vertices, then

$$T'[M(G)] < T'(K_n) + T'(G')$$

where G' is a $2(n-2) - regular$ graph G' with $[n(n-1)/2]$ vertices.
By using theorems 3.5 and 3.11 we obtain the result.

Discussion: Many graph-theoretical parameters have been used in the past to describe the stability of communication networks. Most of these parameters do not take into account what remains after the graph is disconnected. In edge-tenacity (tenacity), the cost takes into account the size of the largest remaining component, since the larger the remaining component, the less successful is the attack. An attacker wishes to make the cost–benefit ratio as small as possible, whereas a network designer wishes to make this ratio as large as possible. We want to design a communication network such that when it begins to lose links (edges) or nodes (vertices), it maintains high stability. The number of the vertices of graphs $M(G)$ and graph G is the same. We can see that the middle graphs $M(G)$ have a higher stability than graphs G. Thus we must select the middle graph of a graph, especially $M(K_7)$, according to the edge-tenacity. In the above graphs, the edge-connectivities of the graph and its middle graph are the same, i.e. $\lambda(G) = \lambda[M(G)]$. However, the orders of their largest components are not equal. Therefore these two graphs must have different stabilities. How can we measure that property? Thus edge-tenacity is a better parameter for measuring the stability of a graph G.

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References

[1] Ayta, A., "On the edge-tenacity of the middle graph of a graph". Int. J. Comput. Math. 82 (2005), no. 5, 551-558.

[2] Bagga, K.S., Beineke, L. W., Lipman M.J., and Pippert, R.E., "On the edge-integrity of graphs", Congr. Numer. 6 (1987), 141-144.

[3] Barefoot, C. A., Entringer R., and Swart, H., "Integrity of trees and powers of cycles", Congr. Numer. 58 (1987), 103-114.

[4] Barefoot, C. A., Entringer R., and Swart, H., "Vulnerability in graphs-a comparative survey", J. Combin. Math. Combin. Comput. 1 (1987), 13-22.

[5] Bundy, J.A., Murty, U.S.R., "Graph Theory with Applications" (The Macmillan press Ltd, 1976).

[6] G. Chartrand and L. Lesniak, Graphs and Digraphs, 2nd ed. Wadsworth, Monterey, CA (1986).

[7] Chartrand, G., A graph-theoretical approach to a communication problem, J.Siam App. Math. 14 (1966) 778-781.

[8] Chartrand, G., Kapoor S.F., and Lick, D.r., "n-Hamiltonian graphs", J. Combin. Theory 9 (1970) 305-312.

[9] Choudum S. A., and Priya, N., "Tenacity of complete graph products and grids", Net-works 34 (1999), no. 3, 192-196.

[10] Choudum S. A., and Priya, N., "Tenacity-maximum graphs", J. Combin. Math. Combin.Comput. 37 (2001), 101-114. USA, Vol. 48.

[11] Chvátal, V., "Tough graphs and Hamiltonian circuits", Discrete Math.5 (1973), 215-228.

[12] Chvátal,V. and Erdös, P., "A note on Hamiltonian Circuits", Discrete Math. 2 (1972),111-113.

[13] Clark, L., Entringer R. C., and Fellows, M. R., "Computational complexity of integrity", Combin. Math. Combin. Comput. 2 (1987), 179-191.

[14] Cozzens, M.B, Moazzami, D., and Stueckle, S., "The tenacity of the Harary Graphs, J. Combin. Math. Combin. Comput. 16 (1994), 33-56.

[15] Cozzens, M.B, Moazzami, D., and Stueckle, S., "The tenacity of a graph", Graph Theory, Combinatorics, and Algorithms (Yousef Alavi and Allen Schwenk eds.) Wiley, New York, (1995), 1111 - 1122.

[16] Cozzens, M.B. and Wu, S.S., "Graphs that are n-edge connected and k-edge critical, Disc. Math., 1989.

[17] Cozzens, M.B., and Wu, s. Y., "On Minimum Critically n-edge-connected Graphs", SIAM, J. Alg. Disc. Math. Vol. 8, No. 4, 1987, 659-669.

[18] Cozzens, M.B. and Wu, s. Y. "Critical neighborhood connectivity", Ars Combinatoria. Vol.29 (1990), 144-160.

[19] Doty, L.L., "A large class of maximally tough graphs", OR Spektrum 13 (1991), 147-151.

[20] Enomoto, H., Jackson, B., Katerinis, P. and Saito, A., "Toughness and the existence of k-factors", J. Graph Theory 9 (1985), 87-95.

[21] Fellows, M.R. and Stueckle, S., "The immersion order, forbidden subgraphs and the complexity of integrity", J. Combin. Math. Combin. Comput.,6, (1989) 23-32.

[22] Goddard, W.D. and swart, H.C., "On the toughness of a graph", Quaestiones Math. 13 (1990), 217-232.

[23] Guichard, D.R., "Binding number of the Cartesian product of two cycles", Ars Combin. 19 (1985), 175-178.

[24] Harary, F. "The maximum connectivity of a graph", **Proc. Nat. Acad. Sci. U.S.A.**, 48 (1962), 1142 - 1146.

[25] Kane,V.G. and Mohanty, S.P., "Binding number, cycles and complete graphs", Lect. Notes in Math. (Combinatorics and Graph Theory, Proceedings, Calcutta 1980) 885, (S.B. Rao, Ed.), Springer, Berlin, (1981), 290-296.

[26] Kane, V. G., Mohanty S. P., and Hales, R. S., "Product graphs and binding number", Ars Combin. 11 (1981), 201-224.

[27] Kane, V. G., Mohanty S. P. and Straus, E. G., Which rational numbers are binding numbers?, J. Graph Theory 5 (1981), 379-384.

[28] Katerinis, P. and Woodall, D. R., Binding numbers of graphs and the existence of k-factors, Quart. J. Math., Oxford, Ser.(2) 38 (1987), 221-228.

[29] Lesniak, L.M. and Pippert, R.E., On the edge-connectivity vector of a graph, Networks, vol. 19, 1989, 667-671.

[30] Lipman, M.J. and Pippert, R.E., Toward a measure of vulnerability; the ratio of disruption, Graph Theory with Applications to Algorithms and Computer Science, (Y. Alavi, et al., Ed.), Wiley, New York, (1985), 507-517.

[31] Li, Y.K., Wang, Q.N., "Tenacity and the maximum network". Gongcheng Shuxue Xue-bao 25 (2008), no. 1, 138-142.

[32] Li, Y.K., Zhang, S.G., Li, X.L., Wu, Y., Relationships between tenacity and some other vulnerability parameters. Basic Sci. J. Text. Univ. 17 (2004), no. 1, 1-4.

[33] Liu, J. and Tian, S., The binding number of Cartesian products of n circuits, J. Shandong Coll. Ocean. 14 (1984), 97-101.

[34] Ma, J.L., Wang, Y.J., Li, X.L., Tenacity of the torus PnCm. (Chinese) Xibei Shifan Daxue Xuebao Ziran Kexue Ban 43 (2007), no. 3, 15-18.

[35] Moazzami, D., Vulnerability in Graphs - a Comparative Survey, J. Combin. Math. Combin. Comput. 30 (1999), 23-31.

[36] Moazzami, D., Stability Measure of a Graph - a Survey, Utilitas Mathematica, 57 (2000), 171-191.

[37] Moazzami, D., On Networks with Maximum Graphical Structure, Tenacity T and number of vertices p, J. Combin.Math. Combin. Comput. 39 (2001).

[38]

[39] Molluzzo, J.C., Toughness, Hamiltonian connectedness and n-Hamiltonicity, in Annals N.Y. Acad. Sci. 319, Proceeding of Second Int'l Conf. on Combin. Math., New York, (1979) (A. Gewirtz, et al., eds.), 402-404.

[40] Piazza, B., Roberts, F., Stueckle, S., Edge-tenacious networks, Networks 25 (1995), no. 1, 7-17.

[41] Piazza, B., Stueckle, S., A lower bound for edge-tenacity, Proceedings of the thirtieth Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1999) Congr. Numer. 137 (1999), 193-196.

[42] Pippert, R.E., On the toughness of a graph, Lect. Notes in Math. (Graph Theory and its Applications) 303, (Yousef. Alavi, et al., ed.), Springer, Berlin, (1972), 225-233.

[43] Pippert, R.E. and Lipman, M.J., Toward a measure of vulnerability, the edge connectivity vector, Graph Theory with Applications to Algorithms and Computer Science, (Y. Alavi, et al., Ed.) Wiley, New York, (1985), 651-657.

[44] Saito, A. and Tian, S.L., The binding number of Line Graphs and Total Graphs, Graphs Combin. 1 (1985), 351-356.

[45] Shi, R., The binding number of a graph and its pancyclicism, Acta Math. Appl. Sin. 3 (1987), 257-269.

[46] Tokushinge, N., Binding number and minimum degree for k-factors, J. Graph Theory 13 (1989), 607-617.

[47] Wang, J., Tian S. and Liu, J., The binding number of product graphs, Lect. Notes in Math.(Graph Theory, Singapore 1983) 1073, (K.M. Koh,et al., Ed.), Springer, Berlin, 1984, 119-128.

[48] Wang, J., Tian S., and Liu, J., The binding number of lexicographic products of graphs, Graph Theory with Applications to Algorithms and Computer Science, (Y. Alavi, et al., Ed.), Wiley, New York, (1985), 761-776.

[49] Wang, Z.P., Ren, G., Zhao, L.C., Edge-tenacity in graphs. J. Math. Res. Exposition 24 (2004), no. 3, 405-410.

[50] Wang, Z.P., Ren, G., A new parameter of studying the fault tolerance measure of communication networksa survey of vertex tenacity theory. (Chinese) Adv. Math. (China) 32 (2003), no. 6, 641-652.

[51] Wang, Z.P., Ren, G., Li, C.R., The tenacity of network graphsoptimization design. I.(Chinese) J. Liaoning Univ. Nat. Sci. 30 (2003), no. 4, 315-316.

[52] Wang, Z.P., Li, C.R., Ren, G., Zhao, L.C., Connectivity in graphsa comparative survey of tenacity and other parameters. (Chinese) J. Liaoning Univ. Nat. Sci. 29 (2002), no. 3, 237-240.

[53] Wang, Z.P., Li, C.R., Ren, G., Zhao, L.C., The tenacity and the structure of networks. (Chinese) J. Liaoning Univ. Nat. Sci. 28 (2001), no. 3, 206210.

[54] Woodall, D. R., The binding number of a graph and its Anderson number, J. Combin. Theory B 15 (1973), 225-255.

[55] Woodall,D.R., Problems 1 to 3, in Combinatorics (Proc. 1972 Oxford Combinatorial Conference) (D.J.Welsh and D.R. Woodall, eds.), Institute of Mathematics and Applications, Southend-Sea, Essex, England, 1972, 359-360.

[56] Woodall, D.R., Abstract No. 20 (The melting-point of a graph, and its Anderson number), Graph Theory News letter 1 (NO. 4) (1972).

[57] Wu, Y., Wei, X.S., Edge-tenacity of graphs. (Chinese) Gongcheng Shuxue Xuebao 21 (2004), no. 5, 704-708. Soc. Lectures in Appl. Math., 11 (1968), 335345